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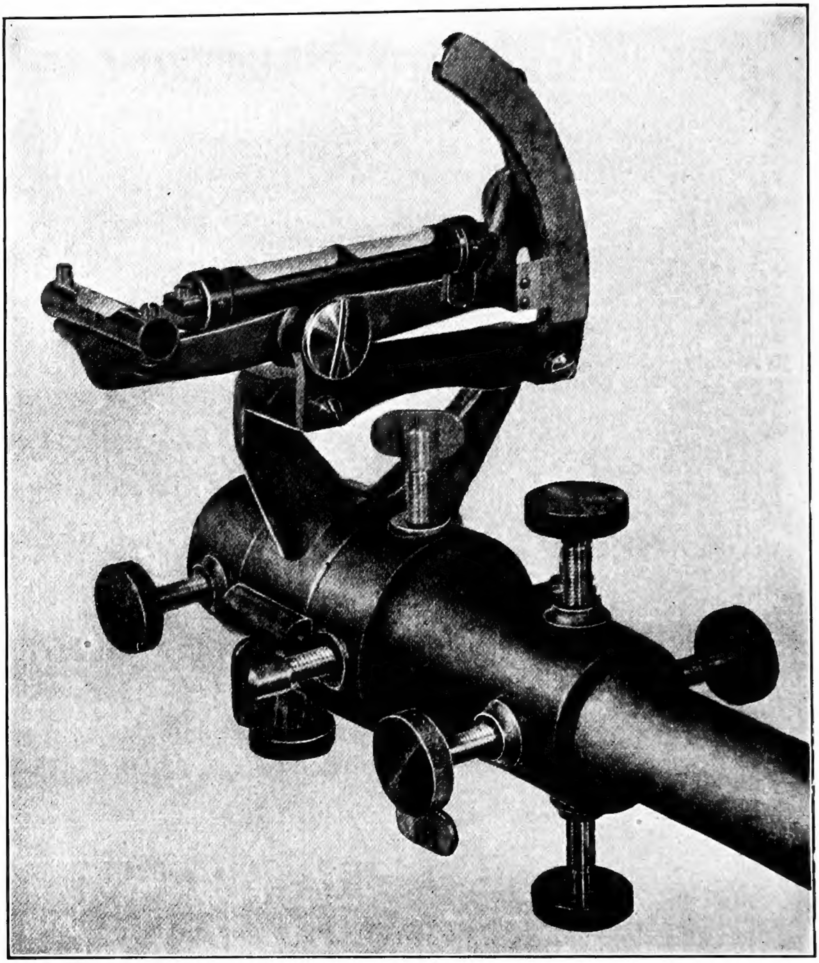

One of the two "**clinometers**" in use in the Testing Laboratory of the College of Civil Engineering at Cornell University (see p. 241). The main barrel or sleeve of the instrument encircles the horizontal shaft or rod (in testing machine) whose angle of torsion is to be obtained, near one extremity of the same. At each end of the barrel are four brass screws having smooth rounded ends where they bear on the shaft. These are used for centering the barrel on the shaft, but do not grip it. The four steel "gripping screws," at the middle of the barrel, are thumb-screws with flat heads and hardened sharp points. They serve to grip the shaft after the centering is completed. After the shaft has thus been gripped at a certain transverse section, the collar carrying the graduated arc is clamped upon the barrel, the plane of the arc and its vernier arm being that of the points of the gripping screws. By taking a reading of the vernier on the arc at any stage of the test (the vernier-arm being adjusted each time so that the bubble of the spirit level carried by this arm is brought to the center of its scale) and subtracting its initial reading, the angle through which the transverse section has turned from its initial position becomes known. The second clinometer is placed at another transverse section, near the other end of the shaft, and serves to measure its turning movement. The difference of these movements is the angle of torsion. The verniers read to single minutes. (The shaft in above figure is 1½ in. in diameter).

Frontispiece.

Mechanics of Engineering

[SOLIDS]

COMPRISING
STATICS AND KINETICS OF SOLIDS, AND THE MECHANICS OF THE MATERIALS OF CONSTRUCTION, OR STRENGTH AND ELASTICITY OF BEAMS, COLUMNS, SHAFTS, ARCHES, ETC.

FOR USE IN TECHNICAL SCHOOLS

BY

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PREFACE

In presenting a revised edition of Parts I, II, and III, of his "Mechanics of Engineering," the writer would call attention to the principal changes that have been made; omissions as well as additions.

The chapter on "Continuous Girders by Graphics" has been omitted in its entirety, while the graphic treatment of the horizontal straight girder, formerly a part of the chapter on "Arch Ribs," has been removed to the Appendix, in which will also be found various paragraphs involving special problems in flexure, once located in the body of the book. Former chapters V and VI in Part III, on beams under oblique forces and on columns, respectively, have been merged in one (Chapter VI), the matter having been largely rewritten and more fully illustrated, with introduction of the more modern formulæ for columns and some treatment of the problem of eccentric loading.

Chapter V in Part III of the revised book, on "Flexure of Reinforced Concrete Beams," is entirely new and presents both theory and numerical illustrations; as also diagrams aiding in practical design. New matter will also be found in an analytical treatment of "Circular Ribs and Hoops," placed at the end of the chapter on "Arch Ribs." Two other new chapters in Part III, are XII, on the flexure of beams treated by a geometrical method (which, however, does not call for the use of drafting instruments) leading to a very simple and available form of the Theorem of Three Moments; and XIII, which gives the analysis of stresses in thick hollow cylinders and spheres. A few pages on the strength of plates have also been added in Chapter III.

Grateful acknowledgment is again due to Dr. H. T. Eddy for the use of his methods * in treating arch ribs.

In Parts I and II numerous additional examples and illustrations are introduced, while many pages have been rewritten throughout the book, aside from the new chapters already referred to. Tables of logarithms, trigonometric functions, and hyperbolic sines and cosines, will be found in the Appendix.

CORNELL UNIVERSITY, ITHACA, N. Y.
May, 1911.

NOTE.—Additional matter involving many examples and forming an appendix to the present work, but too bulky to be incorporated with it, was issued in a separate volume in 1892 and entitled "*Notes and Examples in Mechanics.*" A second edition, revised and enlarged, was published in 1897.

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* See pp. 14 and 25 of "*Researches in Graphical Statics,*" by Prof. H. T. Eddy, C.E., Ph.D., published by D. Van Nostrand, New York, 1878, reprinted from Van Nostrand's Magazine for 1877; or the German translation of the same, "*Neue Constructionen aus der Graphischen Statik,*" published by Teubner u. Cie., Leipsic, 1880.

INTRODUCTORY NOTES.

Preparation.—Prior to the use of this book the student is supposed to have had the usual training given in technical schools in analytical geometry and in the differential and integral calculus; and also a year of college physics.

Gravitation Measure of a Force. Mass and Weight.—Since the gravitation measure of a force is the one almost exclusively used by engineers, a brief résumé of its nature is here given, aside from the paragraph of p. 835, Appendix.

The amount of matter in a certain piece of platinum, kept by the British government, is called by the physicist a pound of *mass*, but the engineer understands by the word "pound" the *force* of gravitation, or weight, exerted by the earth on this piece of metal at London; and if this piece of metal be supported, at London, by a spring balance, the scale of which is so graduated that the pointer now stands at unity, such a balance constitutes a standard instrument with which to measure forces for the purposes of the engineer. According to the indications of such an instrument the same piece of metal, if suspended on the same balance at the equator, at sea-level, would be found to weigh only 0.997 lbs. (force) on account of the diminished intensity of gravitation; the difference, however, being only about three parts in a thousand, or one-third of one per cent. For ordinary engineering problems involving the strength of structures, this difference is of no practical importance.

A unit of force based on this gravitation method is called a *gravitation measure of force*. The *mass* of the piece of platinum, has, of course, suffered no change in the transit from London to the equator, and since the fraction obtained by dividing the weight (obtained from the spring balance) by the acceleration of gravity, g, is constant, regardless of the place where the two quantities are measured, it is convenient (though not essential) for the engineer to give the name "mass" to this fraction when it occurs in the equations of kinetics. For instance, since g (for foot and second) $=32.18$ at London, and 32.09 at the equator (at sea-level), we note that $\frac{1.000}{32.18}=\frac{0.997}{32.09}$ $=0.03108$.

Arithmetic.—In arithmetical operations the student should remember that the degree of refinement attained or employed does not depend on the number of decimal places used, but upon the number of *significant figures*. Thus, each of the quantities 0.0003674 and 510.4 contains four significant figures. For instance, let us suppose that the value of x is to be obtained from the relation $x=a-b$, where $a=0.0000568$ and $b=0.0000421$. Should the student conclude that five decimal places would be accurate enough and thus write 0.00005 for a, and 0.00004 for b, he would obtain $x=0.00001$, containing only one significant figure; whereas the true result is $x=0.0000147$. Hence the former result is seen to be in error to the extent of 47 parts in 147, or 32 parts in 100, i.e., 32 per cent.; which is a very gross and totally unnecessary error. Values obtained from the ordinary 10-inch slide rule usually contain only three significant figures (four if near left of scale).

Logarithms.—The following facts and operations are not usually fresh in the student's mind. The logarithm of a number less than unity is a negative quantity but is usually expressed as the algebraic sum of a positive mantissa (or decimal part) and a negative characteristic which is a whole number; thus, the common logarithm of 0.20 is $\bar{1}.301030\ldots$, that is, $\log. 0.20=+0.301030-1.000000$ (or, $+9.301030-10$). This should be borne in mind in raising such a number to any power. For example: required the value of $x=\left(\frac{11}{13}\right)^{0.71}$.

Solution.—$\frac{11}{13}=0.8461$ and $\log. 0.8461=\bar{1}.9274$, i.e., $=0.9274-1.0000$.

$$\text{Hence}\quad 0.71\times\log. 0.8461=0.71(0.9274-1.0000),=0.6584-0.7100,$$
$$=-0.0516=\bar{1}.9484=\log. 0.8880;\ \text{therefore}\ x=0.8880.$$

Note that, according to the definition of a logarithm, the statement $en=m$ is equivalent to the statement $n=\log_e m$.

MATHEMATICAL DATA.

Trigonometry. $\cos^2 A + \sin^2 A = 1.$ $\quad 2\sin^2 A = 1 - \cos 2A.$

$\cos^2 A - \sin^2 A = \cos 2A \quad 2\cos^2 A = 1 + \cos 2A.$

$\sin 2A = 2\sin A \cos A$

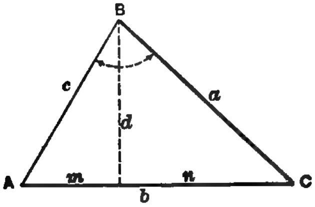
$\cos 2A = \cos^2 A - \sin^2 A$

$\cot \frac{1}{2}A = \frac{\sin A}{1 - \cos A}$

$$\sin A = \frac{\tan A}{\sqrt{1+\tan^2 A}} = \frac{1}{\operatorname{cosec} A}$$

$$\cos A = \frac{1}{\sqrt{1+\tan^2 A}} = \frac{1}{\sec A}$$

Solution of Oblique Triangles, etc.



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

$d = a \sin C \qquad d = c \sin A$

$m = c \cos A \qquad n = a \cos C$

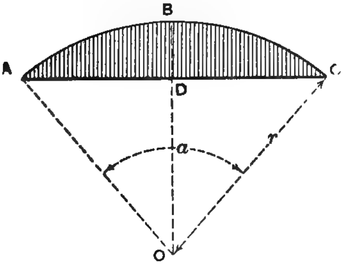
$d = m \tan A \qquad d = n \tan C$

Given a, b, B; to find A: $\quad \sin A = \frac{a \sin B}{b}$

" a, b, C; " A: $\quad \tan A = \frac{a \sin C}{b - a \cos C}$

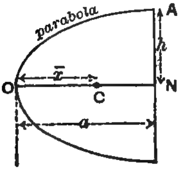
" a, b, C; " c: $\quad c^2 = a^2 + b^2 - 2ab \cos C$

" a, b, c: " C: $\quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$



Mensuration. Area of a circle $= \pi r^2$; circumference $= 2\pi r$. Area of *sector* $ABCOA = \left(\frac{\alpha^\circ}{360^\circ}\right)\pi r^2$; $= \left(\frac{\alpha}{2\pi}\right)\pi r^2$; (the latter α in *radians*). Vol. of sphere $= \frac{4}{3}\pi r^3$. Area of the *segment*, $ABCDA$, of a circle,

$=$ (area of sector $ABCOA$) $-$ (area of triangle ACO).



Area of rightsegment of a *parabola* = two-thirds that of circumscribing rectangle, $= \frac{2}{3}(2ha)$. Equation to curve OA is $\frac{y^2}{h^2} = \frac{x}{a}$. Distance $\overline{OC}$, of center of gravity, is $\bar{x} = \frac{3}{5}a$, from vertex O.

Integral Forms.—(Each integral to be taken between limits, or to have a constant added and determined). (See also p. 480.)

$$\int x^n dx = \frac{x^{n+1}}{n+1}; \qquad \int \frac{dx}{x} = \log_e x; \qquad \int e^x dx = e^x;$$

$$\int \cos x \, dx = \sin x; \qquad \int \sin x \, dx = -\cos x; \qquad \int \frac{dx}{\cos^2 x} = \tan x;$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x; \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x; \qquad \int \frac{dx}{\sqrt{2x-x^2}} = \text{vers} \sin^{-1} x;$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log_e (x + \sqrt{x^2 \pm a^2}); \qquad \int \frac{dx}{\sqrt{a+bx-cx^2}} = \frac{1}{\sqrt{c}} \sin^{-1} \left(\frac{2cx-b}{\sqrt{b^2+4ac}} \right);$$

$$\int \frac{dx}{a-bx^2} = \frac{1}{2\sqrt{ab}} \log_e \frac{\sqrt{ab}+bx}{\sqrt{ab}-bx}; \qquad \int \frac{xdx}{a-bx^2} = -\frac{1}{2b} \log_e (a-bx^2).$$

Numerical Constants.—The acceleration of gravity, g, (for the English foot and second) is 32.16 for the latitude of Philadelphia at sea-level, and for any latitude β, and elevation h above sea-level, is

$$32.1723 - 0.0833 \cos 2\beta - 0.000003h.$$

For ordinary problems in mechanics, however, in the northern United States g may be taken as 32.2, for which value we have

$$\sqrt{2g} = 8.025; \quad \frac{1}{g} = 0.03105; \quad \text{and} \quad \frac{1}{2g} = 0.01553.$$

The ratio $\pi = 3.141592$, or approx. $3\frac{1}{7}$, i.e., $\frac{22}{7}$;

$$\frac{1}{\pi} = 0.31831; \quad \pi^2 = 9.86960; \quad \frac{1}{\pi^2} = 0.10132; \quad \sqrt{\pi} = 1.77245.$$

1° = 0.01745 radians. One radian = 57° 17′ 44.8″.

If n denote any number, then

$\text{Log}_{10}(n) = 0.43429 \times \log_e (n)$; and $\log_e (n) = 2.30258 \times \log_{10} (n)$.

Base of nat. logs. $= e$, $= 2.71828$; base of Briggs system = 10.

GREEK ALPHABET.

Letters.	Names.	Letters.	Names.
Α α	Alpha	Ν ν	Nu
Β β	Bēta	Ξ ξ	Xi
Γ γ	Gamma	Ο ο	Omicron
Δ δ	Delta	Π π	Pi
Ε ϵ	Epsilon	Ρ ρ	Rho
Ζ ζ	Zēta	Σ σ ς	Sigma
Η η	Eta	Τ τ	Tau
Θ ϑ θ	Thēta	Υ υ	Upsilon
Ι ι	Iōta	Φ φ	Phi
Κ κ	Kappa	Χ χ	Chi
Λ λ	Lambda	Ψ ψ	Psi
Μ μ	Mu	Ω ω	Omega

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MECHANICS OF ENGINEERING.

PRELIMINARY CHAPTER.

1. Mechanics treats of the mutual actions and relative motions of material bodies, solid, liquid, and gaseous; and by *Mechanics of Engineering* is meant a presentment of those principles of pure mechanics, and their applications, which are of special service in engineering problems.

2. Kinds of Quantity.—Mechanics involves the following fundamental kinds of quantity: **Space,** of one, two, or three dimensions, i.e., length, surface, or volume, respectively; **time,** which needs no definition here; **force** and **mass,** as defined below; and **abstract numbers,** whose values are independent of arbitrary units, as, for example, a ratio.

3. Force.—A force is one of a pair of equal, opposite, and simultaneous actions between two bodies, by which the state* of their motions is altered or a change of form in the bodies themselves is effected. Pressure, attraction, repulsion, and traction are instances in point. Muscular sensation conveys the idea of force, while a spring-balance gives an absolute measure of it, a beam-balance only a relative measure. In accordance with Newton's third law of motion, that action and reaction are equal, opposite, and simultaneous, forces always occur in pairs; thus, if a pressure of 40 lbs. exists between bodies *A* and *B*, if *A* is considered by itself (i.e., "free"), apart from all other bodies whose actions upon it are called forces, among these forces will be one of 40 lbs. directed from *B* toward *A*. Similarly, if *B* is under consideration, a force of 40 lbs. directed from A toward B takes its place among the forces acting on B. This is the interpretation of Newton's third law.

---

* The state of motion of a small body under the action of no force, or of balanced forces, is either absolute rest, or uniform motion in a right line. If the motion is different from this, the fact is due to the action of an unbalanced force (§ 54).

[NOTE.—In some common phrases, such as "The tremendous force" o. a heavy body in rapid motion, the word force is not used in a technical sense, but signifies energy (as explained in Chap. VI.). The mere fact that a body is in motion, whatever its mass and velocity, does not imply that it is under the action of any force, necessarily. For instance, at any point in the path of a cannon ball through the air, the only forces acting on it are the resistance of the air and the attraction of the earth, the latter having a vertica and downward direction.]

4. **Mass** is the quantity of matter in a body. The masses of several bodies being proportional to their weights at the same locality on the earth's surface, in physics the weight is taken as the mass, but in practical engineering another mode is used for measuring it (as explained in a subsequent chapter), viz.: the mass of a body is equal to its weight divided by the acceleration of gravity in the locality where the weight is taken, or, symbolically, $M = G \div g$. This quotient is a constant quantity, as it should be, since the mass of a body is invariable wherever the body be carried.

5. **Derived Quantities.**—All kinds of quantity besides the fundamental ones just mentioned are compounds of the latter, formed by multiplication or division, such as velocity, acceleration, momentum, work, energy, moment, power, and force-distribution. Some of these are merely names given for convenience to certain combinations of factors which come together not in dealing with first principles, but as a result of common algebraic transformations.

6. **Homogeneous Equations** are those of such a form that they are true for any arbitrary system of units, and in which all terms combined by algebraic addition are of the same kind. Thus, the equation $s = \frac{gt^2}{2}$ (in which $g =$ the acceleration of gravity and t the time of vertical fall of a body in vacuo, from rest) will give the distance fallen through, s, whatever units be adopted for measuring time and distance. But if for

g we write the numerical value 32.2, which it assumes when time is measured in seconds and distance in feet, the equation $s = 16.1t^2$ is true for those units alone, and the equation is not of homogeneous form. Algebraic combination of homogeneous equations should always produce homogeneous equations; if not, some error has been made in the algebraic work. If any equation derived or proposed for practical use is not homogeneous, an explicit statement should be made in the context as to the proper units to be employed.

7. Heaviness.—By heaviness of a substance is meant the weight of a cubic unit of the substance. E.g. the heaviness of fresh water is 62.5, in case the unit of force is the pound, and the foot the unit of space; i.e., a cubic foot of fresh water weighs 62.5 lbs.* In case the substance is not uniform in composition, the heaviness varies from point to point. If the weight of a homogeneous body be denoted by G, its volume by V, and the heaviness of its substance by γ, then $G = V\gamma$.

WEIGHT IN POUNDS OF A CUBIC FOOT (i.e., THE HEAVINESS) OF VARIOUS MATERIALS.

Anthracite, solid	100
“ broken	57
Brick, common hard	125
“ soft	100
Brick-work, common	112
Concrete	125
Earth, loose	72
“ as mud	102
Granite	164 to 172
Ice	58
Iron, cast	450
“ wrought	480
Masonry, dry rubble	138
“ dressed granite or limestone	165
Mortar	100
Petroleum	55
Snow	7
“ wet	15 to 50
Steel	490
Timber	25 to 60
Water, fresh	62.5
“ sea	64.0

8. Specific Gravity is the ratio of the heaviness of a material to that of water, and is therefore an abstract number.

9. A Material Point is a solid body, or small particle, whose dimensions are practically nothing, compared with its range of motion.

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* Or, we may write 62.5 lbs./cub. ft.; or 62.5 lbs./ft.^{3}

10. A Rigid Body is a solid, whose distortion or change of form under any system of forces to be brought upon it in practice is, for certain purposes, insensible.

11. Equilibrium.—When a system of forces applied to a body produces the same effect as if no force acted, so far as the *state of motion* of the body is concerned, they are said to be balanced, or to be in equilibrium. [If no force acts on a material point it remains at rest if already at rest; but if already in motion it continues in motion, and *uniformly* (equal spaces in equal times), in a right line in direction of its original motion. See § 54.]

12. Division of the Subject.—*Statics* will treat of bodies at rest, i.e., of balanced forces or equilibrium; *kinetics*, of bodies in motion; *strength of materials* will treat of the effect of forces in distorting bodies; *hydraulics*, of the mechanics of liquids and gases (thus including *pneumatics*).

13. Parallelogram of Forces.—Duchayla's Proof. To fully determine a force we must have given its amount, its direction, and its point of application in the body. It is generally denoted in diagrams by an arrow. It is a matter of experience that besides the point of application already spoken of any other may be chosen in the line of action of the force. This is called the transmissibility of force; i.e., so far as the *state of motion* of the body is concerned, a force may be applied anywhere in its line of action.

The **Resultant** of two forces (called its components) applied at a point of a body is a single force applied at the same point, which will replace them. To prove that this resultant is given in amount and position by the diagonal of the parallelogram formed on the two given forces (conceived as laid off to some scale, so many pounds to the inch, say), Duchayla's method requires four postulates, viz.: (1) the resultant of two forces must lie in the same plane with them; (2) the resultant of two equal forces must bisect the angle between them; (3) if one of the two forces be increased, the angle between the other force and the resultant will be greater than before; and (4) the transmissibility of force, already mentioned. Granting these, we proceed as follows (Fig. 1): Given the two forces P and $Q =$

$P' + P''$ (P' and P'' being each equal to P, so that $Q = 2P$), applied at O. Transmit P'' to A. Draw the parallelograms OB and AD; OD will also be a parallelogram. By postulate (2), since OB is a rhombus, P and P' at O may be replaced by a single force R' acting through B. Transmit R' to B and replace it by P and P'. Transmit P from B to A, P' from B to D. Similarly P and P'', at A, may be replaced by a single force R'' passing through D; transmit it there and resolve it into P and P''. P' is already at D. Hence P and $P' + P''$, acting at D, are equivalent to P and $P' + P''$ acting at O, in their respective directions. Therefore the resultant of P and $P' + P''$ must lie in the line OD, the diagonal of the parallelogram formed on P and $Q = 2P$ at O. Similarly

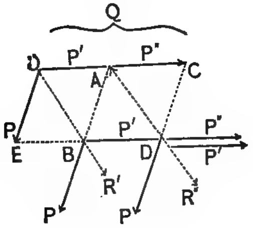

FIG. 1.

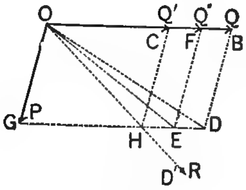

FIG. 2.

this may be proved (that the diagonal gives the *direction* of the resultant) for any two forces P and mP; and for any two forces nP and mP, m and n being any two whole numbers, i.e., for any two commensurable forces. When the forces are incommensurable (Fig. 2), P and Q being the given forces, we may use a *reductio ad absurdum*, thus: Form the parallelogram OD on P and Q applied at O. Suppose for an instant that R the resultant of P and Q does not follow the diagonal OD, but some other direction, as OD'. Note the intersection H, and draw HC parallel to DB. Divide P into equal parts, each less than HD; then in laying off parts equal to these from O along OB, a point of division will come at some point F between C and B. Complete the parallelogram $OFEG$. The force $Q'' = OF$ is commensurable with P, and hence their resultant acts along OE. Now Q is greater than Q'', while R makes a less angle with P than OE, which is contrary to postulate (3); therefore R cannot lie outside of the line OD. Q. E. D.

It still remains to prove that the resultant is represented in *amount*, as well as position, by the diagonal. OD (Fig. 3) is the direction of R the resultant of P and Q; required its amount. If R' be a force equal and opposite to R it will balance P and Q; i.e, the resultant of R' and P must lie in the line QO prolonged (besides being equal to Q). We can therefore determine R' by drawing BA parallel to DO to intersect QO prolonged in A; and then complete the parallelogram BF on BO and BA as sides. Since $OFAB$ and $AODB$ are parallelograms, $\overline{OF}$ must $=\overline{BA}$ and $\overline{BA}$ must $=\overline{OD}$. Hence OF and OD are equal and lie on the same right line. Evidently if R' were any shorter or any longer than OF the resultant of it and $\overline{OB}(=P)$ would not take the direction QOA. Hence R' must $=\overline{OF}$, i.e., $=\overline{OD}$; and hence $R=\overline{OD}$ in *amount*. Q. E. D.

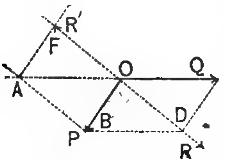

FIG. 3.

Corollary.—The resultant of three forces applied at the same point is the diagonal of the parallelopiped formed on the three forces.

14. Concurrent forces are those whose lines of action intersect in a common point, while **non-concurrent** forces are those which do not so intersect; results obtained for a system of concurrent forces are really derivable, as particular cases, from those pertaining to a system of non-concurrent forces.

15. Resultant.—A single force, the action of which, as regards the *state of motion* of the body acted on, is equivalent to that of a number of forces forming a system, is said to be the **Resultant** of that system, and may replace the system; and conversely a force which is equal and opposite to the resultant of a system will balance that system, or, in other words, when it is combined with that system there will result a new system in equilibrium; this (ideal) force is called the **Anti-resultant.**

In general, as will be seen, a given system of forces can always be replaced by two single forces, but these two can be combined into a single resultant only in particular cases.

15a. Equivalent Systems are those which may be replaced by the same set of two single forces—or, in other words, those which have the same effect, as to state of motion, upon the given body.

15b. Formulæ.—If in Fig. 3 the forces P and Q and the angle $\alpha = POQ$ are given, we have, for the resultant,

$$R = \overline{OD} = \sqrt{P^2 + Q^2 + 2\,PQ\cos\alpha}.$$

(If α is $> 90°$ its cosine is negative.) In general, given any three parts of either plane triangle ODQ, or ODB, the other three may be obtained by ordinary trigonometry. Evidently if $\alpha = 0$, $R = P + Q$; if $\alpha = 180°$, $R = P - Q$; and if $\alpha = 90°$, $R = \sqrt{P^2 + Q^2}$.

15c. Varieties of Forces.—Great care should be used in deciding what may properly be called forces. The latter may be divided into actions *by contact*, and actions *at a distance*. If pressure exists between two bodies and they are perfectly smooth at the surface of contact, the *pressure* (or *thrust*, or *compressive action*), of one against the other constitutes a force, whose direction is normal to the tangent plane at any point of contact (a matter of experience).; while if those surfaces are not smooth there may also exist mutual tangential actions or *friction*. (If the bodies really form a continuous substance at the surface considered, these tangential actions are called *shearing forces*.) Again, when a rod or wire is subjected to tension, any portion of it is said to exert a *pull* or *tensile force* upon the remainder; the ability to do this depends on the property of cohesion. The foregoing are examples of actions by contact.

Actions at a distance are exemplified in the mysterious *attractions*, or *repulsions*, observable in the phenomena of gravitation, electricity, and magnetism, where the bodies concerned are not necessarily in contact. By the term *weight* we shall always mean the *force* of the earth's attraction on the body in question, and not the amount of matter in it.

15d. Example 1.—If $\overline{OD}, = R,$ is given, $= 40$ lbs., while the angle BOD is $110°$ and $QOD = 40°$ (also $= ODB$), find the components P and Q.

Solution.—From the triangle BOD, $\overline{OB}:\overline{OD}::\sin 40°:\sin 30°$; whence

$$P, \text{ or } \overline{OB}, = (40 \times 0.6428) \div 0.5000 = 51.42 \text{ lbs.}$$

Similarly, from triangle BOD, we have $\overline{BD}:\overline{OD}::\sin 110°:\sin 30°$,

$$\therefore\ Q, \text{ or } \overline{BD}, = (40 \times 0.9397) \div 0.05000 = 75.17 \text{ lbs.}$$

Example 2.—Given $P = 20$ lbs., $Q = 30$ lbs., and angle $\alpha (= POQ), = 115°$, find the resultant R in amount and direction. *As to amount*

$$R = \sqrt{(20)^2 + (30) + 2 \times 20 \times 30 \times (-0.4226)} = \sqrt{792.88} = 28.16 \text{ lbs.}$$

As to direction, let β denote the angle $ODB, = QOD$; we then have, from triangle OBD, $20:28.16::\sin\beta:\sin 65°$; whence, solving,

$$\sin\beta = (20 \times 0.9063) \div 28.16 = 0.6437; \ \textit{i.e.}, \text{ angle } \beta = 40°\,4'.$$

PART I.—STATICS.

CHAPTER I.

STATICS OF A MATERIAL POINT.

16. Composition of Concurrent Forces.—A system of forces acting on a material point is necessarily composed of concurrent forces.

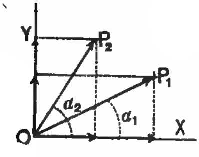

Fig. 4.

Case I.—All the forces in **One Plane.** Let O be the material point, the common point of application of all the forces; P_1, P_2, etc., the given forces, making angles α_1, α_2, etc., with the axis X. By the parallelogram of forces P_1 may be resolved into and replaced by its components, $P_1 \cos \alpha_1$ acting along X, and $P_1 \sin \alpha_1$ along Y. Similarly all the remaining forces may be replaced by their X and Y components. We have now a new system, the equivalent of that first given, consisting of a set of X forces, having the same line of application (axis X), and a set of Y forces, all acting in the line Y. The resultant of the X forces being their algebraic sum (denoted by ΣX) (since they have the same line of application) we have

$$\Sigma X = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \text{etc.} = \Sigma(P \cos \alpha),$$

and similarly

$$\Sigma Y = P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \text{etc.} = \Sigma(P \sin \alpha).$$

These two forces, ΣX and ΣY, may be combined by the parallelogram of forces, giving $R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$ as the single resultant of the whole system, and its direction is determined by the angle α; thus, $\tan \alpha = \frac{\Sigma Y}{\Sigma X}$; see Fig. 5. For equilibrium to exist, R must $= 0$, which requires, *separately*,

$\Sigma X = 0$, and $\Sigma Y = 0$ (for the two squares $(\Sigma X)^2$ and $(\Sigma Y)^2$ can neither of them be negative quantities).

CASE II.—The forces having any directions in space, but all applied at O, the material point. Let P_1, P_2, etc., be the given forces, P_1 making the angles α_1, β_1, and γ_1, respectively, with three arbitrary axes, X, Y, and Z (Fig. 6), at right angles to each other and intersecting at O, the origin. Similarly let α_2, β_2, γ_2, be the angles made by P_2 with these axes, and so on for all the forces. By the parallelopiped of forces, P_1 may be replaced by its components.

$X_1 = P_1 \cos \alpha_1$, $Y_1 = P_1 \cos \beta_1$, and $Z_1 = P_1 \cos \gamma_1$; and similarly for all the forces, so that the entire system is now replaced by the three forces,

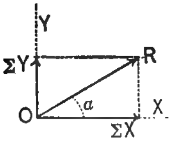

FIG. 5.

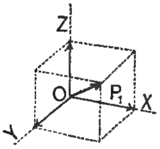

FIG. 6.

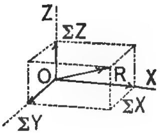

FIG. 7.

$$\Sigma X = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \text{etc};$$
$$\Sigma Y = P_1 \cos \beta_1 + P_2 \cos \beta_2 + \text{etc};$$
$$\Sigma Z = P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \text{etc};$$

and finally by the single resultant

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}.$$

Therefore, for equilibrium we must have **separately**,

$$\Sigma X = 0,\ \Sigma Y = 0, \text{ and } \Sigma Z = 0.$$

R's position may be determined by its direction cosines, viz.,

$$\cos \alpha = \frac{\Sigma X}{R};\ \cos \beta = \frac{\Sigma Y}{R};\ \cos \gamma = \frac{\Sigma Z}{R}.$$

17. Conditions of Equilibrium.—Evidently, in dealing with a system of concurrent forces, it would be a simple matter to replace any two of the forces by their resultant (diagonal formed on them), then to combine this resultant with a third force, and so on until all the forces had been combined, the last resultant being the resultant of the whole system. The foregoing treatment, however, is useful in showing that for equilibrium of concurrent forces in a plane only two conditions are necessary, viz., $\Sigma X = 0$ and $\Sigma Y = 0$; while in space there are three, $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma Z = 0$. In Case I., then, we have conditions enough for determining two unknown quantities; in Case II., three.

18. Problems involving equilibrium of concurrent forces. (A rigid body in equilibrium under no more than three forces may be treated as a material point, since the (two or) three forces are necessarily concurrent.)*

Problem 1.—A body weighing G lbs. rests on a horizontal table: required the pressure between it and the table. Fig. 8. Consider the body **free**, i.e., conceive all other bodies removed (the table in this instance), being replaced by the forces which they exert on the first body. Taking the axis Y vertical and positive upward, and not assuming in advance either the amount or direction of N, the pressure of the table against the body, but knowing that G, the action of the earth on the body, is vertical and downward, we have here a system of concurrent forces in equilibrium, in which the X and Y components of G are known (being 0 and $-G$ respectively), while those, N_X and N_Y, of N are unknown. Putting $\Sigma X = 0$, we have $N_X + 0 = 0$; i.e., N has no horizontal component, $\therefore N$ is vertical. Putting $\Sigma Y = 0$, we have $N_Y - G = 0$, $\therefore N_Y = +G$; or the vertical component of N, i.e., N itself, is positive (upward in this case), and is numerically equal to G.

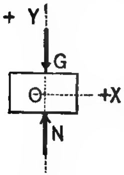

Fig. 8.

Problem 2.—Fig. 9. A body of weight G (lbs.) is moving in a straight line over a rough horizontal table with a uniform velocity v (feet per second) to the right. The tension in an oblique cord by which it is pulled is given, and $= P$ (lbs.), which remains constant, the cord making a given angle of elevation, α, with the path of the body. Required the vertical pressure N (lbs.) of the table, and also its horizontal action F (friction) (lbs.) against the body

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* Three parallel forces form an exception; see §§ 20, 21, etc.

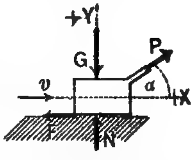

FIG. 9.

Referring by anticipation to Newton's first law of motion, viz., a material point acted on by no force or by balanced forces is either at rest or moving uniformly in a straight line, we see that this problem is a case of balanced forces, i.e., of equilibrium. Since there are only two unknown quantities, N and F, we may determine them by the two equations of Case I., taking the axes X and Y as before. Here let us leave the *direction* of N as well as its amount to be determined by the analysis. As F must evidently point toward the left, treat it as negative in summing the X components; the analysis, therefore, can be expected to give only its numerical value.

$\Sigma X = 0$ gives $P \cos \alpha - F = 0$. $\therefore$ $F = P \cos \alpha$.

$\Sigma Y = 0$ gives $N + P \sin \alpha - G = 0$. $\therefore$ $N = G - P \sin \alpha$.

$\therefore$ N is upward or downward according as G is $>$ or $< P \sin \alpha$. For N to be a downward pressure upon the body would require the surface of the table to be above it. The ratio of the friction F to the pressure N which produces it can now be obtained, and is called the "*coefficient of friction.*" It may vary somewhat with the velocity. (See p. 168.)

This problem may be looked upon as arising from an experiment made to determine the coefficient of friction between the given surfaces at the given uniform velocity.

19. The Free-Body Method.—The foregoing rather labored solutions of very simple problems have been made such to illustrate what may be called the "*free-body method*" of treating any problem involving a body acted on by a system of forces. It consists in conceiving the body isolated from all others which act * on it in any way, those actions being introduced as so many forces known or unknown, in amount and position. The system of forces thus formed may be made to yield certain equations, whose character and number depend on circumstances, such as the behavior of the body, whether the forces are confined to a plane or not, etc., and which are therefore theoretically available for determining an equal number of unknown quantities.

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* That is, in any "*force*-able" way.

10a. Examples.—1. A cast-iron cylinder, with axis horizontal, rests against two *smooth* inclined surfaces, as shown in Fig. 9*a*. Its length, *l*, is 4 ft., diameter, *d*, is 10 in., and "*heaviness*" (p. 3) 480 lbs./cub. ft. Required the pressures (or "*reactions*," or "*supporting forces*"), *P* and *Q* at the two points of contact *A* and *B*. (Points, in the end view.) These pressures on the cylinder are shown pointing normal to the surfaces (*smooth* surfaces) and hence pass through the center of the body, *C*, where we may consider the resultant weight, *G*, of the body to act. These three forces, then, form a concurrent system, and the body is in equilibrium under their action.

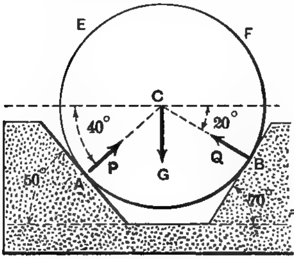

FIG. 9*a*.

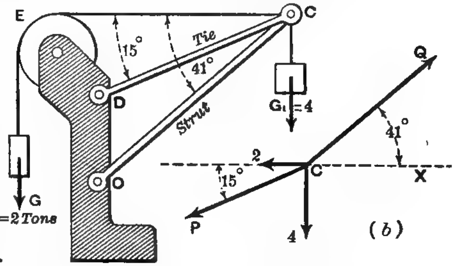

FIG. 9*b*.

Solution.—The weight $G = \frac{\pi d^2 l}{4}\gamma = \frac{\pi}{4}\left(\frac{10}{12}\right)^2 \times 4 \times 480 = 1047.6$ lbs.

$\Sigma X = 0$ gives: $+P \cos 40° - Q \cos 20° + 0 = 0;$ . . . . (1)

$\Sigma Y = 0$ " $+P \sin 40° + Q \sin 20° - G = 0;$ . . . . (2)

that is, numerically, $0.7660P - 0.9396Q = 0;$ . . . . (3)

and $0.6428P + 0.3420Q = 1047.6$ lbs. . . . . (4)

From (3) we have $P = 1.227Q$, which in (4) gives

$$\left.\begin{array}{l}(0.7887 + 0.3420)Q = 1047.6 \text{ lbs.; and hence } Q = 926.4 \text{ lbs.} \\ \text{Therefore } P, = 1.227Q, = 1136.8 \text{ lbs.}\end{array}\right\} \text{Ans.}$$

Example 2.—Fig. 9*b*. The 4-ton weight is suspended on the bolt *C*, which passes through the ends of boom *OC* and tie-rod *DC*. Bolt *C* is also subjected to a horizontal pull toward the left, due to the 2-ton weight, suspended as shown. Find the pull *P* in the tie and the thrust *Q* in the boom. Note that the boom is pivoted at both ends and hence (if we neglect its weight) is under only *two* pressures; both of which, therefore (for the equilibrium of the boom), *must point along its length.* Hence the thrust *Q* on bolt *C* makes an angle of 41° with the horizontal. Similarly, *P*, the action of tie-rod on *C*, is at 15°.

Solution.—At (*b*) we see the bolt as a "*free body*"; in equilibrium under the four concurrent forces.

$\Sigma X = 0$ . . . . $Q \cos 41° - P \cos 15° - G - 0 = 0;$ . . . . . . (5)

$\Sigma Y = 0$ . . . . $Q \sin 41° - P \sin 15° - G_1 - 0 = 0;$ . . . . . . (6)

or, numerically, $0.7547Q - 0.9659P - 2 = 0,$ . . . . (7)

and $0.6560Q - 0.2588P - 4 = 0$ . . . . (8)

From (7), $Q = 2.651 + 1.279P$, which in (8) gives

$$0.6560(2.651 + 1.279P) - 0.2588P = 4;$$

that is, $1.740 + 0.8390P - 0.2588P = 4$; and hence, finally,

$P = 2.260 \div 0.5802 = 3.896$ tons, and $\therefore Q = 7.633$ tons. Ans.

CHAPTER II.

PARALLEL FORCES AND THE CENTRE OF GRAVITY.

20. Preliminary Remarks.—Although by its title this section should be restricted to a treatment of the equilibrium of forces, certain propositions involving the composition and resolution of forces, without reference to the behavior of the body under their action, will be found necessary as preliminary to the principal object in view.

As a rigid body possesses extension in three dimensions, to deal with a system of forces acting on it we require three coordinate axes: in other words, the system consists of "forces in space," and in general the forces are *non-concurrent*. In most problems in statics, however, the forces acting are in one plane: we accordingly begin by considering non-concurrent forces in a plane, of which the simplest case is that of two parallel forces. For the present the body on which the forces act will not be shown in the figure, but must be understood to be there (since we have no conception of forces independently of material bodies). The device will frequently be adopted of introducing into the given system two opposite and equal forces acting in the same line: evidently this will not alter the effect of the given system, as regards the rest or motion of the body.

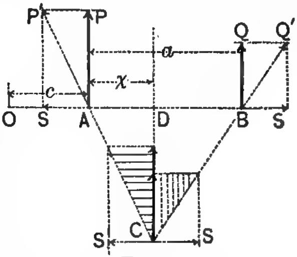

Fig. 10.

21. Resultant of two Parallel Forces.

Case I.—The two forces have the *same direction*. Fig. 10. Let P and Q be the given forces, and AB a line perpendicular to them (P and Q are supposed to have been transferred to the intersections A and B). Put in at A and B two equal and opposite forces S and S, combining them with P and Q to form P'

and Q'. Transfer P' and Q' to their intersection at C, and there resolve them again into S and P, S and Q. S and S annul each other at C; therefore P and Q, acting along a common line CD, replace the P and Q first given; i.e., the resultant of the original two forces is a force $R = P + Q$, acting parallel to them through the point D, whose position must now be determined. The triangle CAD is similar to the triangle shaded by lines, $\therefore P : S :: \overline{CD} : x$; and CDB being similar to the triangle shaded by dots, $\therefore S : Q :: a - x : \overline{CD}$. Combining these, we have* $\frac{P}{Q} = \frac{a-x}{x}$ and $\therefore x = \frac{Qa}{P+Q} = \frac{Qa}{R}$. Now write this $Rx = Qa$, and add Rc, i.e., $Pc + Qc$, to each member, c being the distance of O (Fig. 10), any point in AB produced, from A. This will give $R(x + c) = Pc + Q(a + c)$, in which c, $a + c$, and $x + c$ are respectively the lengths of perpendiculars let fall from O upon P, Q, and their resultant R. Any one of these products, such as Pc, is for convenience (since products of this form occur so frequently in Mechanics as a result of algebraic transformation) called the **Moment** of the force about the arbitrary point O. Hence the resultant of two parallel forces of the same direction is equal to their sum, acts in their plane, in a line parallel to them, and at such a distance from any arbitrary point O in their plane as may be determined by writing its moment about O equal to the sum of the moments of the two forces about O. O is called a *centre of moments*, and each of the perpendiculars a *lever-arm*.

CASE II.—Two parallel forces P and Q of opposite directions. Fig. 11. By a process similar to the foregoing, we obtain $R = P - Q$ and $(P - Q)x = Qa$, i.e., $Rx = Qa$. Subtract each member of the last equation from Rc (i.e., $Pc - Qc$), in which c is the distance, from A, of any arbitrary point O in AB produced. This gives $R(c - x) = Pc - Q(a + c)$. But $(c - x)$, c, and $(a + c)$ are respectively the perpendiculars, from

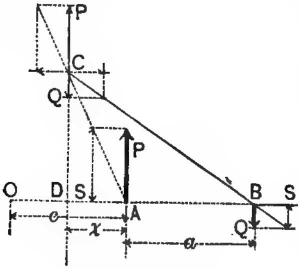

FIG. 11.

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* That is, the resultant of two parallel forces pointing in the same direction *divides the distance between them in the inverse ratio of those forces.*

O, upon R, P, and Q. That is, $R(c - x)$ is the moment of R about O; Pc, that of P about O; and $Q(a + c)$, that of Q about O. But the moment of Q is subtracted from that of P, which corresponds with the fact that Q in this figure would produce a rotation about O opposite in direction to that of P. Having in view, then, this imaginary rotation, we may define the moment of a force as *positive* when the indicated direction about the given point is against the hands of a watch; as *negative* when with the hands of a watch.*

Hence, in general, the resultant of any two parallel forces is, in amount, equal to their algebraic sum, acts in a parallel direction in the same plane, while its moment, about any arbitrary point in the plane, is equal to the algebraic sum of the moments of the two forces about the same point.

Corollary.—If each term in the preceding moment equations be multiplied by the secant of an angle (α, Fig. 12) thus:

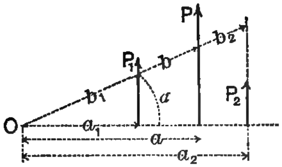

FIG. 12.

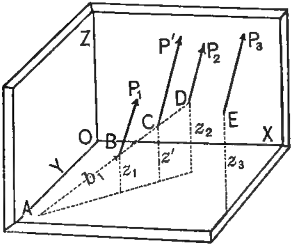

FIG. 13.

(using the notation of Fig. 12), we have

$$Pa \sec \alpha = P_1 a_1 \sec \alpha + P_2 a_2 \sec \alpha, \text{ i.e., } Pb = P_1 b_1 + P_2 b_2,$$

in which b, b_1 and b_2 are the *oblique* distances of the three lines of action from any point O in their plane, and lie on the same straight line; P is the resultant of the parallel forces P_1 and P_2.

22. Resultant of any System of Parallel Forces in Space.—Let P_1, P_2, P_3, etc., be the forces of the system, and x_1, y_1, z_1, x_2, y_2, z_2, etc., the co-ordinates of their points of application as referred to an arbitrary set of three co-ordinate axes X, Y, and Z, perpendicular to each other. Each force is here restricted to a definite point of application in its line of action (with reference to establishing more directly the fundamental equations for the co-ordinates of the centre of gravity of a body). The resultant P' of any two of the forces, as P_1 and P_2, is $= P_1 + P_2$, and may be applied at C, the intersection of its own line of action with a line BD joining the points of application of P_1 and P_2, its components. Produce the latter line to A, where it pierces the plane XY, and let b_1, b', and b_2, respectively, be the distances of B, C, D, from A; then from the corollary of the last article we have

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* These two directions of rotation are often called *counter clockwise*, and *clockwise*, respectively.

$$P'b' = P_1b_1 + P_2b_2;$$

but from similar triangles

$$b' : b_1 : b_2 :: z' : z_1 : z_2, \qquad \therefore P'z' = P_1z_1 + P_2z_2.$$

Now combine P', applied at C, with P_3, applied at E, calling their resultant P'' and its vertical co-ordinate z'', and we obtain

$$P''z'' = P'z' + P_3z_3, \text{ i.e., } P''z'' = P_1z_1 + P_2z_2 + P_3z_3,$$

also

$$P'' = P' + P_3 = P_1 + P_2 + P_3.$$

Proceeding thus until all the forces have been considered, we shall have finally, for the resultant of the whole system,

$$R = P_1 + P_2 + P_3 + \text{etc.};$$

and for the vertical co-ordinate of its point of application, which we may write $\bar{z}$,

$$R\bar{z} = P_1z_1 + P_2z_2 + P_3z_3 + \text{etc};$$

$$\text{i.e., } \bar{z} = \frac{P_1z_1 + P_2z_2 + P_3z_3 \ldots}{P_1 + P_2 + P_3 + \ldots .} = \frac{\Sigma(Pz)}{\Sigma P};$$

and similarly for the other co-ordinates.

$$\bar{x} = \frac{\Sigma(Px)}{\Sigma P} \text{ and } \bar{y} = \frac{\Sigma(Py)}{\Sigma P}.$$

In these equations, in the general case, such products as P_1z_1, etc., cannot strictly be called moments. The point whose co-ordinates are the $\bar{x}$, $\bar{y}$, and $\bar{z}$, just obtained, is called the *Centre of Parallel Forces*, and its position is *independent of the* (*common*) *direction* of the forces concerned.

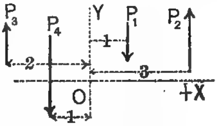

FIG. 14.

Example.—If the parallel forces are contained in one plane, and the axis Y be assumed parallel to the direction of the forces, then each product like P_1x_1 will be a *moment*, as defined in § 21; and it will be noticed in the accompanying numerical example, Fig. 14, that a detailed substitution in the equation

$$R\bar{x} = P_1x_1 + P_2x_2 + \text{etc.}, \quad . \quad . \quad . \quad (1)$$

having regard to the proper sign of each force and of each abscissa, gives the same result as if each product Px were first obtained numerically, and a sign affixed to the product considered as a moment about the point O. Let $P_1 = -1$ lb.; $P_2 = +2$ lbs.; $P_3 = +3$ lbs.; $P_4 = -6$ lbs.; $x_1 = +1$ ft.; $x_2 = +3$ ft.; $x_3 = -2$ ft.; and $x_4 = -1$ ft. Required the amount and position of the resultant R. In amount $R = \Sigma P = -1 + 2 + 3 - 6 = -2$ lbs.; i.e., it is a *downward* force of 2 lbs. As to its position, $R\bar{x} = \Sigma(Px)$ gives $(-2)\bar{x} = (-1) \times (+1) + 2 \times 3 + 3 \times (-2) + (-6) \times (-1) = -1 + 6 - 6 + 6$. Now from the figure, by inspection, it is evident that the moment of P_1 about O is negative (*with* the hands of a watch), and is numerically $= 1$, i.e., its moment $= -1$; similarly, by inspection, that of P_2 is seen to be positive, that of P_3 negative, that of P_4 positive; which agree with the results just found, that $(-2)\bar{x} = -1 + 6 - 6 + 6 = +5$ ft. lbs. (Since a moment is a product of a force (lbs.) by a length (ft.), it may be called so many foot-pounds.) Next, solving for $\bar{x}$, we obtain $\bar{x} = (+5) \div (-2) = -2.5$ ft.; i.e., the resultant of the given forces is a downward force of 2 lbs. acting in a vertical line 2.5 ft. to the left of the origin. Hence, if the body in question be a horizontal rod whose weight has been already included in the statement of forces, a support placed 2.5 ft. to the left of O and capable of resisting at least 2 lbs. downward pressure will preserve equilibrium; and the pressure which it exerts against the rod must be an upward force, P_6, of 2 lbs., i e. the equal and opposite of the resultant of P_1, P_2, P_3, P_4.

Fig. 15 shows the rod as a free body in equilibrium under the five forces. $P_6 = +2$ lbs. $=$ the *reaction* of the support. Of course P_5 is one of a pair of equal and opposite forces; the other one is the pressure of the rod against the support, and would take its place among the forces acting on the support.

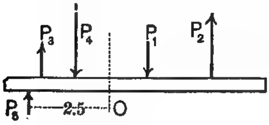

FIG. 15.

23. Centre of Gravity.—Among the forces acting on any rigid body at the surface of the earth is the so-called attraction of the latter (i.e., gravitation), as shown by a spring-balance, which indicates the *weight* of the body hung upon it. The weights of the different particles of any rigid body constitute a system of parallel forces (practically so, though actually slightly convergent). The point of application of the resultant of these forces is called the *centre of gravity* of the body, and may also be considered the *centre of mass*, the body being of very small dimensions compared with the earth's radius.

If $\bar{x}$, $\bar{y}$, and $\bar{z}$ denote the co-ordinates of the centre of gravity of a body referred to three co-ordinate axes, the equations derived for them in § 22 are directly applicable, with slight changes in notation.

Denote the weight of any particle * of the body by dG, its volume by dV, by γ its *heaviness* (rate of weight, see § 7) and its co-ordinates by x, y, and z; then, using the integral sign as indicating a summation of like terms for all the particles of the body, we have, for heterogeneous bodies (see also p. 119, *Notes*).

$$\bar{x} = \frac{\int \gamma x dV}{\int \gamma dV}; \quad \bar{y} = \frac{\int \gamma y dV}{\int \gamma dV}; \quad \bar{z} = \frac{\int \gamma z dV}{\int \gamma dV}; \quad . \quad . \quad (1)$$

while, if the body is homogeneous, γ is the same for all its elements, and being therefore placed outside the sign of summation, is cancelled out, leaving for *homogeneous* bodies (V denoting the total volume)

$$\bar{x} = \frac{\int x dV}{V}; \quad \bar{y} = \frac{\int y dV}{V}; \quad \text{and } \bar{z} = \frac{\int z dV}{V}. \quad . \quad . \quad (2)$$

---

* Any subdivision of the body may be adopted for use of equations (1) and (2), etc.; but it must be remembered that the x (or y, or z) in each term of the summations, or integrals, is the co-ordinate of the *center of gravity of the subdivision employed.*

Corollary.—It is also evident that if a homogeneous body is for convenience considered as made up of several finite parts, whose volumes are V_1, V_2, etc., and whose gravity co-ordinates are $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$; $\bar{x}_2$, $\bar{y}_2$, $\bar{z}_2$; etc., we may write

$$\bar{x} = \frac{V_1\bar{x}_1 + V_2\bar{x}_2 + \dots}{V_1 + V_2 + \dots} \quad . \quad . \quad . \quad . \quad (3)$$

If the body is heterogeneous, put G_1 (weights), etc., instead of V_1, etc., in equation (3).

If the body is an infinitely thin *homogeneous shell* of uniform thickness $= h$, then $dV = hdF$ (dF denoting an element, and F the whole area of one surface) and equations (2) become, after cancellation,

$$\bar{x} = \frac{\int x dF}{F}; \quad \bar{y} = \frac{\int y dF}{F}; \quad \bar{z} = \frac{\int z dF}{F}. \quad . \quad . \quad . \quad (4)$$

For a thin homogeneous plate, or shell, of uniform thickness, and composed of several finite parts, of area F_1, F_2, etc., with gravity co-ordinates $\bar{x}_1$, $\bar{x}_2$, etc., we may write

$$\bar{x} = \frac{F_1\bar{x}_1 + F_2\bar{x}_2 + \dots}{F_1 + F_2 + \dots} \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

Similarly, for a *homogeneous wire* of constant small cross-section (i.e., a geometrical line, having weight), its length being s, and an element of length ds, we obtain

$$\bar{x} = \frac{\int x ds}{s}; \quad \bar{y} = \frac{\int y ds}{s}; \quad \bar{z} = \frac{\int z ds}{s}. \quad . \quad . \quad . \quad (5)$$

24. Symmetry.—Considerations of symmetry of form often determine the centre of gravity of homogeneous solids without analysis, or limit it to a certain line or plane. Thus the centre of gravity of a sphere, or any regular polyedron, is at its centre of figure; of a right cylinder, in the middle of its axis; of a thin plate of the form of a circle or regular polygon, in the centre of figure; of a straight wire of uniform cross-section, in the middle of its length.

Again, if a homogeneous body is symmetrical about a plane, the centre of gravity must lie in that plane, called a plane of gravity; if about a line, in that line called a line of gravity; if about a point, in that point.

25. By considering certain modes of subdivision of a homogeneous body, lines or planes of gravity are often made apparent. E.g., a line joining the middle of the bases of a trapezoidal plate is a line of gravity, since it bisects all the strips of uniform width determined by drawing parallels to the bases; similarly, a line joining the apex of a triangular plate to the middle of the opposite side is a line of gravity. Other cases can easily be suggested by the student.

26. Problems.—(1) Required the position of the centre of gravity of a *fine homogeneous wire of the form of a circular arc, AB*, Fig. 16. Take the origin O at the centre of the circle, and the axis X bisecting the wire. Let the length of the wire, $s, = 2s_1$; $ds =$ element of arc. We need determine only the $\bar{x}$, since evidently $\bar{y} = 0$. Equations (5), § 23, are applicable here, i.e., $\bar{x} = \frac{\int x ds}{s}$.

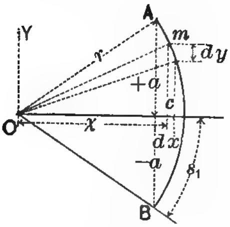

FIG. 16.

From similar triangles we have *

$$ds : dy :: r : x; \therefore ds = \frac{rdy}{x};$$

$$\therefore \bar{x} = \frac{r}{2s_1}\int_{y=-a}^{y=+a} dy = \frac{2ra}{2s_1},$$ i.e., = chord × radius ÷ length of wire. For a semicircular wire, this reduces to $\bar{x} = 2r \div \pi$.

PROBLEM 2. *Centre of gravity of trapezoidal (and triangular) thin plates, homogeneous, etc.*—Prolong the non-parallel sides of the trapezoid to intersect at O, which take as an origin, making the axis X perpendicular to the bases b and b_1. We may here use equations (4), § 23, and may take a vertical strip for our element of area, dF, in determining $\bar{x}$; for each point of such a strip has the same x. Now $dF = (y + y')dx$, and from similar triangles $y + y' = \frac{b}{h}x$. Now $F_1 = \frac{1}{2}(bh - b_1h_1)$,* can be written $\frac{1}{2}\frac{b}{h}(h^2 - h_1^2)$, and $\bar{x} = \frac{\int x dF}{F}$ becomes

---

* The two triangles meant (m being *any* point of the wire) are the finite triangle Omc, and the infinitely small one at m formed by the infinitesimal lengths dy, dx, and ds.

$\frac{y+y_1}{x} =$

$$= \left[\frac{b}{h}\int_{h_1}^{h} x^2 dx\right] \div \frac{1}{2}\frac{b}{h}(h^2 - h_1^2) = \frac{2}{3}\frac{h^3 - h_1^3}{h^2 - h_1^2}$$

for the trapezoid.

For a triangle $h_1 = 0$, and we have $\bar{x} = \frac{2}{3}h$; that is, the centre of gravity of a triangle is one third the altitude from the base. The centre of gravity is finally determined by knowing that a line joining the middles of b and b_1 is a line of gravity; or joining O and the middle of b in the case of a triangle.

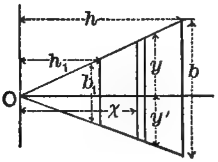

FIG. 17.

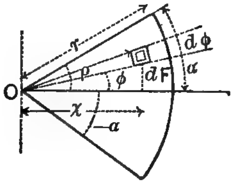

FIG. 18.

PROBLEM 3. *Sector of a circle. Thin plate,* etc.—Let the notation, axes, etc., be as in Fig. 18. Angle of sector $= 2\alpha$; $\bar{x} = ?$ Using polar co-ordinates, the element of area dF (a small rectangle) $= \rho d\varphi \,.\, d\rho$, and its $x = \rho \cos \varphi$; hence the total area =

$$F = \int_{-\alpha}^{+\alpha}\left[\int_0^r \rho d\rho\right] d\varphi = \int_{-\alpha}^{+\alpha} \frac{1}{2} r^2 d\varphi = \frac{r^2}{2}\left[\varphi\right]_{-\alpha}^{+\alpha};$$

i.e., $F = r^2\alpha$. From equations (4), § 23, we have

$$\bar{x} = \frac{1}{F}\int x dF$$

$$= \frac{1}{F}\int\int \cos \varphi\rho^2 d\rho d\varphi = \frac{1}{F}\int_{-\alpha}^{+\alpha}\left[\cos\varphi \int_0^r \rho^2 d\rho\right] d\varphi.$$

---

* Note that $b_1 : h_1 :: b : h$, so that $b_1h_1 = (b \div h)h_1^2$.

(*Note on double integration.*—The quantity

$$\left[\cos\varphi \int_0^r \rho^2\, d\rho\right] d\varphi,$$

is that portion of the summation $\int\int \cos\varphi\rho^2 d\rho d\varphi$ which belongs to a single elementary sector (triangle), since all its elements (rectangles), from centre to circumference, have the same φ and $d\varphi$.)

That is,

$$\bar{x} = \frac{1}{F}\cdot\frac{r^3}{3}\int_{-\alpha}^{+\alpha}\cos\varphi d\varphi = \frac{r^3}{3r^2\alpha}\Big[_{-\alpha}^{+\alpha}\sin\varphi = \frac{2}{3}\cdot\frac{r\sin\alpha}{\alpha};$$

or, putting $\beta = 2\alpha$ = total angle of sector, $\bar{x} = \frac{4}{3}\,\frac{r\sin\frac{1}{2}\beta}{\beta}$.

For a semicircular plate this reduces to $\bar{x} = \frac{4r}{3\pi}$.

[*Note.*—In numerical substitution the arcs α and β used above (unless sin or cos is prefixed) are understood to be expressed in circular measure (π-measure); e.g., for a quadrant, $\beta = \frac{\pi}{2} = 1.5707$*+; for 30°, $\beta = \frac{\pi}{6}$; or, in general, if β in degrees $= \frac{180^\circ}{n}$, then β in π-measure $= \frac{\pi}{n}$.]

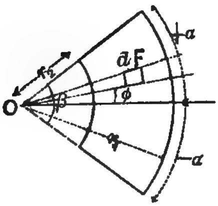

Fig. 19.

Problem 4. *Sector of a flat ring*; thin plate, etc.—Treatment similar to that of Problem 3, the difference being that the limits of the interior integrations are $\Big[_{r_2}^{r_1}$ instead of $\Big[_{o}^{r}$. Result,

$$\bar{x} = \frac{4}{3}\cdot\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\cdot\frac{\sin\frac{1}{2}\beta}{\beta}.$$

---

* "Radians."

Problem 5.—*Segment of a circle;* thin plate, etc.—Fig. 20. Since each rectangular element of any vertical strip has the same x, we may take the strip as dF in finding $\bar{x}$, and use y as the half-height of the strip. $dF = 2ydx$, and from similar triangles $x : y :: (-dy) : dx$,* i.e., $xdx = -ydy$. Hence from eq. (4), § 23,

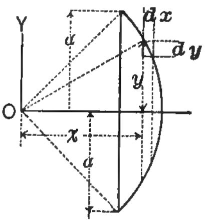

Fig. 20.

$$\bar{x} = \frac{\int xdF}{F} = \frac{\int x2ydx}{F} = \frac{-2\int_a^0 y^2dy}{F} = \frac{2}{3F}\Big[_a^0 (-y^3) = \frac{2}{3}\cdot\frac{a^3}{F};$$

but $a =$ the half-chord, hence, finally, $\bar{x} = \frac{(chord)^3}{12F}$.

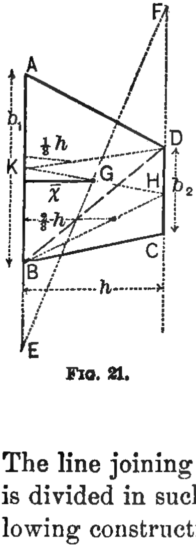

Fig. 21.

Problem 6.—*Trapezoid;* thin plate, etc., by the method in the corollary of § 23; equation (4*a*). Required the distance $\bar{x}$ from the base AB. Join DB, thus dividing the trapezoid $ABCD$ into two triangles $ADB = F_1$ and $DBC = F_2$, whose gravity x's are, respectively, $x_1 = \frac{1}{3}h$ and $x_2 = \frac{2}{3}h$. Also, $F_1 = \frac{1}{2}hb_1$, $F_2 = \frac{1}{2}hb_2$, and F (area of trapezoid) $= \frac{1}{2}h(b_1 + b_2)$. Eq. (4*a*) of § 23 gives $F\bar{x} = F_1x_1 + F_2x_2$; hence, substituting,

$$(b_1 + b_2)\bar{x} = \tfrac{1}{3}b_1h + \tfrac{2}{3}b_2h.$$

$$\therefore\ \bar{x} = \frac{h}{3}\cdot\frac{(b_1 + 2b_2)}{b_1 + b_2}.$$

The line joining the middles of b_1 and b_2 is a line of gravity, and is divided in such a ratio by the centre of gravity that the following construction for finding the latter holds good: Prolong each base, in opposite directions, an amount equal to the other base; join the two points thus found: the intersection with the other line of gravity is the centre of gravity of the trapezoid. Thus, Fig. 21, with $BE = b_2$ and $DF = b_1$, join FE, etc.

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* The minus sign is used for dy since, as we progress from left to right in bringing into account all the various strips, x increases while y diminishes; i.e., dx is an increment and dy a *decrement*. At the point of beginning of the summation, on left, $y = +a$; while at the extreme right, $y = 0$.

PROBLEM 7. *Homogeneous oblique cone or pyramid.*—Take the origin at the vertex, and the axis X perpendicular to the base (or bases, if a frustum). In finding $\bar{x}$ we may put $dV =$ volume of any lamina parallel to YZ. F being the base of such a lamina, each point of the lamina having the same x. Hence, (equations (2), § 23), (see also Fig. 22).

$$\bar{x} = \frac{1}{V}\int x\,dV, \quad V = \int dV = \int F\,dx;$$

but, from the geometry of *similar* plane figures,

$$F : F_2 :: x^2 : h_2^2, \quad \therefore F = \frac{F_2}{h_2^2}x^2,$$

and

$$V = \frac{F_2}{h_2^2}\int x^2 dx = \frac{F_2}{h_2^2}\left[\frac{x^3}{3}\right. ; \int x\,dV = \frac{F_2}{h_2^2}\int x^3 dx = \frac{F_2}{h_2^2}\left[\frac{x^4}{4}\right. .$$

For a frustum, $\bar{x} = \frac{3}{4}\cdot\frac{h_2^4 - h_1^4}{h_2^3 - h_1^3}$; while for a pyramid, h_1, being $= 0$, $\bar{x} = \frac{3}{4}h$. Hence the centre of gravity of a pyramid is one fourth the altitude from the base. It also lies in the line joining the vertex to the centre of gravity of the base.

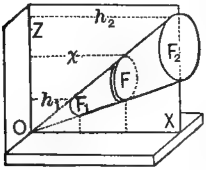

FIG. 22.

PROBLEM 8.—If the heaviness of the material of the above cone or pyramid varied directly as x, γ_2 being its heaviness at the base F_2, we should use equations (1), § 23, putting $\gamma = \frac{\gamma_2}{h_2}x$; and finally, for the frustum,

$$\bar{x} = \frac{4}{5}\cdot\frac{h_2^5 - h_1^5}{h_2^4 - h_1^4},$$

and for a complete cone $\bar{x} = \frac{4}{5}h_2$.

27. The Centrobaric Method.—If an elementary area dF be revolved about an axis in its plane, through an angle $\alpha < 2\pi$, the distance from the axis being $= x$, the volume generated is $dV = \alpha x dF$, and the total volume generated by all the dF's of a finite plane figure whose plane contains the axis and which lies entirely on one side of the axis, will be $V = \int dV = \alpha\int x dF$. But from § 23, $\alpha\int x dF = \alpha F\bar{x}$; $\alpha\bar{x}$ being the length of path described by the centre of gravity of the plane figure, we may write: *The volume of a solid of revolution generated by a plane figure, lying on one side of the axis, equals the area of the figure multiplied by the length of curve described by the centre of gravity of the figure.*

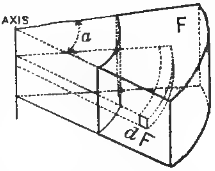

FIG. 23.

A corresponding statement may be made for the surface generated by the revolution of a line. The arc α must be expressed in π measure in numerical work.

27a. Centre of Gravity of any Quadrilateral.—Fig. 23*a*. *Construction;* $ABGD$ being any quadrilateral. Draw the diagonals. On the long segment DK of DB lay off $DE = BK$, the shorter, to determine E; similarly, determine N on the other diagonal, by making $GN = AK$. Bisect EK in H and KN in M. The intersection of EM and NH is the centre of gravity, C.

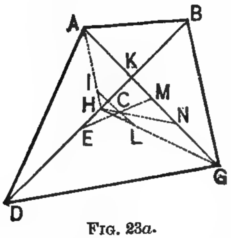

FIG. 23*a*.

Proof.—H being the middle of DB, and AH and HG having been joined, I the centre of gravity of the triangle ABD is found on AH, by making $HI = \frac{1}{3}AH$; similarly, by making $HL = \frac{1}{3}HG$, L is the centre of gravity of triangle BDG. $\therefore$ IL is parallel to AG and is a gravity-line of the whole figure; and the centre of gravity C may be found on it if we can make $CL : CI ::$ area ABD : area BDG (§ 21). But since these triangles have a common base DB, their areas are proportional to the slant heights (equally inclined to DB) AK and KG, i.e., to GN and NA. Hence HN, which divides IL in the required ratio, contains C, and *is* $\therefore$ *a gravity-line.* By similar reasoning, using the other diagonal, AG, and the two triangles into which it divides the whole figure, we may prove EM to be a gravity-line also. Hence the construction is proved.

27b. Examples.—1. Required the volume of a sphere by the centrobaric method.

A sphere may be generated by a semicircle revolving about its diameter through an arc $\alpha = 2\pi$. The length of the path described by its centre of gravity is $= 2\pi \frac{4r}{3\pi}$ (see Prob. 3, § 26), while the area of the semicircle is $\frac{1}{2}\pi r^2$. Hence by § 27,

$$\text{Volume generated} = 2\pi \cdot \frac{4r}{3\pi} \cdot \tfrac{1}{2}\pi r^2 = \frac{4}{3}\pi r^3.$$

2. Required the position of the centre of gravity of the sector of a flat ring in which $r_1 = 21$ feet, $r_2 = 20$ feet, and $\beta = 80°$ (see Fig. 19, and § 26, Prob. 4).

$\sin\frac{\beta}{2} = \sin 40° = 0.64279$, and β in *circular measure* $= \frac{80}{180}\pi = \frac{4}{9}\pi = 1.3962$ *radians*. By using r_1 and r_2 in feet, $\bar{x}$ will be obtained in feet.

$$\therefore \bar{x} = \frac{4}{3} \cdot \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2} \cdot \frac{\sin\frac{\beta}{2}}{\beta} = \frac{4}{3} \cdot \frac{1261}{41} \cdot \frac{0.64279}{1.3962} = 18.87 \text{ feet.}$$

3. Find the height ($\bar{z}$, $=\overline{OC}$) of the center of gravity of the plane figure in Fig. 23*b* above its base OX.

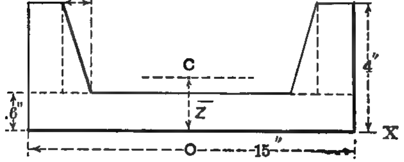

Fig. 23*b*.

This figure is bounded by straight lines and is an approximation to the shape of the cross-section of a steel "channel" (see p. 275).

Dividing it into three rectangles and two triangles (see dotted lines in figure) and applying eq. (4a) of p. 19, we have

$$\bar{z} = \frac{15 \times .6 \times .3 + 2[3.4 \times .6 \times 2.3] + 2\left[1.7 \times .5 \times \frac{5.2}{3}\right]}{15 \times 0.6 + 2[3.4 \times 0.6] + 2[1.7 \times 0.5]} = 1.017 \text{ in.}$$

(The student should carefully verify these numerical details.)

CHAPTER III.

STATICS OF A RIGID BODY.

28. Couples.—On account of the peculiar properties and utility of a system of two equal forces acting in parallel lines and in opposite directions, it is specially considered, and called a **Couple.** The *arm* of a couple is the perpendicular distance between the forces; its *moment*, the product of this arm, by one of the forces. The axis of a couple is an imaginary line drawn perpendicular to its plane on that side from which the rotation appears *positive* (against the hands of a watch). (An ideal rotation is meant, suggested by the position of the arrows; any actual rotation of the rigid body is a subject for future consideration.) In dealing with two or more couples the lengths of their axes are made proportional to their moments; in fact, by selecting a proper scale, numerically equal to these moments. E.g., in Fig. 24, the moments of the two couples there shown are Pa and Qb; their axes p and q so laid off that $Pa : Qb :: p : q$, and that the ideal rotation may appear positive, viewed from the outer end of the axis.

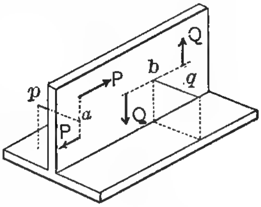

FIG. 24.

For example, if each force P of a couple is 60 lbs., and the arm is $a=6$ ft., its moment is 360 *foot-pounds;* or 0.180 *foot-tons;* or 4320 *inch-pounds;* or 2.16 *inch-tons.*

29. *No single force can balance a couple.*—For suppose the couple P, P, could be balanced by a force R', then this, acting at some point C, ought to hold the couple in equilibrium. Draw CO through O, the centre of symmetry of the couple, and make $OD = OC$. At D put in two opposite and equal forces, S and T, equal and parallel to R'. The supposed equilibrium is undisturbed. But if R', P, and

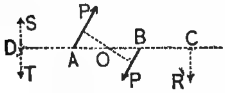

FIG. 25.

P are in equilibrium, so ought (by symmetry about O) S, P, and P to be in equilibrium, and they may be removed without disturbing equilibrium. But we have left T and R', which are evidently not in equilibrium; $\therefore$ the proposition is proved by this *reductio ad absurdum.* Conversely a couple has no single resultant.

30. *A couple may be transferred anywhere in its own plane.*—First, it may be turned through any angle α, about any point of its arm, or of its arm produced. Let (P, P') be a couple, G any point of its arm (produced), and α any angle. Make $GC = GA$, $CD = AB$, and put in at C, P_1 and P_2 equal to P (or P'), opposite to each other and perpendicular to GC; and P_3 and P_4 similarly at D. Now apply and combine P and P_1 at O, P' and P_4 at O'; then evidently R and R' neutralize each other, leaving P_3 and P_2 equivalent to the original couple (P, P'). The arm $CD = AB$. Secondly, if G be at infinity, and $\alpha = 0$, the same proof applies, i.e., a couple may be moved parallel to itself in its own plane. Therefore, by a combination of the two transferrals, the proposition is established for any transferral in the plane.

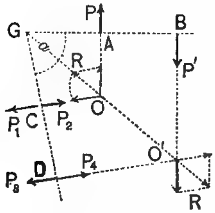

FIG. 26.

31. *A couple may be replaced by another of equal moment in a parallel plane.*—Let (P, P') be a couple.[*] Let CD, in a parallel plane, be parallel to AB. At D put in a pair of equal and opposite forces, S_3 and S_4, parallel to P and each $= \frac{\overline{AE}}{\overline{ED}}P$. Similarly at C, S_1 and S_2 parallel to P and each $= \frac{\overline{BE}}{\overline{EC}}P$. But, from similar triangles,

$$\frac{AE}{ED} = \frac{BE}{EC}; \quad \therefore S_1 = S_2 = S_3 = S_4.$$

---

[*] See Fig. 27, which is a *perspective view.* The arm of the couple (P, P') is AB, in the background. The length of CD, which is in the foreground, may be anything whatever.

[NOTE.—The above values are so chosen that the intersection point E may be the point of application of $(P' + S_2)$, the resultant of P' and S_2; and also of $(P + S_3)$, the resultant of P and S_3, as follows from § 21; thus (Fig. 28), R, the resultant of the two parallel forces P and S_3, is $= P + S_3$, and its moment about any centre of moments, as E, its own point of application, should equal the (algebraic) sum of the moments of its components about E; i.e., $R \times \text{zero} = P . \overline{AE} - S_3 . \overline{DE}; \therefore S_3 = \frac{\overline{AE}}{\overline{DE}} . P$.]

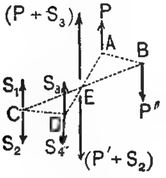

FIG. 27.

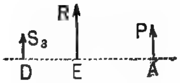

FIG. 28.

Replacing P' and S_2 by $(P' + S_2)$, and P and S_3 by $(P + S_3)$, the latter resultants cancel each other at E, leaving the couple (S_1, S_4) with an arm CD, equivalent to the original couple P, P' with an arm AB. But, since $S_1 = \frac{\overline{BE}}{\overline{EC}} . P = \frac{\overline{AB}}{\overline{CD}} . P$, we have $S_1 \times \overline{CD} = P \times \overline{AB}$; that is, their moments are equal.

32. Transferral and Transformation of Couples.—In view of the foregoing, we may state, in general, that a couple acting on a rigid body may be transferred to any position in any parallel plane, and may have the values of its forces and arm changed in any way so long as its moment is kept unchanged, and still have the same effect on the rigid body (as to rest or motion, not in distorting it).

Corollaries.—A couple may be replaced by another in any position so long as their axes are equal and parallel and similarly situated with respect to their planes.

A couple can be balanced only by another couple whose axis is equal and parallel to that of the first, and dissimilarly situated. For example, Fig. 29, Pa being $= Qb$, the rigid body AB (here supposed without weight) is in equilibrium in each case shown. By "reduction of a couple to a certain arm a" is meant that for the original couple whose arm is a', with forces each $= P'$, a new couple is substituted whose arm shall be $= a$, and the value of whose forces P and P must be computed from the condition

$$Pa = P'a', \quad \text{i.e.,} \quad P = P'a' \div a.$$

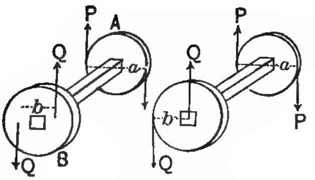

Fig. 29.

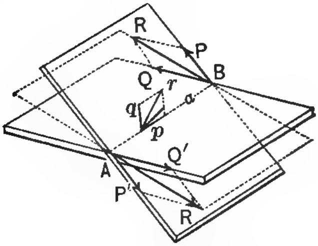

Fig. 30.

33. Composition of Couples.—Let (P, P') and (Q, Q') be two couples in different planes *reduced to the same arm* $\overline{AB} = a$, which is a portion of the line of intersection of their planes. That is, whatever the original values of the individual forces and arms of the two couples were, they have been transferred and replaced in accordance with § 32, so that $P.\overline{AB}$, the moment of the first couple, and the direction of its axis, p, have remained unchanged; similarly for the other couple. Combining P with Q and P' with Q', we have a resultant couple (R, R') whose arm is also $\overline{AB}$. The axes p and q of the component couples are proportional to $P.\overline{AB}$ and $Q.AB$, i.e., to P and Q, and contain the same angle as P and Q. Therefore the parallelogram $p \ldots q$ is similar to the parallelogram $P \ldots Q$; whence $p : q : r :: P : Q : R$, or $p : q : r :: Pa : Qa : Ra$. Also r is evidently perpendicular to the plane of the resultant couple (R, R'), whose moment is Ra. Hence r, the diagonal of the parallelogram on p and q, is the axis of the resultant couple. To combine two couples, therefore, we have only to combine their axes, as if they were forces, by a parallelogram, the diagonal being the axis of the resultant couple; the plane of this couple will be perpendicular to the axis just found, and its moment bears the same relation to the moments of the component couples as the diagonal axis to the two component axes. Thus, if two couples, of moments Pa and Qb, lie in planes perpendicular to each other, their resultant couple has a moment $Rc = \sqrt{(Pa)^2 + (Qb)^2}$.

It three couples in different planes are to be combined, the axis of their resultant couple is the diagonal of the parallelopiped formed on the axes, laid off to the same scale and *pointing in the proper directions*, the proper *direction* of an axis being *away* from the plane of its couple, on the side from which the couple appears of positive rotation.

34. If several couples lie in the same plane their axes are parallel and the axis of the resultant couple is their algebraic sum; and a similar relation holds for the moments: thus, in Fig. 24, the resultant of the two couples has a moment $= Qb - Pa$, which shows us that a convenient way of combining couples, when all in one plane, is to call the moments positive or negative, according as the ideal rotations are against, or with, the hands of a watch, as seen from the *same* side of the plane; the sign of the algebraic sum will then show the ideal rotation of the resultant couple.

35. Composition of Non-concurrent Forces in a Plane.—Let P_1, P_2, etc., be the forces of the system; x_1, y_1, x_2, y_2, etc., the co-ordinates of their points of application; and α_1, α_2, . . . etc., their angles with the axis X. Replace P_1 by its components X_1 and Y_1, parallel to the arbitrary axes of reference. At the origin put in two forces, opposite to each other and equal and parallel to X_1; similarly for Y_1. (Of course $X_1 = P_1 \cos \alpha$ and $Y_1 = P_1 \sin \alpha$.) We now have P_1 replaced by two forces X_1

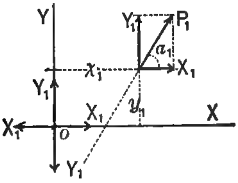

Fig. 31.

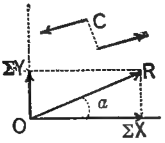

Fig. 32.

and Y_1 *at the origin*, and two couples, in the same plane, whose moments are respectively $-X_1y_1$ and $+Y_1x_1$, and are therefore (§ 34) equivalent to a single couple, in the same plane with a moment $= (Y_1x_1 - X_1y_1)$.

Treating all the remaining forces in the same way, the whole system of forces is replaced by

the force $\Sigma(X) = X_1 + X_2 + \ldots$ at the origin, along the axis X;
the force $\Sigma(Y) = Y_1 + Y_2 + \ldots$ at the origin, along the axis Y;

and the couple whose mom. $G = \Sigma\,(Yx - Xy)$, which may be called the couple C (see Fig. 32), and may be placed anywhere in the plane. Now $\Sigma(X)$ and $\Sigma(Y)$ may be combined into a force R; i.e.,

$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$ and its direction-cosine is $\cos\alpha = \frac{\Sigma X}{R}$.

Since, then, the whole system reduces to C and R, we must have for equilibrium $R = 0$, and $G = 0$; i.e., **for equilibrium**

$$\Sigma X = 0,\ \Sigma Y = 0, \text{ and } \Sigma(Yx - Xy) = 0. \quad . \text{ eq. (1)}$$

If R alone $= 0$, the system reduces to a couple whose moment is $G = \Sigma(Yx - Xy)$; and if G alone $= 0$ the system reduces to a single force R, applied at the origin. If, in general, neither R nor $G = 0$, the system is still equivalent to a single force, but not applied at the origin (as could hardly be expected, since the origin is arbitrary); as follows (see Fig. 33):

Replace the couple C by one of equal moment, G, with each force $= R$. Its arm will therefore be $\frac{G}{R}$. Move this couple in the plane so that one of its forces R may cancel the R already at the origin, thus leaving a **single resultant** R for the whole system, applied in a line at a perpendicular distance, $c = \frac{G}{R}$, from the origin, and making an angle α whose cosine $= \frac{\Sigma X}{R}$, with the axis X. It is easily proved that the "*moment,*" Rc, of the single resultant, about the origin O, *is equal to the algebraic sum of those of its "components"* (i.e., the forces of the system.

36. More convenient form for the equations of equilibrium of non-concurrent forces in a plane.—In (I.), Fig. 34, O being any point and a its perpendicular distance from a force P; put in at O two equal and opposite forces P and $P' =$ and $\parallel$ to P, and we have P replaced by an equal single force P' at O, and a couple whose moment is $+ Pa$. (II.) shows a similar construction, dealing with the X and Y components of P, so that in (II.) P is replaced by single forces X' and Y' at O (and they are equivalent to a resultant P', at O, as in (I.), and two couples whose moments are $+ Yx$ and $- Xy$.

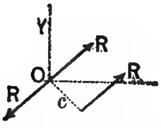

FIG. 33.

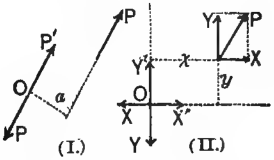

FIG. 34.

Hence, O being the same point in both cases, the couple Pa is equivalent to the two last mentioned, and, their axes being parallel, we must have $Pa = Yx - Xy$. Equations (1), § 35, for equilibrium, may now be written*

$$\Sigma X = 0, \ \Sigma Y = 0, \text{ and } \Sigma(Pa) = 0. \quad . \quad . \quad (2)$$

In problems involving the equilibrium of non-concurrent forces in a plane, we have *three independent conditions*, or *equations*, and can determine at most three unknown quantities. For practical solution, then, the rigid body having been made *free* (by conceiving the actions of all other bodies as represented by forces), and being in equilibrium (which it must be if at rest), we apply equations (2) literally; i.e., assuming an origin and two axes, equate the sum of the X components of all the forces to zero; similarly for the Y components; and then for the "moment-equation," having dropped a perpendicular from the origin upon each force, write the algebraic sum of the products (*moments*) obtained by multiplying each force by its perpendicular, or "*lever-arm*," equal to zero, calling each product + or — according as the ideal rotation appears against, or with, the hands of a watch, as seen from the same side of the plane. (The converse convention would do as well.)

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* Another proof is given on p. 15 of the "*Notes and Examples in Mechanics.*"

Sometimes it is convenient to use three moment equations, taking a new origin each time, and then the $\Sigma X = 0$ and $\Sigma Y = 0$ are superfluous, as they would not be independent equations.

37. Problems involving Non-concurrent Forces in a Plane.—*Remarks.* The weight of a rigid body is a vertical force through its centre of gravity, downwards.

If the surface of contact of two bodies is *smooth* the action (pressure, or force) of one on the other is perpendicular to the surface at the point of contact. If a cord must be imagined cut, to make a body free, its tension must be inserted in the line of the cord, and in such a direction as to keep *taut* the small portion still fastened to the body. In case the pin of a hinge must be removed, to make the body free, its pressure against the ring being unknown in *direction* and *amount*, it is most convenient to represent it by its unknown components X and Y, in *known* directions. In the following problems there is supposed to be no friction. If the line of action of an unknown force is known, but not its direction (forward or backward), *assume a direction for it* and adhere to it in all the three equations, and if the assumption is correct the value of the force, after elimination, will be positive; if incorrect, negative.*

Problem 1.—Fig. 35. Given an oblique rigid rod, with two loads G_1 (its own weight) and G_2; required the reaction of the *smooth* vertical wall at A, and the direction and amount of the *hinge*-pressure at O. The reaction at A must be horizontal; call it X'. The pressure at O, being unknown in direction, will have both its X and Y components unknown. The three unknowns, then, are X_0, X', and Y_0, while G_1, G_2, a_1, a_2, and h are known. The figure shows the rod as a *free body*, all the forces acting on it have been put in, and, since the rod is at rest, constitute a system of non-concurrent forces in a plane, ready for the conditions of equilibrium. Taking origin and axes as in the figure,

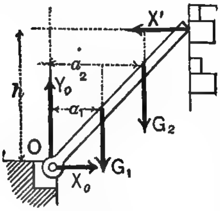

FIG. 35.

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* That is, the force must point in a direction opposite to that first assumed for it.

$\Sigma X = 0$ gives $+X_0 - X' = 0$; $\Sigma Y = 0$ gives $+Y_0 - G_1 - G_2 = 0$; while $\Sigma(Pa) = 0$, about O, gives $+X'h - G_1a_1 - G_2a_2 = 0$. (The moments of X_0 and Y_0 about O are, each, $=$ zero.) By elimination we obtain $Y_0 = G_1 + G_2$; $X_0 = X' = [G_1a_1 + G_2a_2] \div h$; while the pressure at $O = \sqrt{X_0^2 + Y_0^2}$, and makes with the horizontal an angle whose tan $= Y_0 \div X_0$.

[N.B. A special solution for this problem consists in this, that the resultant of the two known forces G_1 and G_2 intersects the line of X' in a point which is easily found by § 21. The hinge-pressure must pass through this point, since three forces in equilibrium must be concurrent.]

Note that the line of action of the pressure at 0, i.e., of the resultant of X_0 and Y_0, *does not coincide with the axis of the rod;* the rod being subjected to more than the two forces at its extremities. The case therefore differs from that presented by the boom in Ex. 2 of p. 12.

Problem 2.—Given two rods with loads, three hinges (or "pin-joints"), and all dimensions: required the three hinge-pressures; i.e., there are six unknowns, viz., three X and three Y components. We obtain three equations from each of the two free bodies in Fig. 37. The student may fill out the details. Notice the application of the principle of action and reaction at B (see § 3).

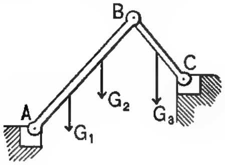

FIG. 36.

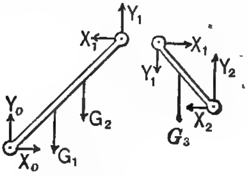

FIG. 37.

Problem 3.—A Warren bridge-truss rests on the horizontal *smooth* abutment-surfaces in Fig. 38. It is composed of equal isosceles triangles; no piece is continuous beyond a joint, each of which is a *pin connection.* All loads are considered as acting at the joints, so that each piece will be subjected to a simple tension or compression. "*Two-force* pieces; see p. 18, *Notes.*)

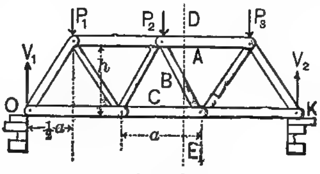

FIG. 38.

First, required the reactions of the supports V_1 and V_2; these and the loads are called the *external* forces. $\Sigma(Pa)$ about $O = 0$ gives (the whole truss is the free body)

$$V_2 3a - P_1 \cdot \tfrac{1}{2}a - P_2 \cdot \tfrac{3}{2}a - P_3 \cdot \tfrac{5}{2}a = 0;$$

while $\Sigma(Pa)$ about $K = 0$ gives

$$- V_1 \cdot 3a + P_3 \cdot \tfrac{1}{2}a + P_2 \tfrac{3}{2}a + P_1 \tfrac{5}{2}a = 0;$$
$$\therefore V_1 = \tfrac{1}{6}[5P_1 + 3P_2 + P_3];$$
and $$V_2 = \tfrac{1}{6}[P_1 + 3P_2 + 5P_3].$$

Secondly, required the stress (thrust or pull, compression or tension) in each of the pieces A, B, and C cut by the imaginary line DE. The stresses in the pieces are called *internal* forces. These appear in a system of forces acting on a free body only when a portion of the truss or frame is conceived separated from the remainder in such a way as to expose an internal plane of one or more pieces. Consider as a free body the portion on the left of DE (that on the right would serve as well, but the pulls or thrusts in A, B, and C would be found to act in directions opposite to those they have on the other portion; see § 3). Fig. 39. The arrows (forces) A, B, and C, are assumed to point, respectively, in the directions shown in the figure. They, with V_1, P_1, and P_2, form a system holding the body in equilibrim.

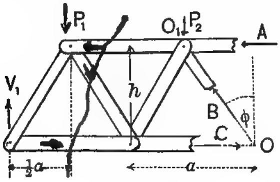

FIG. 39.

For this system, $\Sigma(Pa)$ about $O = 0$ gives

$$0 + Ah - V_1 2a + P_1 \cdot \tfrac{3}{2}a + P_2 \cdot \tfrac{1}{2}a = 0;$$

and hence $$A = (\tfrac{1}{2}a \div h)[4V_1 - 3P_1 - P_2],$$

which is *positive;* since, (see above), $4V_1$ is $> 3P_1 + P_2$.

Therefore the assumption that A points to the left is confirmed and A is a thrust, or compression; (its value as above.)

Again, taking moments about O_1 (intersection of A and B), we have an equation in which the only unknown is C, viz.,

$$Ch - V_1 \cdot \tfrac{3}{2}a + P_1 a = 0; \quad \therefore C = (\tfrac{1}{2}a \div h)[3V_1 - 2P_1],$$

a positive value since $3V_1$ is $>2P_1$; $\therefore C$ must point to the right as assumed; i.e., is a tension, and$=\frac{a}{2h}[3V_1-2P_1]$.

Finally, to obtain B, put Σ(vert. comps.)$=0$; i.e.

$$B \cos \phi + V_1 - P_1 - P_2 = 0.$$

$\therefore B \cos \phi = P_1 + P_2 - V_1$; but, (see foregoing value of V_1) we may write

$$V_1 = (P_1 + P_2) - (\tfrac{1}{6}P_1 + \tfrac{1}{2}P_2) + \tfrac{1}{6}P_3.$$

$\therefore B \cos \varphi$ will be $+$ (upward) or $-$ (downward), and B will be compression or tension, as $\frac{1}{6}P_3$ is $<$ or $>$ $[\frac{1}{6}P_1 + \frac{1}{2}P_2]$.

$$B = [P_1 + P_2 - V_1] \div \cos \varphi = \frac{\sqrt{h^2 + \frac{1}{4}a^2}}{h}[P_1 + P_2 - V_1].$$

Problem 4.—Given the weight G_1 of rod, the weight G_2, and all the geometrical elements (the student will assume a convenient notation); required the tension in the cord, and the amount and direction of pressure on hinge-pin. Fig. 41.

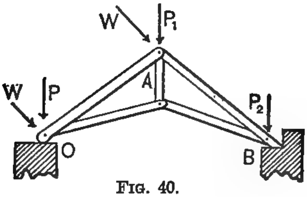

FIG. 40.

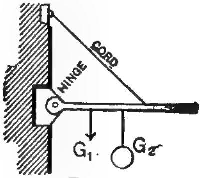

FIG. 41.

Problem 5.—Roof-truss; pin-connection; all loads at joints; wind-pressures W and W_1 normal to OA; required the three reactions or supporting forces (of the two horizontal surfaces and one vertical surface), and the stress in each piece. All geometrical elements are given; also P, P_1, P_2, W (Fig. 40).

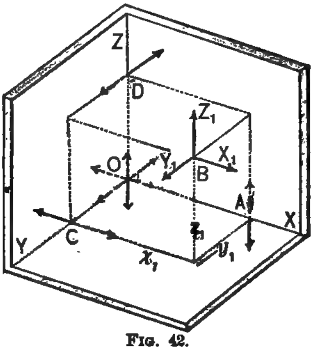

FIG. 42.

38. Composition of Non-concurrent Forces in Space.—Let P_1, P_2, etc., be the given forces, and x_1, y_1, z_1, x_2, y_2, z_2, etc., their points of application referred to an arbitrary origin and axes; α_1, β_1, γ_1, etc., the angles made by their lines of application with X, Y, and Z.

Considering the first force* P_1, replace it by its three components parallel to the axes, $X_1 = P_1 \cos \alpha_1$; $Y_1 = P_1 \cos \beta_1$; and $Z_1 = P_1 \cos \gamma_1$ (P_1 itself is not shown in the figure). At O, and also at A, put a pair of equal and opposite forces, each equal and parallel to Z_1; Z_1 is now replaced by a single force Z_1 acting upward at the origin, and two couples, one in a plane parallel to YZ and having a moment $= - Z_1y_1$ (as we see it looking toward O from a remote point on the axis $+ X$), the other in a plane parallel to XZ and having a moment $= + Z_1x_1$ (seen from a remote point on the axis $+ Y$). Similarly at O and C put in pairs of forces equal and parallel to X_1, and we have X_1, at B, replaced by the single force X_1 at the origin, and the couples, one in a plane parallel to XY, and having a moment $+ X_1y_1$, seen from a remote point on the axis $+ Z$, the other in a plane parallel to XZ, and of a moment $= - X_1z_1$, seen from a remote point on the axis $+ Y$; and finally, by a similar device, Y_1 at B is replaced by a force Y_1 at the origin and two couples, parallel to the planes XY and YZ, and having moments $- Y_1x_1$ and $+ Y_1z_1$, respectively. (In Fig. 42 the single forces at the origin are broken lines, while the two forces constituting any one of the six couples may be recognized as being equal and parallel, of opposite directions, and both continuous, or both dotted.) We have, therefore, replaced the force P_1 by three forces X_1, Y_1, Z_1, at O, and six couples (shown more clearly in Fig. 43; the couples have been transferred to symmetrical positions). Combining each two couples whose axes are parallel to X, Y, or Z, they can be reduced to three, viz.,

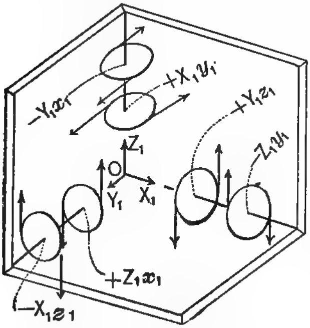

Fig. 43.

one with an X axis and a moment $= Y_1z_1 - Z_1y_1$;
one with a Y axis and a moment $= Z_1x_1 - X_1z_1$;
one with a Z axis and a moment $= X_1y_1 - Y_1x_1$.

---

* This "first force," P_1, is applied at the point B, whose co-ordinates are x_1, y_1, and z_1, and is typical of all the other forces of the system.

Dealing with each of the other forces P_2, P_3, etc., in the same manner, the whole system may finally be replaced by three forces ΣX, ΣY, and ΣZ, at the origin and three couples whose moments are, respectively, (ft-lbs., for example)

$$L = \Sigma(Yz - Zy) \text{ with its axis parallel to } X;$$
$$M = \Sigma(Zx - Xz) \text{ with its axis parallel to } Y;$$
$$N = \Sigma(Xy - Yx) \text{ with its axis parallel to } Z.$$

The "axes" of these couples, being parallel to the respective co-ordinate axes X, Y, and Z, and proportional to the moments L, M, and N, respectively, the axis of their resultant C, whose moment is G, must be the diagonal of a parallelopipedon constructed on the three component axes (proportional to) L, M, and N. Therefore, $G = \sqrt{L^2 + M^2 + N^2}$, while the resultant of ΣX, ΣY, and ΣZ is

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}$$

acting at the origin. If α, β, and γ are the direction-angles of R, we have $\cos\alpha = \frac{\Sigma X}{R}$, $\cos\beta = \frac{\Sigma Y}{R}$, and $\cos\gamma = \frac{\Sigma Z}{R}$; while if λ, μ, and ν are those of the *axis* of the couple C, we have $\cos\lambda = \frac{L}{G}$, $\cos\mu = \frac{M}{G}$, and $\cos\nu = \frac{N}{G}$.

For equilibrium we have both $G = 0$ and $R = 0$; i.e., separately, *six conditions*, viz.,

$$\Sigma X = 0,\ \Sigma Y = 0,\ \Sigma Z = 0; \quad \text{and} \quad L = 0,\ M = 0,\ N = 0 \quad . \quad (1)$$

Now, noting that $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma(Xy - Yx) = 0$ are the conditions for equilibrium of the system of non-concurrent forces which would be formed by projecting each force of our actual system upon the plane XY, and similar relations for the planes YZ and XZ, we may restate equations (1) in another form, more serviceable in practical problems, viz.:

Note.—*If a system of non-concurrent forces in space is in equilibrium, the plane systems formed by projecting the given system upon each of three arbitrary co-ordinate planes will each be in equilibrium.* But we can obtain *only six independent* *equations* in any case, available for six unknowns. If R alone $= 0$, we have the system equivalent to a couple C, whose moment $= G$; if G alone $= 0$, the system has a single resultant R applied at the origin. In general, neither R nor G being $= 0$, we cannot further combine R and C (as was done with non-concurrent forces in a plane) to produce a single resultant unless R and C happen to be in parallel planes; in which case the system may be reduced to a single resultant by use of the device explained near foot of p. 32.

Remark.—In general, R and C not being in parallel planes, the system may be reduced to *two single forces* not in the same plane, by assigning any value we please to P, one of the forces of the couple C, computing the corresponding arm $a = G \div P$, transferring C until one of the P's has the same point of application as R, and then combining these two forces into a single resultant. This last force and the second P are, then, the equivalent of the original system, but are not in the same plane. (See §§ 15 and 15*a*.)

Again, if a reference plane be chosen at right angles to R, and the couple C be decomposed into two couples, one in the reference plane and the other in a plane at right angles to it, this second couple and R may be replaced by a single force (as on p. 32) and we then have the whole system replaced by a *single force* and *a couple situated in a plane perpendicular to that force;* (and this may be called a "screwdriver action.")

Example.—A shaft, with crank and drum attached and supported horizontally on two *smooth* cylindrical bearings, constitutes a hoisting device. See Fig. 43*a*. A force P is to be applied to the crank handle at 30° with the horizontal (and ⏋ to the crank), and acting in a plane at right angles to the shaft; and is to be of such value as to preserve equilibrium when the weight of 800 lbs. is sustained, as shown. The weight of the shaft, etc., is 200 lbs., and its center of gravity is at C in the axis of the shaft. (Counterpoise for crank not shown.) The reactions at the two bearings will lie in planes ⏋ to the axis of the shaft (*smooth* cylindrical surfaces), making unknown angles with the vertical; and will be represented by their X and Z components,

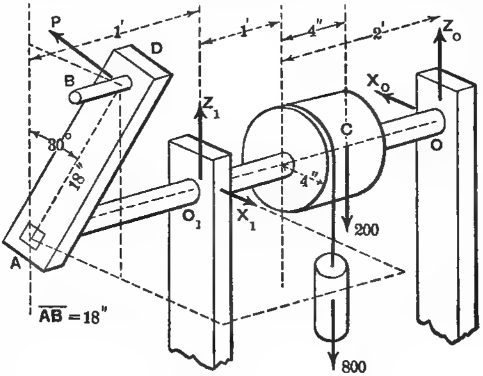

FIG. 43*a*.

as shown. It is required to find the proper value for P and the amount and position of the two reactions.

Solution.—The seven forces shown in the figure (of which five are unknown) constitute a non-concurrent system of forces in space; in equilibrium. Since there are no Y-components the condition $\Sigma Y=0$ is already satisfied. Let us now apply the statement of the "note" on p. 39, first projecting the forces on the plane ZX (vertical plane ⊥ to the shaft). (That is, we take an "*end-view*" of the system.) Each of the seven forces projects in full length, or value, since they are all parallel to that plane. Treating the plane system so formed as in equilibrium and taking moments about the point O, we find (feet and lbs.)

$$+P\times 1.5-800\times \tfrac{1}{3}+0=0; \quad \therefore\ P=177.77 \text{ lbs.} \quad . \quad . \quad . \quad (1)$$

Next projecting on the vertical plane ZY, containing axis of shaft (i.e., taking a "*side-view*" of the system) we note that the projection of P is $P \sin 30°$ and those of X_0 and X_1 each zero, while Z_0, Z_1, and the 200 and 800 lbs., project in full length; hence taking moments about O we have

$$+200\times 1\tfrac{2}{3}+800\times 2-Z_1\times 3-P\times 0.50\times 4+0=0 \quad . \quad . \quad . \quad (2)$$

while moms. about O_1 gives $+Z_0\times 3-P\times 0.5\times 1-200\times 1\tfrac{1}{3}-800\times 1=0$ (3)

Finally, projecting on the horizontal plane XY ("top-view"), the forces in this projection are $P \cos 30°$, X_0, and X_1; so taking moms. about point O_1,

$$+X_0\times 3-P\times 0.8660\times 1=0; \quad \therefore\ X_0=+51.34 \text{ lbs.}; \quad . \quad (4)$$

while from $\Sigma X=0$, $X_1-X_0-P\times 0.866=0$; i.e., $X_1=205.3$ lbs. . . (5)

From (2) and (3) we have $Z_1=+525.93$ lbs., and $Z_0=+385.18$ lbs. All these + signs show that the arrows for X_0, X_1, Z_0, and Z_1 have been correctly assumed (Fig. 43*a*) as to direction. Combining results, we find that the pressure or reaction at O is R_0, $=\sqrt{X_0^2+Z_0^2}$, $=388.6$ lbs. and makes an angle whose tang. is $X_0 \div Z_0$, (i.e., 0.1333), viz., 7° 36′, on the left of the vertical; also that the pressure or reaction at O_1 is R_1, $=\sqrt{X_1^2+Z_1^2}=564.6$ lbs., at an angle on the *right* of the vertical whose tang., $=X_1 \div Z_1$, $(=0.3905)$; i e., 21° 20′.

39. Problem (Somewhat similar to the foregoing.)—Given all geometrical elements (including α, β, γ, angles of P), also the weight of Q, and weight of apparatus G; A being a hinge whose pin is in the axis Y, O a *ball-and-socket* joint: required the *amount* of P (lbs.) to preserve equilibrium, also the pressures (*amount* and *direction*) at A and O; no friction. Replace P by its X, Y, and Z components. The pressure at A will have Z and X components; that at O, X, Y, and Z components. [Evidently there are six unknowns; Y_0 will come out negative.

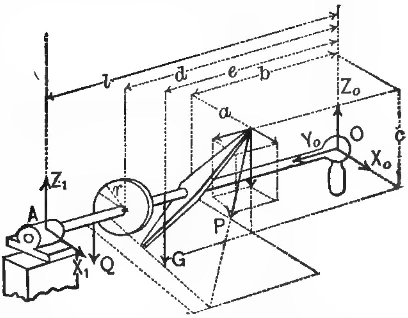

FIG. 44.

CHAPTER IV.

STATICS OF FLEXIBLE CORDS.

40. Postulate and Principles.—The cords are perfectly flexible and inextensible. All problems will be restricted to one plane. Solutions of problems are based on two principles, viz.:

PRIN. I.—The strain or tension, in a cord at any point can act only along the cord, or along the tangent if it be curved.

PRIN. II.—We may apply to flexible cords in equilibrium all the conditions for the equilibrium of rigid bodies; since, if the system of cords became rigid, it would still, with greater reason, be in equilibrium.

41. The Simple Pulley.—A "simple pulley" is one that is acted on by only *one* cord (or belt) and the reaction of the bearing supporting its axle (or "journal").

A cord in equilibrium over a simple pulley whose axle is smooth is under equal tensions on both sides; for, Fig. 46, considering the pulley and its portion of cord free $\Sigma(Pa) = 0$ about the centre of axle gives $P'r = Pr$, i.e., $P' = P =$ tension in the cord. Hence the pressure R at the axle *bisects the angle* α, and therefore if a weighted pulley rides upon a cord ABC, Fig. 47, its position of equilibrium, B, may be found by cutting the vertical through A by an arc of radius $CD =$ length of cord, and centre at C, and drawing a horizontal through the middle of AD to cut CD in B. A smooth ring would serve as well as the pulley; this would be a *slip knot*. From Fig. 46, $R = 2P \cos \frac{1}{2}\alpha$.

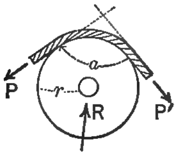

FIG. 46.

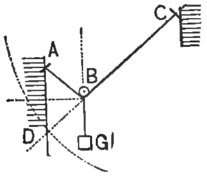

FIG. 47.

42. If three cords meet at a *fixed knot*, and are in equilibrium, the tension in any one is the equal and opposite of the resultant of those in the other two.

43. Tackle.—If a cord is continuous over a number of sheaves in blocks forming a tackle, neglecting the weight of the cord and blocks and friction of any sort, we may easily find the ratio between the cord-tension P and the weight to be sustained. E.g., Fig. 48, regarding all the straight cords as vertical and considering the block B free, we have, Fig. 49 (from $\Sigma Y = 0$), $4P - G = 0$, $\therefore P = \frac{G}{4}$. The stress on the support C will $= 5P$.

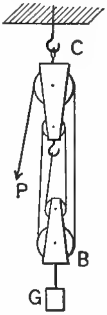

FIG. 48.

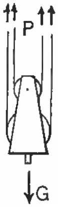

FIG. 49.

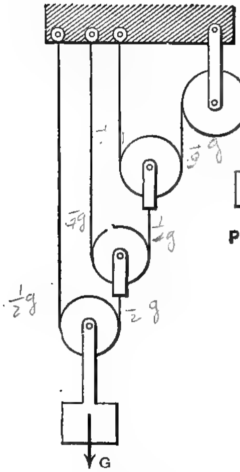

FIG. 49*a*.

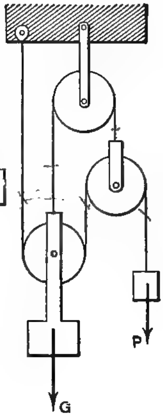

FIG. 49*b*.

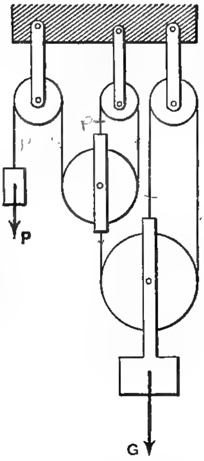

FIG. 49*c*.

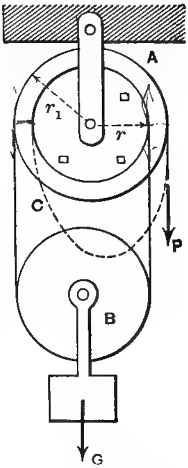

FIG. 49*d*.

Other designs of tackle are presented in Figs. 49*a*, 49*b*, 49*c*, and 49*d*, and should be worked out as exercises by the student. In each case the weight G is supposed to be given and a value of the smaller weight (or pull) P must be determined for the equilibrium of the tackle. Friction, and the weights of the pulleys and cords, are neglected and all straight parts of cords (or chains) are considered vertical.

All of the pulleys shown are "simple pulleys," *except the one at A* in Fig. 49*d*, which represents a "*differential pulley*" tackle. Pulley A consists of two ordinary pulleys fastened together, the groove in each being so rough, or furnished with "sprocket-teeth" in case a chain is used, that *slipping of the cord or chain is prevented.* The chain or cord is endless, the loop C being slack. B is a simple pulley. In this case, for equilibrium the pull P must $= \frac{1}{2}G(r_1 - r) \div r_1$. The other results are $P = \frac{1}{8}G$ for Fig. 49*a*; $\frac{1}{4}G$ for 49*b*; and $\frac{1}{9}G$ for 49*c*.

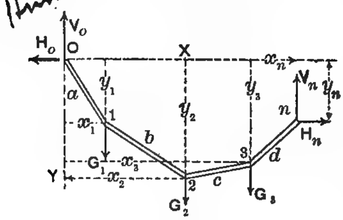

FIG. 50.

44. Weights Suspended at Fixed Knots.—Given all the geometrical elements in Fig. 50, and one weight, G_1; required the remaining weights and the forces H_0, V_0, H_n and V_n, at the points of support, that equilibrium may obtain. H_0 and V_0 are the horizontal and vertical components of the tension in the cord at O; similarly H_n and V_n those at n. There are $n+2$ unknowns. (The solution of this problem is deferred. See p. 420.)

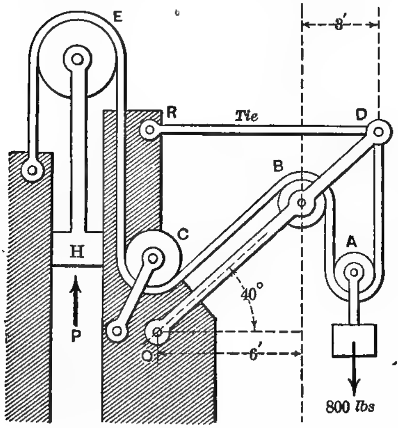

FIG. 50a.

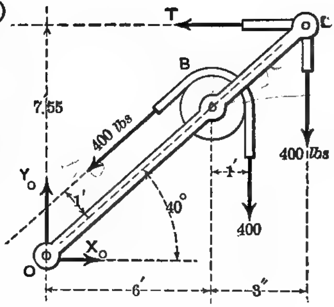

FIG. 50b.

45. Example.—The boom OD, tie-rod RD, with four simple pulleys and a cable, form a crane as shown in Fig. 50a. Find the necessary vertical force P to be exerted on the piston at H, that the load of 800 lbs. may be sustained. Also find the pressure of pulley B on its bearing, the pull T in the tie-rod and the pressure P_0 (amount and position) at pin O; neglecting all friction and rigidity (p. 192) and the weights of the members. Dimensions as in figure. Since all pulleys are "simple" the tension in cable is the same at all points; and is $=400$ lbs. since the straight parts of cable adjoining pulley A are parallel. For a similar reason $P=800$ lbs.

Pressure at B bisects the angle (50°) between adjoining straight parts of cable; i.e., is 25° with vertical, and $=2\times400\times\cos 25°=725$ lbs., (§ 41).

Next take the free body in Fig. 50b (boom and pulley B together with a part of cable.) Three unknowns and three equations.

$\Sigma(\text{moms.})_0=0$; $\therefore\ +T\times9\times\tan 40°+400\times1-400\times9-400\times7=0$ (1)

i.e., $T\times9\times0.8391=6000$ ft.-lbs.; $\therefore\ T=794.3$ lbs., (tension in tie-rod.)

$\Sigma X=0$, $\therefore X_0-400\cos 40°-T=0$; $\therefore X_0=794.3+400\times.766=1100.7$ lbs.

$\Sigma Y=0$, $\therefore\ Y_0-400\sin 40°-400-400=0$;

$\therefore\ Y_0=400\times.6428+800=1057.12$ lbs.

Hence P_0, $=\sqrt{X_0^2+Y_0^2}$, $=1526$ lbs. at $\tan^{-1}Y_0/X_0$, or 43° 50′, with horiz.

Note.—If the weight 800 lbs. were attached directly to cable on right of pulley B, the value of P would need to be 1600 lbs.

46. Loaded Cord as Parabola.—If the weights are equal and infinitely small, and are intended to be uniformly spaced *along the horizontal*, when equilibrium obtains, the cord having no weight, it will form a parabola. Let q = weight of loads per horizontal linear unit, O be the vertex of the curve in which the cord hangs, and m any point. We may consider the portion Om as a **free body**, if the reactions of the contiguous portions of the cord are put in, H_0 and T, and these (from Prin. I.) must act along the tangents to the curve at O and m, respectively; i.e., H_0 is horizontal, and T makes some angle φ (whose tangent $= \frac{dy}{dx}$, etc.) with the axis X. Applying Prin. II.,

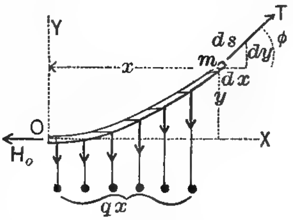

FIG. 51.

$$\Sigma X = 0 \text{ gives } T\cos\varphi - H_0 = 0;\ \text{i.e., } T\frac{dx}{ds} = H_0; \quad . \quad . \quad (1)$$

$$\Sigma Y = 0 \text{ gives } T\sin\varphi - qx = 0;\ \text{i.e., } T\frac{dy}{ds} = qx. \quad . \quad . \quad (2)$$

Dividing (2) by (1), member by member, we have $\frac{dy}{dx} = \frac{qx}{H_0}$;

$\therefore dy = \frac{q}{H_0}xdx$, the differential equation of the curve;

$y = \frac{q}{H_0}\int_0^x xdx = \frac{q}{H_0}\cdot\frac{x^2}{2}$; or $x^2 = \frac{2H_0}{q}\cdot y$, the equation of a **parabola** whose vertex is at O and axis vertical.

NOTE.—The same result, $\frac{dy}{dx} = \frac{qx}{H_0}$, may be obtained by considering that we have here (Prin. II.) a *free rigid body* acted on by three forces, T, H_0, and $R = qx$, acting vertically through the middle of the abscissa x; the resultant of H_0 and R must be equal and opposite to T, Fig. 52. $\therefore$ $\tan\varphi = \frac{R}{H_0}$, or $\frac{dy}{dx} = \frac{qx}{H_0}$. Evidently also the tangent-line bisects the abscissa x. (Try moments about m.)

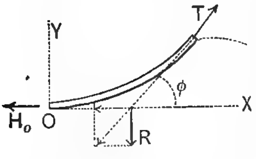

FIG. 52.

Example.—Let q=800 lbs. "per foot run" and x=100 ft., with y=20 ft. Then we have, for the value of the tension at the vertex O of the parabola,

$$H = qx^2 \div 2y = 800\times(100)^2 \div 40 = 200{,}000 \text{ lbs.}$$

47. Problem under § 46. [Case of a suspension-bridge in which the suspension-rods are vertical, the weight of roadway is uniform per horizontal foot, and large compared with that of the cable and rods. Here the roadway is the only load: it is generally furnished with a stiffening truss to avoid deformation under passing loads.]—Given the span $= 2b$, Fig. 53, the deflection $= a$, and the rate of loading $= q$ lbs. per horizontal foot; required the tension in the cable at O, also at m; and the length of cable needed. From the equation of the parabola $qx^2 = 2H_0y$, putting $x = b$ and $y = a$, we have $H_0 = qb^2 \div 2a =$ the tension at O. From $\Sigma Y = 0$ we have $V_1 = qb$, while $\Sigma X = 0$ gives $H_1 = H_0$; $\therefore$ the tension at $m = \sqrt{H_1^2 + V_1^2} = \frac{1}{2a}[qb\sqrt{4a^2 + b^2}]$.

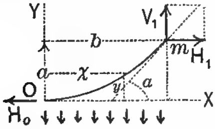

FIG. 53.

The semi-length, Om, of cable (from p. 88, Todhunter's Integral Calculus) is (letting n denote $H_0 \div 2q$, $= b^2 \div 4a$)

$$Om = \sqrt{na + a^2} + n \,.\, \log_e [(\sqrt{a} + \sqrt{n + a}) \div \sqrt{n}].$$

48. The Catenary.*—A flexible, inextensible cord or chain, of uniform weight per unit of length, hung at two points, and supporting *its own weight alone*, forms a curve called the **catenary.** Let the tension H_0 at the lowest point or vertex be represented (for algebraic convenience) by the weight of an imaginary length, c, of similar cord weighing q lbs. per unit of length, i.e., $H_0 = qc$; an actual portion of the cord, of length s, weighs qs lbs. Fig. 54 shows as *free* and in equilibrium a portion of the curve of any length s, reckoning from O the vertex. Required the equation of the curve. The load is uniformly spaced *along the curve*, and not horizontally, as in §§ 46 and 47.

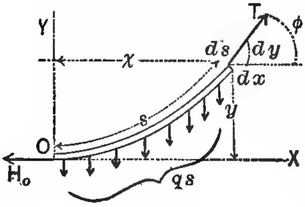

FIG. 54.

$\Sigma Y = 0$ gives $T\frac{dy}{ds} = qs$; while $\Sigma X = 0$ gives $T\frac{dx}{ds} = qc$. Hence, by division, $c\,dy = s\,dx$, and squaring, $c^2dy^2 = s^2dx^2$. . . . . (1)

---

* For the "transformed catenary," see p. 395.

Put $dy^2 = ds^2 - dx^2$, and we have, after solving for dx

$$dx = \frac{cds}{\sqrt{s^2+c^2}}. \quad \therefore x = c\int_0^s \frac{ds}{\sqrt{s^2+c^2}} = c\Big[_0^s \log_e (s + \sqrt{s^2+c^2})$$

and
$$x = c \,.\, \log_e [(s + \sqrt{s^2+c^2}) \div c], \quad . \quad . \quad . \quad (2)$$

a relation between the horizontal abscissa and length of curve

Again, in eq. (1) put $dx^2 = ds^2 - dy^2$, and solve for dy. This gives $dy = \frac{sds}{\sqrt{c^2+s^2}} = \frac{1}{2} \cdot \frac{d(c^2+s^2)}{(c^2+s^2)^{\frac{1}{2}}}$. Therefore

$$y = \tfrac{1}{2}\int_0^s (c^2 + s^2)^{-\frac{1}{2}} d(c^2+s^2) = \tfrac{1}{2}\Big[_0^s 2(c^2+s^2)^{\frac{1}{2}},$$ and finally

$$y = \sqrt{s^2+c^2} - c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Clearing of radicals and solving for c, we have

$$c = (s^2 - y^2) \div 2y. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now T, the tension at any point, $= \sqrt{(qs)^2 + (qc)^2}$, and from (3) we obtain Know how to derive.

$$T = q(y+c). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

Example.—A 40-foot chain weighs 240 lbs.,' and is so hung from two points at the same level that the deflection is 10 feet. Here, for $s = 20$ ft., $y = 10$; hence eq. (4) gives the *parameter*, $c = (400-100) \div 20 = 15$ feet. $q = 240 \div 40 = 6$ lbs. per foot. ∴ the tension at the middle is $H_0 = qc = 6 \times 15 = 90$ lbs.; while the greatest tension is at either support and $= \sqrt{90^2 + 120^2} = 150$ lbs.

Knowing $c = 15$ feet, and putting $s = 20$ feet = half length of chain, we may compute the corresponding value of x from eq. (2); this will be the half-span. That is,

$$x = 15 \,.\, \log_e 3 = 15 \times 2.303 \times 0.4771 = 16.48 \text{ ft.}$$

To derive s in terms of x, transform eq. (2) in the way that $n = \log_e m$ may be transformed into $e^n = m$, clear of radicals and solve for s, obtaining *

$$s = \tfrac{1}{2}c\left[e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right]; \quad \text{or,} \quad s = c \,.\, \sinh\left(\frac{x}{c}\right). \quad . \quad . \quad (5)$$

Again, eliminate s from (2) by substitution from (3), transform as above, clear of radials, and solve for $y + c$, whence

$$y + c = \tfrac{1}{2}c\left[e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right]; \quad \text{or,} \quad y + c = c \,.\, \cosh\left(\frac{x}{c}\right). \quad (6)$$

which is the equation of a catenary with axes as in Fig. 54. If the horizontal axis be taken a distance $= c$ below the vertex, the new ordinate $z=y+c$, while x remains the same; the last equation is simplified. See figure below.

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* sinh and cosh denote "*hyperbolic*" sine and cosine; see table, appendix.

If the span and length of chain are given, or if the span and deflection are given, c can be determined from (5) or (6) only by successive assumptions and approximations.

48a. Catenary (Chain or Cable) with Supports at Different Levels.—Given the span k, the difference of elevation d of the two supports, and the whole length of chain, l; it is required to find x' and y' (see Fig. 54a) and thus determine the position of the vertex, or lowest point, O, of the catenary. By applying the equations of p. 47 to parts $A'O$ and $B'O$, in turn, and combining, we may finally deduce (see p. 179 of Rankine's Applied Mechanics)

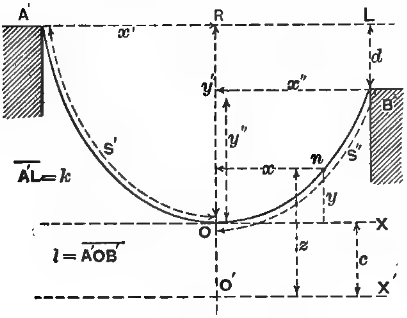

FIG. 54a.

$$\sqrt{l^2-d^2}=c\left[e^{\frac{k}{2c}}-e^{-\frac{k}{2c}}\right]; \qquad \text{i.e.,} \quad \sqrt{l^2-d^2}=2c\,.\sinh\left(\frac{k}{2c}\right). \quad . \quad (7)$$

and also the relation $$x'-x''=c\cdot\log_e\left[\frac{l+d}{l-d}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From (7) we find the "*parameter*," c, by trial; then the value of $x'-x''$ from (8); whence, finally, we obtain x' and x'' separately (since $x'+x''=k$). With x' known y' is found from (6); i.e.,

$$y'+c=\tfrac{1}{2}c\left[e^{\frac{x'}{c}}+e^{-\frac{x'}{c}}\right]; \quad \text{or,} \quad y'+c=c\cdot\cosh\left(\frac{x'}{c}\right). \quad . \quad . \quad . \quad (9)$$

Thus the position of the vertex is located. The greatest tension will be at the highest point A', viz., $T_{A'}=q(y'+c)$. . . . . . . (10)

[The expression $\frac{1}{2}[e^u-e^{-u}]$ is called the "*hyperbolic sine*" of the number u, or sinh (u); and $\frac{1}{2}[e^u+e^{-u}]$ the "*hyperbolic cosine*" of u, or cosh (u): e being the Naperian base 2.71828 ... Tables of sinh u and cosh u will be found in the appendix.]

Example.—A chain 100 ft. long is supported at two points 80 ft. apart horizontally and 30 ft. vertically; find the position of its lowest point. That is, (Fig. 54a) given $l=100$ ft., $k=80$ ft., $d=30$ ft.

Solution. $\sqrt{l^2-d^2}=\sqrt{9100}=95.4$ ft., the left-hand member of (7). Assuming $c=20$ ft. as a first trial, we find $k\div 2c=2.00$ and sinh (2.00) $=3.6269$, so that $2c$ sinh (2.00) is 145.176, which is much larger than 95.4. Next, with c assumed as 40, 39, and 38.3 ft., we find $2c$ sinh ($k\div 2c$) to be $2\times 40\times 1.1745=93.96$; $2\times 39\times 1.2153=94.8$; and $2\times 38.3\times 1.2444=95.3$, respectively; and hence conclude that $c=38.2$ ft. will satisfy eq. (7) with sufficient accuracy. Eq. (8) now becomes

$$x'-x''=38.2\times 2.303\times 0.2689=23.66 \text{ ft.};$$

and finally we obtain $x'=51.83$ and $x''=28.17$ ft. From eq. (9) we now have $y'+38.2=38.2\times 2.0713=79.10$ ft. and $\therefore\ y'=40.9$ ft. With $q=1.5$ lbs. per foot, the tension at $A'=1.5\times 79.1=118.6$ lbs.

PART II.—KINETICS.

CHAPTER I.

RECTILINEAR MOTION OF A MATERIAL POINT.

49. Uniform Motion implies that the moving point passes over equal distances in equal times; **variable motion,** that unequal distances are passed over in equal times. In uniform motion the distance passed over in a unit of time, as one second, is called the **velocity** ($= v$), which may also be obtained by dividing the length of *any portion* ($= s$) of the path by the time ($= t$) taken to describe that portion, however small or great; in variable motion, however, the velocity varies from point to point, its value at any point being expressed as the quotient of ds (an infinitely small distance containing the given point) by dt (the infinitely small portion of time in which ds is described).

49*a*. By **acceleration** is meant the rate at which the velocity of a variable motion is changing at any point, and may be a *uniform acceleration*, in which case it equals the total change of velocity between any two points, however far apart, divided by the time of passage; or a *variable acceleration*, having a different value at every point, this value then being obtained by dividing the velocity-increment, dv, or gain of velocity in passing from the given point to one infinitely near to it, by dt, the time occupied in acquiring the gain.* (Acceleration must be understood in an algebraic sense, a negative acceleration implying a decreasing velocity, or else that the velocity in a negative direction is increasing.) The foregoing applies to motion in a path or line of any form whatever, the distances mentioned being portions of the path, and therefore measured along the path. (See p. 43 in the "Notes," etc.

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* See addendum on p. 836.

50. Rectilinear Motion, or motion in a straight line.—The general relations of the quantities involved may be thus stated (see Fig. 55): Let $v =$ velocity of the body at any instant; then $dv =$ gain of velocity in an instant of time dt. Let $t =$ time elapsed since the body left a given fixed point, which will be taken as an origin, O. Let $s =$ distance (+ or —) of the body, at any instant, from the origin O; then $ds =$ distance traversed in a time dt. Let $p =$ acceleration $=$ *rate at which v is increasing* at any instant. All these may be variable; and t is taken as the **independent variable,** i.e., time is conceived to elapse by *equal* small increments, each $= dt$; hence two consecutive ds's will not in general be equal, their difference being called d^2s. Evidently d^2t is $=$ zero, i.e., dt is constant.

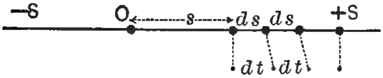

FIG. 55.

Since $\frac{1}{dt}$ = number of instants in one second, the velocity at any instant (i.e., the distance which *would* be described *at that rate* in one second) is $v = ds \cdot \frac{1}{dt}$; $\therefore v = \frac{ds}{dt}$. . . . . (I.)

Similarly, $p = dv \cdot \frac{1}{dt}$, and $\left(\text{since } dv = d\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt}\right)$.

$$\therefore p = \frac{dv}{dt} = \frac{d^2s}{dt^2}. \quad \ldots\ldots \quad \text{(II.)}$$

Eliminating dt, we have also $vdv = pds$. . . . . . (III.)

These are the fundamental differential formulæ of rectilinear motion (for curvilinear motion we have these and some in addition) as far as kinematics, i.e., as far as space and time, is concerned. The consideration of the mass of the material point and the forces acting upon it will give still another relation (see § 55).

Example.—If we have given $s = [6t^3 + 3t^2 + 2t]$ *ft.*, for a certain motion, then the velocity, v, at any time, $= ds \div dt$, $= [18t^2 + 6t + 2]$ *ft. per sec.;* and the acceleration, p, $= dv \div dt$, $= [36t + 6]$ *ft. per sec. per sec.*

51. Rectilinear Motion due to Gravity.—If a material point fall freely in vacuo, no initial direction other than vertical having been given to its motion, many experiments have shown that this is a uniformly accelerated rectilinear motion in a vertical line having an acceleration (called the *acceleration of gravity*) equal to 32.2 feet per square second,* or 9.81 metres per square second; i.e., the velocity increases at this constant rate in a downward direction, or decreases in an upward direction.

[NOTE.—By "square second" it is meant to lay stress on the fact that an acceleration (being $= d^2s \div dt^2$) is in quality equal to one dimension of length divided by two dimensions of time. E.g., if instead of using the foot and second as units of space and time we use the foot and the minute, g will $= 32.2 \times 3600$; whereas a velocity of say six feet per second would $= 6 \times 60$ feet per minute. The value of $g = 32.2$ implies the units foot and second, and is sufficiently exact for practical purposes.]

52. Free Fall in Vacuo.—Fig. 56. Let the body start at O with an initial downward velocity $= c$. The acceleration is constant and $= + g$. Reckoning both time and distance (+ downwards) from O, required the values of the variables s and v after any time t. From eq. (II.), § 50, we have $+ g = dv \div dt$; $\therefore dv = gdt$, in which the two variables are separated.

Hence $\int_c^v dv = g\int_0^t dt$; i.e., $\Big[_c^v v = g\Big[_0^t t$; or $v - c = gt - 0$; and finally, $v = c + gt$. . . . . (1)

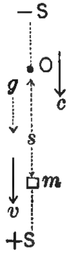

FIG. 56.

(Notice the correspondence of the limits in the foregoing operation; when $t = 0$, $v = + c$.)

From eq. (I.), § 50, $v = ds \div dt$; $\therefore$ substituting from (1), $ds = (c + gt)dt$, in which the two variables s and t are separated.

$$\therefore \int_0^s ds = c\int_0^t dt + g\int_0^t tdt; \text{ i.e., } \Big[_0^s s = c\Big[_0^t t + g\Big[_0^t \frac{t^2}{2},$$

or
$$s = ct + \tfrac{1}{2}gt^2. \quad . \ . \ . \ . \ . \ . \ . \quad (2)$$

Again, eq. (III.), § 50, $vdv = gds$, in which the variables v and s are already separated.

$$\therefore \int_c^v vdv = g\int_0^s ds; \text{ or } \Big[_c^v \tfrac{1}{2}v^2 = g\Big[_0^s s; \text{ i.e., } \tfrac{1}{2}(v^2 - c^2) = gs,$$

or
$$s = \frac{v^2 - c^2}{2g}. \quad . \ . \ . \ . \ . \ . \ . \quad (3)$$

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* Or, 32.2 "feet per second per second."

If the initial velocity $=$ zero, i.e., if the body falls from rest, eq. (3) gives $s = \frac{v^2}{2g}$ and $v = \sqrt{2gs}$. [From the frequent recurrence of these forms, especially in hydraulics, $\frac{v^2}{2g}$ is called the "height due to the velocity v," i.e., the vertical height through which the body must fall from rest to acquire the velocity v; while, conversely, $\sqrt{2gs}$ is called the velocity due to the height or "*head*" s.]

By eliminating g between (1) and (3), we may derive another formula between three variables, s, v, and t, viz.,

$$s = \tfrac{1}{2}(c + v)t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Example.—A leaden ball occupies 4.6 seconds in falling from the eaves of a tall building to the sidewalk; initial velocity zero. Find the height fallen through, $=s'$. We have from eq. (2)

$$s' = 0 + \tfrac{1}{2}(32.2)(4.6)^2 = 341 \text{ ft.}$$

53. Upward Throw.—If the initial velocity were in an upward direction in Fig. 56 we might call it $-c$, and introduce it with a negative sign in equations (1) to (4), just derived; but for variety let us call the upward direction $+$, in which case an upward initial velocity would $= +c$, while the acceleration $= -g$, constant, as before. (The motion is supposed confined within such a small range that g does not sensibly vary.) Fig. 57. From $p = dv \div dt$ we have $dv = -gdt$ and

$$\int_c^v dv = -g\int_0^t dt; \therefore v - c = -gt; \text{or } v = c - gt. \quad (1)a$$

From $v = ds \div dt$, $ds = cdt - gtdt$,

$$\text{i.e.,} \int_0^s ds = c\int_0^t dt - g\int_0^t tdt; \text{ or } s = ct - \tfrac{1}{2}gt^2. \quad (2)a$$

$vdv = pds$ gives $\int_c^v vdv = -g\int_0^s ds$, whence

$$\tfrac{1}{2}(v^2 - c^2) = -gs, \text{ or finally, } s = \frac{c^2 - v^2}{2g}. \quad . \quad (3)a$$

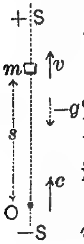


FIG. 57.

And by eliminating g from (1)a and (3)a,

$$s = \tfrac{1}{2}(c + v)t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)a$$

The following is now easily verified from these equations: the body passes the origin again ($s = 0$) with a velocity $= -c$, after a lapse of time $= 2c \div g$. The body comes to rest (for an instant) (put $v = 0$) after a time $= c \div g$, and at a distance $s = c^2 \div 2g$ ("height due to velocity c") from O. For $t > c \div g$, v is negative, showing a downward motion; for $t > 2c \div g$, s is negative, i.e., the body is below the starting-point while the rate of change of v is constant and $= -g$ at all points.

Example.—Let c be 40 ft./sec. Then in a time $= 40 \div 32.2$, $= 1.24$ sec., the body will reach its maximum height, $(40)^2 \div 2 \times 32.2 = 24.84$ ft. above the start. After 3 sec. the body will be found a distance $s_3 = 40 \times 3 - \frac{1}{2}(32.2)(3)^2 = -24.9$ ft. from the origin, i.e., *below* it.

54. Newton's Laws.—As showing the relations existing in general between the motion of a material point and the actions (forces) of other bodies upon it, experience furnishes the following three laws or statements as a basis for kinetics:

(1) A material point under no forces, or under balanced forces, remains in a state of rest or of uniform motion in a right line. (This property is often called *Inertia.*)

(2) If the forces acting on a material point are unbalanced, an acceleration of motion is produced, proportional to the resultant force and in its direction.

(3) Every action (force) of one body on another is always accompanied by an equal, opposite, and simultaneous reaction. (This was interpreted in § 3.)

As all bodies are made up of material points, the results obtained in Kinetics of a Material Point serve as a basis for the Kinetics of a Rigid Body, of Liquids, and of Gases.

55. Mass.—If a body is to continue moving in a right line, the resultant force P at all instants must be directed along that line (otherwise it would have a component deflecting the body from its straight course). (See addendum on p. 835.)

In accordance with Newton's second law, denoting by p the acceleration produced by the resultant force (G being the body's weight), we must have the proportion $P : G :: p : g$; i.e.,

$$P = \frac{G}{g} \cdot p \, \ldots\ldots\ldots, \quad \text{or } P = Mp. \quad . \quad . \quad \text{(IV.)}$$

Eq. IV. and (I.), (II.), (III.) of § 50 are the fundamental equations of Kinetics. Since the quotient $G \div g$ is invariable, wherever the body be moved on the earth's surface (G and g changing in the same ratio), it will be used as the measure of the mass M or quantity of matter in the body. In this way it will frequently happen that the quantities G and g will appear in problems where the weight of the body, i.e., the force of the earth's attraction upon it, and the acceleration of gravity have no direct connection with the circumstances. No name will be given to the unit of mass, it being always understood that the fraction $G \div g$ will be put for M before any numerical substitution is made. From (IV.) we have, in words,

accelerating force = mass × acceleration;
also, *acceleration = accelerating force ÷ mass.*

56. Uniformly Accelerated Motion.—If the resultant force is **constant** as time elapses, the acceleration must be constant (from eq. (IV.), since of course M is constant) and $= P \div M$. The motion therefore will be uniformly accelerated, and we have only to substitute $+ p_1$, (*constant*), for g in eqs. (1) to (4) of § 52 for the equations of this motion, the initial velocity being $= c$ (in the line of the force).

$$v = c + p_1 t \quad . \quad . \quad . \quad (1); \qquad s = ct + \tfrac{1}{2} p_1 t^2; \quad . \quad . \quad . \quad (2)$$

$$s = \frac{(v^2 - c^2)}{2p_1}; \quad . \quad . \quad . \quad (3), \quad \text{and } s = \tfrac{1}{2}(c + v)t \quad . \quad . \quad . \quad (4)$$

If the force is in a negative direction, the acceleration will be negative, and may be called a *retardation;* the initial velocity should be made negative if its direction requires it.

57. Examples of Unif. Acc. Motion.—*Example* 1. Fig. 58. A small block whose weight is ½ lb. has already described a distance $Ao = 48$ inches over a smooth portion of a horizontal table in two seconds; at O it encounters a rough portion, and a consequent constant friction of 2 oz. Required the distance described beyond O, and the time occupied in coming to rest. Since we shall use 32.2 for g, times must be in seconds, and distances in feet; as to the unit

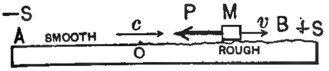


FIG. 58.

of force, as that is still arbitrary, say ounces. Since AO was smooth, it must have been described with a uniform motion (the resistance of the air being neglected); hence with a velocity $= 4$ ft. $\div$ 2 sec. $= 2$ ft. per sec. The initial velocity for the retarded motion, then, is $c = +2$ at O. At any point beyond O the acceleration $=$ force $\div$ mass $= (-2 \text{ oz.}) \div (8 \text{ oz.} \div 32.2) = -8.05$ ft. per square second, i.e., $p = -8.05 =$ constant; hence the motion is uniformly accelerated (retarded here), and we may use the formulæ of § 56 with $c = +2$, $p_1 = -8.05$. At the end of the motion v must be zero, and the corresponding values of s and t may be found by putting $v = 0$ in equations (3) and (1), and solving for s and t respectively: thus from (3), $s = \frac{1}{2}(-4) \div (-8.05)$, i.e., $s = 0.248 +$, which must be feet; while from (1), $t = (-2) \div (-8.05) = 0.248 +$, which must be seconds.

Example 2. (Algebraic.)—Fig. 59. The two masses $M_1 = G_1 \div g$ and $M = G \div g$ are connected by a flexible, inextensible cord. Table smooth. Required the acceleration common to the two rectilinear motions, and the tension in the string S, there being no friction under G_1, none at the pulley, and *no mass* in the latter or in the cord. At any instant of the motion consider G_1 free (Fig. 60), N being the pressure of the table against G_1. Since the motion is in a horizontal right line Σ(vert. compons.)$= 0$, i.e., $N - G_1 = 0$, which determines N. S, the only horizontal force (and resultant of all the forces) $= M_1 p$, i.e.,

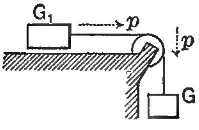


FIG. 59.

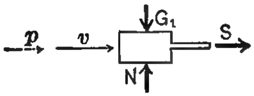


FIG. 60.

$$S = G_1 p \div g. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

At the same instant of the motion consider G free (Fig. 61); the tension in the cord is the same value as above $= S$. The accelerating force is $G - S$, and

$$\therefore = \text{mass} \times \text{acc., or } G - S = (G \div g)p. \quad . \quad (2)$$

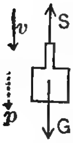


FIG. 61.

From equations (1) and (2) we obtain $p = (Gg) \div (G + G_1)$ = a constant; hence each motion is *uniformly accelerated*, and we may employ equations (1) to (4) of § 56 to find the velocity and distance from the starting-points, at the end of any assigned time t, or *vice versâ*. The initial velocity must be known, and may be zero. Also, from (1) and (2) of this article,

$$S = (GG_1) \div (G + G_1) = \text{constant.}$$

Example 3.—A body of $2\frac{3}{4}$ (short) tons weight is acted on during $\frac{1}{2}$ minute by a constant force P. It had previously described $316\frac{2}{3}$ yards in 180 seconds under no force; and subsequently, under no force, describes 9900 inches in $\frac{1}{40}$ of an hour. Required the value of P. Ans. $P = 22.1$ lbs.

Example 4.—A body of 1 ton weight, having an initial velocity of 48 inches per second, is acted on for $\frac{1}{4}$ minute by a force of 400 avoirdupois ounces. Required the final velocity. Ans. 10.037 ft. per sec.

Example 5.—Initial velocity, 60 feet per second; body weighs 0.30 of a ton. A resistance of $112\frac{1}{2}$ lbs. retards it for $\frac{2}{15}$ of a minute. Required the distance passed over during this time. Ans. 286.8 feet.

Example 6.—Required the time in which a force of 600 avoirdupois ounces will increase the velocity of a body weighing $1\frac{1}{2}$ tons from 480 feet per minute to 240 inches per second. Ans. 30 seconds.

Example 7.—What distance is passed over by a body of (0.6) tons weight during the overcoming of a constant resistance (friction), if its velocity, initially 144 inches per sec., is reduced to zero in 8 seconds. Required, also, the friction. Ans. 48 ft. and 55 lbs.

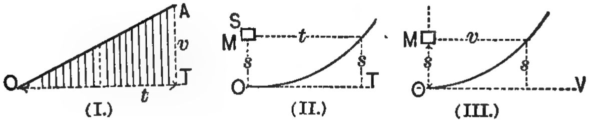
Example 8.—Before the action of a force (value required) a body of 11 tons had described uniformly 950 ft. in 12 minutes. Afterwards it describes 1650 feet uniformly in 180 seconds. The force acts 30 seconds. $P = ?$ Ans. $P = 178$ lbs.

58. Graphic Representations. Unif. Acc. Motion.—With the initial velocity $= 0$, the equations of § 56 become

$$v = p_1 t, \quad . \ . \ . \ . \ . \quad (1) \qquad s = \tfrac{1}{2} p_1 t^2, \quad . \ . \ . \ . \ . \quad (2)$$

$$s = v^2 \div 2p_1, \quad . \ . \ . \quad (3) \quad \text{and} \quad s = \tfrac{1}{2} vt. \quad . \ . \ . \ . \ . \quad (4)$$

Eqs. (1), (2), and (3) contain each two variables, which may graphically be laid off to scale as co-ordinates and thus give a curve corresponding to the equation. Thus, Fig. 62, in (I.), we have a right line representing eq. (I.); in (II.), a parabola with axis parallel to s, and vertex at the origin for eq. (2); also a parabola similarly situated for eq. (3). Eq. (4) contains three variables, s, v, and t. This relation can be shown in (I.), s being represented by the *area* of the shaded triangle $= \frac{1}{2} vt$. (II.) and (III.) have this advantage, that the axis OS may be made the actual path of the body. [Let the student determine how the origin shall be moved in each case to meet the supposition of an initial velocity $= + c$ or $- c$.] (See Notes, p. 120.)



(I.) (II.) (III.)

FIG. 62.

59. Variably Accelerated Motions.—We here restate the equations (differential)

$$v = \frac{ds}{dt} \ . \ . \ (\text{I.}); \quad p = \frac{dv}{dt} = \frac{d^2s}{dt^2} \ . \ . \ (\text{II.}); \quad vdv = pds \ . \ . \ (\text{III.});$$

and resultant force

$$= P = Mp, \quad . \ . \ . \ . \ . \ . \quad (\text{IV.});$$

which are the only ones for *general use* in rectilinear motion and involve the *five variables, s, t, v, p,* and *P*.

PROBLEM 1.—In pulling a mass M along a smooth, horizontal table, by a horizontal cord, the tension is so varied that $s = 4t^3$ (*not a homogeneous equation;* the units are, say, the foot and second). Required by what law the tension varies.

From (I.) $v = \frac{ds}{dt} = \frac{d(4t^3)}{dt} = 12t^2$; from (II.), $p = \frac{d(12t^2)}{dt} = 24t$; and (IV.) the tension $= P = Mp = 24Mt$, i.e., varies directly as the time.

PROBLEM 2. "Harmonic Motion," Fig. 63.—A small block on a smooth horizontal table is attached to two horizontal elastic cords (and they to pegs) in such a way that when the block is at O, each cord is straight but not tense; in any other position, as m, one cord is tense, the other slack. The cords are alike in every respect, and, as with springs, the tension varies directly with the elongation ($= s$ in figure). If for an elongation s_1 the tension is T_1, then for any elongation s it is $T = T_1 s \div s_1$. If the block be given an initial velocity $= c$ at O, it begins to execute an oscillatory motion on both sides of O; m is any point of its motion. The tension T is the accelerating force; *variable* and always pointing toward O. The acceleration at any point m, then, is $p = -(T \div M) = -(T_1 s \div M s_1)$, which for brevity put $p = -as$, a being a constant. Required the equations of motion, the initial velocity being $= +c$, at O. From eq. (III.)

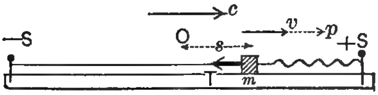


FIG. 63

$$vdv = -asds; \therefore \int_c^v vdv = -a\int_0^s sds,$$

$$\text{i.e., } \tfrac{1}{2}(v^2 - c^2) = -\tfrac{1}{2}as^2; \quad \text{or, } v^2 = c^2 - as^2. \quad . \quad (1)$$

$$\left.\begin{array}{r}\text{From (I.), } dt = ds \div v;\\ \text{hence from (1),}\end{array}\right\} \int_0^t dt = \int_0^s [ds \div \sqrt{c^2 - as^2}]; \quad \text{or,}$$

$$t = \frac{1}{\sqrt{a}}\int_0^s \frac{d[s\sqrt{a} \div c]}{\sqrt{1-(s\sqrt{a} \div c)^2}} = \frac{1}{\sqrt{a}}\Big[\sin^{-1}\left(\frac{s\sqrt{a}}{c}\right)\Big]_0^s$$

$$= \frac{1}{\sqrt{a}}\sin^{-1}\left(\frac{s\sqrt{a}}{c}\right). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Inverting (2), we have $s = (c \div \sqrt{a}) \sin (t\sqrt{a})$, . . . (3)

Again, by differentiating (3), see (I.), $v = c \cos (t\sqrt{a})$ (4)

Differentiating (4), see (II.), $p = -c\sqrt{a} \sin (t\sqrt{a})$. . . (5)

These are the relations required, each between *two* of the four variables, s, t, v, and p; but the peculiar property of the motion is made apparent by inquiring the time of passing from O to a state of rest; i.e., put $v = 0$ in equation (4), we obtain $t = \frac{1}{2}\pi \div \sqrt{a}$, or $\frac{3}{2}\pi \div \sqrt{a}$, or $\frac{5}{2}\pi \div \sqrt{a}$, and so on, while the corresponding values of s (from equation (3)), are $+(c \div \sqrt{a})$, $-(c \div \sqrt{a})$, $+(c \div \sqrt{a})$, and so on. This shows that the body vibrates equally on both sides of O in a cycle or period whose duration $= 2\pi \div \sqrt{a}$, and is *independent of the initial velocity given it at O.* Each time it passes O the velocity is either $+c$, or $-c$, the acceleration $= 0$, and the time since the start is $= n\pi \div \sqrt{a}$, in which n is any whole number. At the extreme point $p = \mp c\sqrt{a}$, from eq. (5). If then a different amplitude be given to the oscillation by changing c, the duration of the period is still the same, i.e., the vibration is *isochronal.** The motion of an ordinary pendulum is nearly, that of a cycloidal pendulum exactly, harmonic.

If the crank-pin of a reciprocating engine moved uniformly in its circular path, the piston would have a harmonic motion if the connecting-rod were infinitely long, or if the design in Fig. 64 were used. (Let the student prove this from eq. (3).) Let $2r$ = length of stroke, and c = the uniform velocity of the crank-pin, and M = mass of the piston and rod AB. Then the velocity of M at mid-stroke must $= c$, at the dead-points, zero; its acceleration at mid-stroke zero; at the dead-points the acc. $= c\sqrt{a}$, and $s = r = c \div \sqrt{a}$ (from eq. (3)); $\therefore \sqrt{a} = c \div r$, and the acc. at a dead point (the maximum acc.)

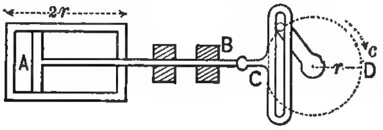


FIG. 64.

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* See illustrations and example on pp. 47, 48, of the "Notes."

$= c^2 \div r$. Hence on account of the acceleration (or retardation) of M in the neighborhood of a dead-point a pressure will be exerted on the crank-pin, equal to mass × acc. $= Mc^2 \div r$ at those points, independently of the force transmitted due to steam-pressure on the piston-head, and makes the resultant pressure on the pin at C smaller, and at D larger than it would be if the "*inertia*" of the piston and rod were not thus taken into account. We may prove this also by the free-body method, considering *AB free* immediately after passing the dead-point C, neglecting all friction. See Fig. 65. The forces acting are: G, the weight; N, the pressures of the guides; P, the known effective steam-pressure on piston-head; and P', the unknown pressure of crank-pin on side of slot. There is no change of motion vertically; $\therefore N' + N - G = 0$, and the resultant force is $P - P' =$ mass × accel. $= Mc^2 \div r$, hence $P' = P - Mc^2 \div r$. Similarly at the other dead-point we would obtain $P' = P + Mc^2 \div r$. In high-speed engines with heavy pistons, etc., $Mc^2 \div r$ is no small item. [The upper half-revol., alone, is here considered.] (See example on p. 68, "*Notes.*")

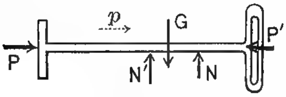

FIG. 65.

PROBLEM 3.—Supposing the earth at rest and *the resistance of the air to be null*, a body is given an initial upward vertical velocity $= c$. Required the velocity at any distance s from the centre of the earth, whose attraction varies inversely as the square of the distance s.

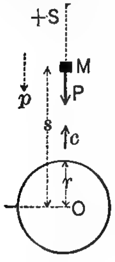

FIG. 66.

See Fig. 66.—The attraction on the body at the surface of the earth where $s = r$, the radius, is its weight G; at any point m it will be $P = G(r^2 \div s^2)$,* while its mass $= G \div g$.

Hence the acceleration at $m = p = (-P) \div M = -g(r^2 \div s^2)$. Take equation III., $vdv = pds$, and we have

$$vdv = -gr^2 s^{-2} ds; \ \therefore$$

$$\int_c^v vdv = -gr^2 \int_r^s s^{-2} ds; \text{ or, } \Big[_c^v \tfrac{1}{2}v^2 = -gr^2 \Big[_r^s -\frac{1}{s},$$

$$\text{i.e., } \tfrac{1}{2}(v^2 - c^2) = -gr^2 \left(\frac{1}{r} - \frac{1}{s}\right). \quad . \quad . \quad . \quad (1)$$

---

* That is, the force of attraction, (P, lbs.) at any distance, s, from O is to the force at the surface (viz., G lbs.) as r^2 is to s^2.

Evidently v decreases, as it should. Now inquire how small a value c may have that the body shall *never return*; i.e., that v shall not $= 0$ until $s = \infty$. Put $v = 0$ and $s = \infty$ in (1) and solve for c; and we have

$$c = \sqrt{2gr} = \sqrt{2 \times 32.2 \times 21000000},$$

$=$ about 36800 ft. per sec. or nearly 7 miles per sec. Conversely, if a body be allowed to fall, from rest, toward the earth, the velocity with which it would strike the surface would be less than seven miles per second through whatever distance it may have fallen.

If a body were allowed to fall through a straight opening in the earth passing through the centre, the motion would be harmonic, since the attraction and consequent acceleration now vary directly with the distance from the centre. See Prob. 2. This supposes the earth homogeneous.

PROBLEM 4.—Steam working expansively and raising a weight.

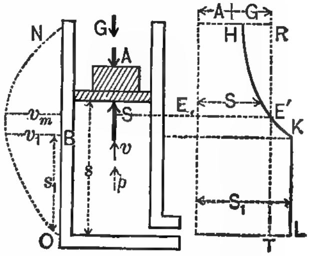

FIG. 67.

Fig. 67.—A piston works without friction in a vertical cylinder. Let $S =$ total steam-pressure on the underside of the piston; the weight G, of the mass $G \div g$ (which includes the piston itself) and an atmospheric pressure $= A$, constitute a constant back-pressure. Through the portion $OB = s_1$, of the stroke, S is constant $= S_1$, while beyond B, boiler communication being "cut off," S diminishes with Boyle's law, i.e., in this case, for any point above B, we have, neglecting the "clearance", F being the cross-section of the cylinder, *

$$S : S_1 :: Fs_1 : Fs; \quad \text{or} \quad S = S_1 s_1 \div s.$$

(Which gives S as a function of s at any point *above* B.)

Full length of stroke $= ON = s_n$. Given, then, the forces S_1 and A, the distances s_1 and s_n, and the velocities at O and at N both $= 0$ (i.e., the mass $M = G \div g$ is to start from rest at O. and to *come to rest at* N). required the proper weight G to fulfil these conditions, S varying as already stated. The acceleration at any point will be

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* See p. 627 for meaning of "clearance."

$$p = [S - A - G] \div M. \quad . \quad . \quad . \quad . \quad (1)$$

Hence (eq. III.) $Mvdv = [S - A - G]ds$, and ∴ for the whole stroke

$$M\int_0^0 vdv = \int_0^N [S - A - G]ds; \text{ i.e.,}$$

$$0 = S_1\int_0^{s_1} ds + S_1s_1\int_{s_1}^{s_n}\frac{ds}{s} - A\int_0^{s_n} ds - G\int_0^{s_n} ds,$$

or
$$S_1s_1\left[1 + \log_e \frac{s_n}{s_1}\right] = As_n + Gs_n. \quad . \quad . \quad . \quad (2)$$

Since $S = S_1$ = constant, from O to B, and variable, $= S_1s_1 \div s$, from B to N, we have had to write the summation $\int_0^N Sds$ in two parts.

From (2), G becomes known, and ∴ M also ($= G \div g$).

Required, further, the time occupied in this upward stroke. From O to B (the point of cut-off) the motion is uniformly accelerated, since p is constant (S being $= S_1$ in eq. (1)), with the initial velocity zero; hence, from eq. (3), § 56, the velocity at $B = v_1 = \sqrt{2[S_1 - A - G]s_1 \div M}$ is known; ∴ the time $t_1 = 2s_1 \div v_1$ becomes known (eq. (4), § 56) of describing OB. At any point beyond B the velocity v may be obtained thus: From (III.) $vdv = pds$, and eq. (1) we have, summing between B and any point above,

$$M\int_{v_1}^{v} vdv = S_1s_1\int_{s_1}^{s}\frac{ds}{s} - (A + G)\int_{s_1}^{s} ds; \text{ i.e.,}$$

$$\frac{G}{g}\frac{(v^2 - v_1^2)}{2} = S_1s_1 \log_e \frac{s}{s_1} - (A + G)(s - s_1).$$

This gives the relation between the two variables v and s anywhere between B and N; if we solve for v and insert its value in $dt = ds \div v$, we shall have dt = a function of s and ds, which is not integrable. Hence we may resort to approximate methods for the time from B to N. Divide the space BN into an uneven number of equal parts, say five; the distances of the points of division from O will be s_1, s_2, s_3, s_4, s_5, and s_n. For these values of s compute (from above equation) v_1 (already known), v_2, v_3, v_4, v_5, and v_n (known to be zero). To the first four spaces apply Simpson's Rule,* and we have the time from B to the end of s_5,

$$\Big[_1^5 t = \int_1^5 \frac{ds}{v};\ \text{approx.} = \frac{s_5 - s_1}{12}\left[\frac{1}{v_1} + \frac{4}{v_2} + \frac{2}{v_3} + \frac{4}{v_4} + \frac{1}{v_5}\right];$$

while regarding the motion from 5 to N as uniformly retarded (approximately) with initial velocity $= v_5$ and the final $=$ zero, we have (eq. (4), § 56),

$$\Big[_5^N t = 2(s_n - s_5) \div v_5.$$

By adding the three times now found we have the whole time of ascent. In Fig. 67 the dotted curve on the left shows by horizontal ordinates the variation in the velocity as the distance s increases; similarly on the right are ordinates showing the variation of S. The point E, where the velocity is a maximum $= v_m$, may be found by putting $p = 0$, i.e., for $S = A + G$, the accelerating force being $= 0$, see eq. (1). Below E the accelerating force, and consequently the acceleration, is positive; above, negative (i.e., the back-pressure exceeds the steam-pressure). The horizontal ordinates between the line $HEKL$ and the right line RT are proportional to the accelerating force. If by condensation of the steam a vacuum is produced below the piston on its arrival at N, the accelerating force is downward and $= A + G$. [Let the student determine how the detail of this problem would be changed, if the cylinder were horizontal instead of vertical.]

60. Direct Central Impact.—Suppose two masses M_1 and M_2 to be moving in the same right line so that their distance apart continually diminishes, and that when the collision or impact takes place the line of action of the mutual pressure coincides with the line joining their centres of gravity, or centres of mass, as they may be called in this connection. This is called a direct central impact, and the motion of each mass is variably accelerated and rectilinear during their contact, the only force being the pressure of the other body. The whole mass of each body will be considered concentrated in the centre of mass, on the supposition that all its particles undergo simultaneously the same change of motion in parallel directions. (This is not strictly true; the effect of the pressure being gradually felt, and transmitted in vibrations. These vibrations endure to some extent after the impact.) When the centres of mass cease to approach each other the pressure between the bodies is a maximum and the bodies have a common velocity; after this, if any capacity for restitution of form (elasticity) exists in either body, the pressure still continues, but diminishes in value gradually to zero, when contact ceases and the bodies separate with different velocities. Reckoning the time from the first instant of contact, let t' = duration of the first period, just mentioned; t'' that of the first + the second (restitution). Fig. 68. Let M_1 and M_2 be the masses, and at *any instant during the contact* let v_1 and v_2 be simultaneous values of the velocities of the mass-centres respectively (reckoning velocities positive toward the right), and P the pressure (variable). At any instant the acceleration of M_1 is $p_1 = -(P \div M_1)$, while at the same instant that of M_2 is $p_2 = +(P \div M_2)$; M_1 being retarded, M_2 accelerated, in velocity. Hence (eq. II., $p = dv \div dt$) we have

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* See p. 13 of the "*Notes and Examples.*"

$$M_1 dv_1 = -Pdt; \quad \text{and} \quad M_2 dv_2 = +Pdt. \quad . \quad . \quad (1)$$

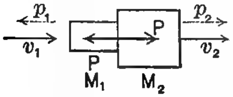

FIG. 68.

Summing all similar terms for the first period of the impact, we have (calling the velocities before impact c_1 and c_2, and the common velocity at instant of maximum pressure C)

$$M_1 \int_{c_1}^{C} dv_1 = -\int_0^{t'} Pdt, \text{ i.e., } M_1(C - c_1) = -\int_0^{t'} Pdt; \quad (2)$$

$$M_2 \int_{c_2}^{C} dv_2 = +\int_0^{t'} Pdt, \text{ i.e., } M_2(C - c_2) = +\int_0^{t'} Pdt. \quad . \quad (3)$$

The two integrals* are identical, numerically, term by term, since the pressure which at any instant accelerates M_2 is numerically equal to that which retards M_1; hence, though we do not know how P varies with the time, we can eliminate the definite integral between (2) and (3) and solve for C. If the impact is *inelastic* (i.e., no power of restitution in either body, either on account of their total inelasticity or damaging effect of the pressure at the surfaces of contact), they continue to move with this common velocity, which is therefore their final velocity. Solving, we have

$$C = \frac{M_1c_1 + M_2c_2}{M_1 + M_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Next, supposing that the impact is *partially elastic*, that the bodies are of the same material, and that the summation $\int_{t'}^{t''} Pdt$ for the second period of the impact bears a ratio, e, to that $\int_0^{t'} Pdt$, already used, a ratio peculiar to the material, if the impact is not too severe, we have, summing equations (1) for the second period (letting V_1 and V_2 = the velocities after impact),

$$M_1 \int_C^{V_1} dv_1 = -\int_{t'}^{t''} Pdt, \text{ i.e., } M_1(V_1 - C) = -e\int_0^{t'} Pdt; \quad (5)$$

$$M_2 \int_C^{V_2} dv_2 = +\int_{t'}^{t''} Pdt, \text{ i.e., } M_2(V_2 - C) = +e\int_0^{t'} Pdt. \quad (6)$$

e is called the coefficient of restitution.

Having determined the value of $\int_0^{t} Pdt$ from (2) or (3) in terms of the masses and initial velocities, substitute it and that of C, from (4), in (5), and we have (for the final velocities)

$$V_1 = [M_1c_1 + M_2c_2 - eM_2(c_1 - c_2)] \div [M_1 + M_2]; \quad . \quad (7)$$

and similarly

$$V_2 = [M_1c_1 + M_2c_2 + eM_1(c_1 - c_2)] \div [M_1 + M_2]. \quad . \quad (8)$$

For $e = 0$, i.e., for *inelastic impact*, $V_1 = V_2 = C$ in eq. (4); for

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* That is, the right-hand members of eqs. (2) and (3).

$e = 1$, or *elastic impact*, (7) and (8) become somewhat simplified.

To determine e experimentally, let a ball (M_1) of the substance fall upon a very large slab (M_2) of *the same substance*, noting both the height of fall h_1, and the height of rebound H_1. Considering M_2 as $= \infty$, with

$$c_1 = \sqrt{2gh_1}, \quad V_1 = -\sqrt{2gH_1}, \text{ and } c_2 = o,$$

eq. (7) gives

$$-\sqrt{2gH_1} = -e\sqrt{2gh_1}\,;\ \therefore e = \sqrt{H_1 \div h_1}.$$

Let the student prove the following from equations (2), (3), (5), and (6):

(*a*) For any direct central impact whatever,

$$M_1c_1 + M_2c_2 = M_1V_1 + M_2V_2.$$

[The product of a mass by its velocity being sometimes called its *momentum*, this result may be stated thus:

In any direct central impact the sum of the momenta before impact is equal to that after impact (or at any instant during impact). This principle is called the *Conservation of Momentum*. The present is only a particular case of a more general proposition.

It can be proved that C, eq. (4), is the velocity of the centre of gravity of the two masses before impact; the conservation of momentum, then, asserts that this velocity is unchanged by the impact, i.e., by the mutual actions of the two bodies.]

(*b*) The loss of velocity of M_1, and the gain of velocity of M_2, are twice as great when the impact is elastic as when inelastic.

(*c*) If $e = 1$, and $M_1 = M_2$, then $V_1 = +c_2$, and $V_2 = c_1$.

Example.—Let M_1 and M_2 be perfectly elastic, having weights = 4 and 5 lbs. respectively, and let $c_1 = 10$ ft. per sec. and $c_2 = -6$ ft. per sec. (i. e., before impact *M_2 is moving in a direction contrary to that of M_1*). By substituting in eqs. (7) and (8), with $e = 1$, $M_1 = 4 \div g$, and $M_2 = 5 \div g$, we have

$$V_1 = \frac{1}{9}\Big[4 \times 10 + 5 \times (-6) - 5\Big(10 - (-6)\Big)\Big] = -7.7 \text{ ft. per sec.}$$

$$V_2 = \frac{1}{9}\Big[4 \times 10 + 5 \times (-6) + 4\Big(10 - (-6)\Big)\Big] = +8.2 \text{ ft. per sec.}$$

as the velocities after impact. Notice their directions, as indicated by their signs

CHAPTER II.

"VIRTUAL VELOCITIES."

61. Definitions.—If a material point is moving in a direction not coincident with that of the resultant force acting (as in curvilinear motion in the next chapter), and any element of its path, *ds*, projected upon this **force**;* the length of this projection, *du*, Fig. 69, is called the "VIRTUAL VELOCITY" of the force, since $du \div dt$ may be considered the velocity of the force at this instant, just as $ds \div dt$ is that of the point. The product of the force by its *du* will be called its *virtual moment*, reckoned + or − according as the direction from O to D is the same as that of the force or opposite.

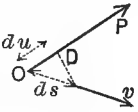

FIG. 69.

62. Prop. I.—*The virtual moment of a force equals the algebraic sum of those of its components.* Fig. 70. Take the direction of *ds* as an axis X; let P_1 and P_2 be components of P; α_1, α_2, and α their angles with X. Then (§ 16) $P \cos \alpha = P_1 \cos \alpha_1 + P_2 \cos \alpha_2$. Hence $P(ds \cos \alpha) = P_1(ds \cos \alpha_1) + P_2(ds \cos \alpha_2)$. But $ds \cos \alpha$ = the projection of *ds* upon P, i.e., $= du$; $ds \cos \alpha_1 = du_1$, etc.; $\therefore Pdu = P_1du_1 + P_2du_2$. If in Fig. 70 α_1 were $> 90°$, evidently we would have $Pdu = -P_1du_1 + P_2du_2$, i.e., P_1du_1 would then be negative, and OD_1 would fall behind O; hence the definition of + and − in § 61. For any number of components the proof would be similar, and is equally applicable whether they are in one plane or not.

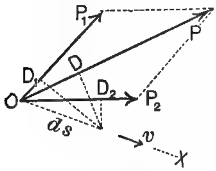

FIG. 70.

63. Prop. II.—*The sum of the virtual moments equals zero, for concurrent forces in equilibrium.*

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* We should rather say "projected on the line of action of the force;" but the phrase used may be allowed, for brevity.

(If the forces are balanced, the material point is moving in a straight line if moving at all.) The resultant force is zero. Hence, from § 62, $P_1du_1 + P_2du_2 +$ etc. $= 0$, having proper regard to sign, i.e., $\Sigma(Pdu) = 0$.

64. Prop. III.—*The sum of the virtual moments equals zero for any small displacement or motion of a rigid body in equilibrium under non-concurrent forces in a plane; all points of the body moving parallel to this plane.* (Although the kinds of motion of a given rigid body which are consistent with balanced non-concurrent forces have not yet been investigated, we may imagine any slight motion for the sake of the algebraic relations between the different du's and forces.)

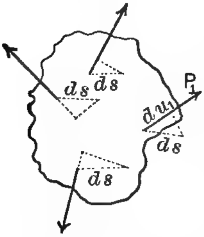

FIG. 71.

First, let the motion be a *translation*, all points of the body describing equal parallel lengths $= ds$. Take X parallel to ds; let α_1, etc., be the angles of the forces with X. Then (§ 35) $\Sigma(P \cos \alpha) = 0$; $\therefore ds\Sigma(P \cos \alpha) = 0$; but $ds \cos \alpha_1 = du_1$; $ds \cos \alpha_2 = du_2$; etc.; $\therefore \Sigma(Pdu) = 0$. Q. E. D.

Secondly, let the motion be a *rotation* through a small angle $d\theta$ in the plane of the forces about any point O in that plane, Fig. 72. With O as a pole let ρ_1 be the radius-vector of the point of application of P_1, and a_1 its lever-arm from O; similarly for the other forces. In the rotation each point of application describes a small arc, ds_1, ds_2, etc., proportional to ρ_1, ρ_2, etc., since $ds_1 = \rho_1 d\theta$, $ds_2 = \rho_2 d\theta$, etc. From § 36, $P_1a_1 +$ etc. $= 0$; but from similar triangles $ds_1 : du_1 :: \rho_1 : a_1$; $\therefore a_1 = \rho_1 du_1 \div ds_1 = du_1 \div d\theta$; similarly $a_2 = du_2 \div d\theta$, etc. Hence we must have $[P_1du_1 + P_2du_2 + \ldots] \div d\theta = 0$, i.e., $\Sigma(Pdu) = 0$. Q. E. D.

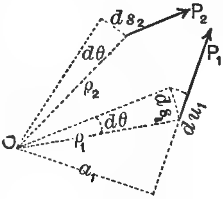

FIG. 72.

Now since any slight displacement or motion of a body may be conceived to be accomplished by a small translation followed by a rotation through a small angle, and since the foregoing deals only with projections of paths, the proposition is established and is called the *Principle of Virtual Velocities.*

[A similar proof may be used for any slight motion whatever in space when a system of non-concurrent forces is balanced.] Evidently if the path (ds) of a point of application is perpendicular to the force, the virtual velocity (du), and consequently the virtual moment (Pdu) of the force are zero. Hence we may frequently make the displacement of such a character in a problem that one or more of the forces may be excluded from the summation of virtual moments.

65. Connecting-Rod by Virtual Velocities.—Let the effective steam-pressure P be the means, through the connecting-rod and crank (i.e., two links), of raising the weight G *very slowly;* neglect friction and the weight of the links themselves. Consider AB as *free* (see (b) in Fig. 73), BC also, at (c); let the “small displacements” of both be *simultaneous* small portions of their ordinary motion in the apparatus. A has moved to A_1 through dx; B to B_1, through ds, a small arc; C has not moved. The forces acting on AB are P (steam-pressure), N (vertical reaction of guide), and N' and T (the tangential and normal components of the crank-pin pressure). Those on BC are N' and T (reversed), the weight G, and the oblique pressure of bearing P'. The motion being slow (or rather the acceleration being small), each of these two systems will be considered as balanced. Now put $\Sigma(Pdu) = 0$ for AB, and we have

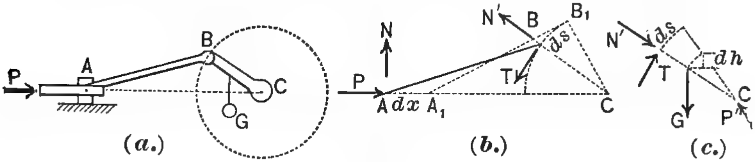

FIG. 73.

$$Pdx + N \times 0 + N' \times 0 - Tds = 0. \quad . \quad . \quad (1)$$

For the simultaneous and corresponding motion of BC, $\Sigma(Pdu) = 0$ gives

$$N' \times 0 + Tds - Gdh + P' \times 0 = 0, \quad . \quad . \quad (2)$$

dh being the vertical projection of G's motion.

From (1) and (2) we have, easily, $Pdx - Gdh = 0$, . (3) which is the same as we might have obtained by putting $\Sigma(Pdu) = 0$ *for the two links together, regarded collectively as a free body*, and describing a small portion of the motion they really have in the mechanism, viz., (Fig. 74,)

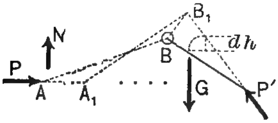

FIG. 74.

$$Pdx + N \times 0 - Gdh + P' \times 0 = 0. \quad . \quad . \quad (4)$$

We may therefore announce the—

66. Generality of the Principle of Virtual Velocities.—*If any mechanism of flexible inextensible cords, or of rigid bodies jointed together, or both, at rest, or in motion with very small accelerations, be considered free collectively* (*or any portion of it*), *and all the external forces put in; then* (*disregarding mutual frictions*) *for a small portion of its prescribed motion,* $\Sigma(Pdu)$ *must* $= 0$, *in which the* du, *or virtual velocity, of each force,* P, *is the projection of the path of the point of application upon the force* (the product, Pdu, being $+$ or $-$ according to § 61).

67. *Example.*—In the problem of § 65, having given the weight G, required the proper steam-pressure (effective) P to hold G in equilibrium, or to raise it uniformly, if already in motion, for a *given position of the links.* That is, Fig. 75, given a, r, c, α, and β, required the ratio $dh : dx$; for, from equation (3), § 65, $P = G(dh : dx)$. The projections of dx and ds upon AB will be equal, since $AB = A_1B_1$, and makes an (infinitely) small angle with A_1B_1, i.e., $dx \cos \alpha = ds \cos (\beta - \alpha)$. Also, $dh = (c : r)ds \sin \beta$.

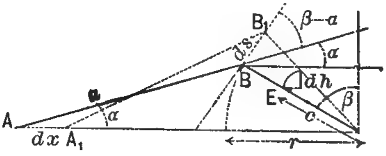

FIG. 75.

Eliminating ds, we have,

$$\frac{dh}{dx}=\frac{c\sin\beta\cos\alpha}{r\cos(\beta-\alpha)};\qquad \therefore P=G\frac{c\sin\beta\cos\alpha}{r\cos(\beta-\alpha)}.$$

68. When the acceleration of the parts of the mechanism is not practically zero, $\Sigma(Pdu)$ will not $=0$, but a function of the masses and velocities to be explained in the chapter on Work, Energy, and Power. If friction occurs at moving joints, enough "free bodies" should be considered that no free body extend beyond such a joint; it will be found that this friction cannot be eliminated in the way T and N' were, in § 65.

69. Additional Problems; to be solved by "virtual velocities." PROBLEM 1.—Find relations between the forces acting on a straight lever in equilibrium; also, on a bent lever.

PROBLEM 2.—When an ordinary copying-press is in equilibrium, find the relation between the force applied horizontally and tangentially at the circumference of the wheel, and the vertical resistance under the screw-shaft. See Fig. 75*a*.

Solution.—Considering free the rigid body consisting of the wheel and screw-shaft, let R be the resistance at the point of the shaft (pointing along the axis of the shaft), and P the required horizontal tangential force at edge of wheel. Let radius of wheel be r. Besides R and P there are also acting on this body certain pressures, or "supporting forces," consisting of the reactions of the collars, and reactions of the threads of nut against the threads of screw. Denote by s the "*pitch*" of the screw, i.e., the distance the shaft would advance for a full turn of the wheel. Then if we imagine the wheel to turn through a small angle $d\theta$, the corresponding advance, ds, of the shaft would be $\frac{sd\theta}{2\pi}$, from the proportion $s:ds::2\pi:d\theta$.

The path of the point of application of P is AB, a small portion of a helix, the projection of which on the line of P is $rd\theta$, $=\overline{Ac}$; ds projects in its full length on the line of the force R. In the case of each of the other forces, however, the path of the point of application is *perpendicular* to the line of the force (which is normal to the rubbing surfaces, friction being disregarded). Hence, substituting in $\Sigma(Pdu)=0$, we have

$$+P\,.\,rd\theta-R\,.\,ds+0+0=0;$$

whence

$$P=\frac{ds}{rd\theta}\,.\,R=\frac{s}{2\pi r}\,.\,R.$$

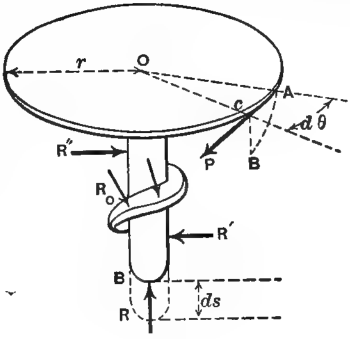

FIG. 75*a*.

CHAPTER III.

CURVILINEAR MOTION OF A MATERIAL POINT.

[Motion in a plane, only, will be considered in this chapter.]

70. Parallelogram of Motions.—It is convenient to regard the curvilinear motion of a point in a plane as compounded, or made up, of two independent rectilinear motions parallel respectively to two co-ordinate axes X and Y, as may be explained thus: Fig. 76. Consider the drawing-board CD as fixed, and let the head of a T-square move from A toward B along the edge according to any law whatever, while a pencil moves from M toward Q along the blade. The result is a curved line on the board, whose form depends on the character of the two X and Y *component motions*, as they may be called. If in a time t_1 the T-square head has moved an X distance $= MN$, and the pencil simultaneously a Y distance $= MP$, by completing the parallelogram on these lines, we obtain R, the position of the point on the board at the end of the time t_1. Similarly, at the end of the time t_1' we find the point at R'.

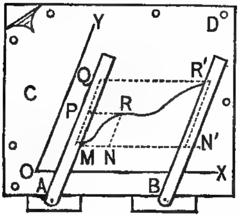

FIG. 76.

71. Parallelogram of Velocities.—Let the X and Y motions be *uniform*, required the resulting motion. Fig. 77. Let c_x and c_y be the constant uniform X and Y velocities. Then in any time, t, we have $x = c_x t$ and $y = c_y t$; whence we have, eliminating t, $x \div y = c_x \div c_y =$ constant, i.e., x is proportional to y, i.e., the path is a straight line. Laying off $OA = c_x$, and $AB = c_y$, B is a point of the path, and OB is the distance described by the point in the first

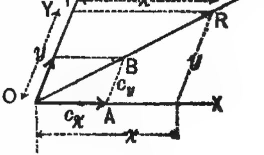

FIG. 77.

second. Since by similar triangles $\overline{OR} : x :: \overline{OB} : c_x$, we have also $\overline{OR} = \overline{OB} \,.\, t$; hence the resultant motion is uniform, and its velocity, $\overline{OB} = c$, *is the diagonal of the parallelogram formed on the two component velocities.*

Corollary.—If the resultant motion is curved, the direction and velocity of the motion at any point will be given by the diagonal formed on the component velocities at that instant. The direction of motion is, of course, a tangent to the curve.

72. Uniformly Accelerated X and Y Motions.—The *initial velocities of both being zero.* Required the resultant motion. Fig. 78. From § 56, eq. (2) (both c_x and c_y being $= 0$), we have $x = \frac{1}{2}p_x t^2$ and $y = \frac{1}{2}p_y t^2$, whence $x \div y = p_x \div p_y =$ constant, and the resultant motion is in a straight line. Conceive lines laid off from O on X and Y to represent p_x and p_y to scale, and form a parallelogram on them. From similar triangles (OR being the distance described in the resultant motion in any time t), $\overline{OR} : x :: \overline{OB} : p_x$; $\therefore \ \overline{OR} = \frac{1}{2}\overline{OB}t^2$. Hence, from the form of this last equation, the resultant motion is uniformly accelerated, and its acceleration is $\overline{OB} = p_1$, (on the same scale as p_x and p_y).

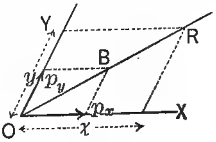

FIG. 78.

This might be called the parallelogram of accelerations, but is really a parallelogram of forces, if we consider that a free material point, initially at rest at O, and acted on simultaneously by constant forces P_x and P_y (so that $p_x = P_x \div M$ and $p_y = P_y \div M$), must begin a uniformly accelerated rectilinear motion in the direction of the resultant force, (*having no initial velocity* in any direction.)

73. In general, considering the point hitherto spoken of as a *free material point*, under the action of one or more forces, in view of the foregoing, and of Newton's second law, given the initial velocity in amount and direction, the starting-point, the initial amounts and directions of the acting forces and the laws of their variation if they are not constant, we can resolve them into a single X and a single Y force at any instant, determine the X and Y motions independently, and afterwards the resultant motion.

Note.—The resultant force *is never in the direction of the tangent to the path* (except at a point of inflection). The relations which its amount and position at any instant bear to the velocity, rates of change of that velocity (i.e., accelerations), both as to amount and position, and the radius of curvature of the path, will now be treated (§ 74).

In Fig. 79, A, B, and C are three "consecutive" positions of the moving point, AB and BC being two short chords of the curve. When dt is taken smaller and smaller (position B remaining unchanged) and finally becomes zero, the points A and C merge into B and the chords AB and BC becomes tangents at B; and hence the results to be obtained *only apply to a single point, B.* But note that before dt becomes zero each equation [except (7)] is divided through by dt (or dt^2) and therefore the individual terms do not necessarily become zero also.

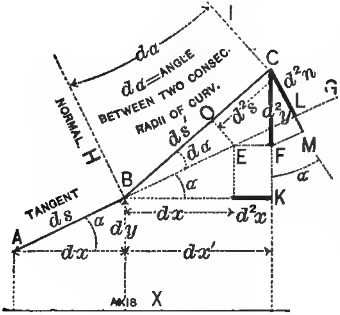

FIG. 79.

74. General Equations for the curvilinear motion of a material point in a plane.—The motion will be considered resulting from the composition of independent X and Y motions, X and Y being perpendicular to each other. Fig. 79. In two consecutive **equal** times (each $= dt$), let dx and $dx' =$ small spaces due to the X motion; and dy and $CK = dy'$, due to the Y motion. Then ds and ds' are two consecutive elements of the curvilinear motion. Prolong ds, making $BE = ds$; then $EF = d^2x$, $CF = d^2y$, and $CO = d^2s$ (EO being perpendicular to BC). Also draw CL perpendicular to BG and call CL d^2n. Call the velocity of the X motion v_x, its acceleration p_x; those of the Y motion, v_y and p_y. Then,

$$v_x = \frac{dx}{dt}; \quad v_y = \frac{dy}{dt}; \quad p_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}; \quad \text{and } p_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}.$$

For the velocity along the curve (i.e., tangent) $v = ds \div dt$, we shall have, since $ds^2 = dx^2 + dy^2$,

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = v_x^2 + v_y^2. \quad . \quad . \quad (1)$$

Hence v is the diagonal formed on v_x and v_y (as in § 71). Let p_t = the acceleration of v, i.e., the *tangential acceleration*. then $p_t = d^2s \div dt^2$, and, since d^2s = the sum of the projections of EF and CF on BC, i.e., $d^2s = d^2x \cos\alpha + d^2y \sin\alpha$, we have

$$\frac{d^2s}{dt^2} = \frac{d^2x}{dt^2}\cos\alpha + \frac{d^2y}{dt^2}\sin\alpha;\ \text{i.e.,}\ p_t = p_x \cos\alpha + p_y \sin\alpha. \quad (2)$$

By **Normal Acceleration** we mean the rate of change of the velocity in the direction of the normal. In describing the element $AB = ds$, no progress has been made in the direction of the normal BH i.e., there is *no velocity* in the direction of the normal; but in describing BC (on account of the new direction of path) the point has progressed a distance CL (call it d^2n) in the direction of the old normal BH (though none in that of the new normal CI). Hence, just as the tang. acc. $= \dfrac{ds' - ds}{dt^2} = \dfrac{d^2s}{dt^2}$, so the normal accel. $= \dfrac{CL - \text{zero}}{dt^2} = \dfrac{d^2n}{dt^2}$. It now remains to express this normal acceleration ($= p_n$) in terms of the X and Y accelerations. From the figure, $\overline{CL} = \overline{CM} - ML$, i.e.,

$$d^2n = d^2y \cos\alpha - d^2x \sin\alpha \ \{\text{since } EF = d^2x\};$$

$$\therefore \frac{d^2n}{dt^2} = \frac{d^2y}{dt^2}\cos\alpha - \frac{d^2x}{dt^2}\sin\alpha.$$

Hence $$p_n = p_y \cos\alpha - p_x \sin\alpha. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The norm. acc. may also be expressed in terms of the tang. velocity v, and the radius of curvature r, as follows:

$ds' = r\,d\alpha$, or $d\alpha = ds' \div r$; also $d^2n = ds'd\alpha, = ds'^2 \div r$,

$$\text{i.e.,}\ \frac{d^2n}{dt^2} = \left(\frac{ds'}{dt}\right)^2 \frac{1}{r}, \quad \text{or} \quad p_n = \frac{v^2}{r}. \quad . \quad . \quad . \quad . \quad (4)$$

If now, Fig. 80, we resolve the forces $X = Mp_x$ and $Y = Mp_y$, which at this instant account for the X and Y accelerations (M = mass of the material point), into components along the tangent and normal to the curved path, we shall have, as *their equivalent*, a tangential force

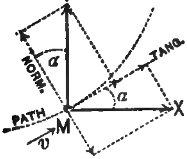

FIG. 80.

$$T = Mp_x \cos\alpha + Mp_y \sin\alpha,$$

and a normal force

$$N = Mp_y \cos \alpha - Mp_x \sin \alpha.$$

But [see equations (2), (3), and (4)] we may also write

$$T = Mp_t = M\frac{dv}{dt}; \quad \text{and} \quad N = Mp_n = M\frac{v^2}{r}. \quad . \quad (5)$$

Hence, if a free material point is moving in a curved path, the sum of the tangential components of the acting forces must equal (the mass) × tang. accel.; *that of the normal components,* = (the mass) × normal accel. = (mass) × (square of veloc. in path) ÷ (rad. curv.).

It is evident, therefore, that the resultant force (= diagonal on T and N or on X and Y, Fig. 80) *does not act along the tangent* at any point, but toward the concave side of the path; unless $r = \infty$.

Radius of curvature.—From the line above eq. (4) we have $d^2n = ds'^2 \div r$; hence (line above eq. (3)), $ds'^2 \div r = d^2y \cos \alpha - d^2x \sin \alpha$; but $\cos \alpha = dx \div ds$, and $\sin \alpha = dy \div ds$,

$$\therefore \frac{ds'^2}{r} = d^2y\frac{dx}{ds} - d^2x\frac{dy}{ds}; \text{ or } \frac{ds'^2ds}{r} = dx^2\left[\frac{dxd^2y - dyd^2x}{dx^2}\right];$$

$$\text{i.e., } \frac{ds'^2ds}{r} = dx^2d\left[\frac{dy}{dx}\right] = dx^2d\,(\tan \alpha),$$

$$\therefore r = \left(\frac{ds'^2ds}{dt^3}\right) \div \left[\left(\frac{dx}{dt}\right)^2 \frac{d\tan\alpha}{dt}\right];$$

or,

$$r = v^3 \div \left[v_x^2\frac{d\tan\alpha}{dt}\right]. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which is equally true if, for v_x and $\tan \alpha$, we put v_y and $\tan (90° - \alpha)$, respectively.

Change in the velocity square.—Since the tangential acceleration $\frac{dv}{dt} = p_t$, we have $ds\frac{dv}{dt} = p_tds$; i.e.,

$$\frac{ds}{dt}dv = p_tds, \quad \text{or} \quad vdv = p_tds \quad \text{and} \quad \therefore \frac{v^2 - c^2}{2} = \int p_tds. \ (7)$$

having integrated between any initial point of the curve where $v = c$, and any other point where $v = v$. This is nothing more than equation (III.), of § 50.

75. Normal Acceleration. Another Method.—Fig. 81. Consider a material point m describing a circular path ABC, with constant velocity $=v$; the center of the curve being at O and the radius $=r$. The velocity v is always tangent to the curve. Let the linear arc BC be described in the small time dt, the angle at the center being $d\alpha$. At B the velocity is directed along the tangent BT, while at C it is ⌉ to OC and makes an angle $d\alpha$ with a line parallel to BT. As m moves along the curve from B to C, the point n, which is the foot of the ⌉ dropped from any position of m upon the normal BO, moves from B toward D; while the foot, a, of the ⌉ let fall from m upon the tangent BT, moves along BT with an average velocity $=v'$, a little less than v. Now the motion of m may be regarded as compounded of these two motions, viz., that of n and that of a. The motion of n is called the "motion of m *along the normal*." The velocity of n is zero at B, where v is ⌉ to the normal, and is $v \sin d\alpha$ at the point D; hence in the time dt the gain of n's velocity is $v \sin d\alpha - 0$, and the *rate* of gain, or acceleration, is $p_n = v \sin d\alpha \div dt$. But $\sin d\alpha = \overline{CD} \div r$ and $\overline{CD} = BC' = v'dt$. Substituting, we have $p_n = vv' \div r$.

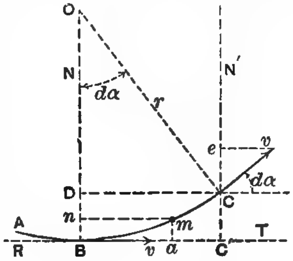

FIG. 81.

Now make dt equal to zero and we have $v' = v$; and finally $p_n = v^2 \div r$, as the value of the normal acceleration, just at the point B.

76. Uniform Circular Motion. Centripetal Force.—The velocity being constant, p_t must be $= 0$, and $\therefore T$ (or ΣT if there are several forces) must $= 0$. The resultant of all the forces, therefore, must be a normal force $= (Mc^2 \div r) =$ a constant (eq. 5, § 74). This is called the "deviating force," or "centripetal force;" without it the body would continue in a straight line. Since forces always occur in pairs (§ 3), a "centrifugal force," equal and opposite to the "centripetal" (one being the reaction of the other), will be found among the forces acting on the body to whose constraint the deviation of the first body from its natural straight course is due. For example, the attraction of the earth on the moon acts as a centripetal or deviating force on the latter, while the equal and opposite force *acting on the earth* may be called the centrifugal. If a small block moving on a smooth horizontal table is gradually turned from its straight course AB by a fixed circular guide, tangent to AB at B, the pressure of the guide against the block is the centripetal force $Mc^2 \div r$ directed *toward* the centre of curvature, while

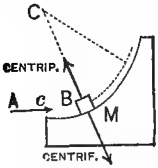

FIG 82.

the centrifugal force $Mc^2 \div r$ is the pressure of the block against the *guide*, directed *away* from that centre.

Note.—One is not justified, therefore, in saying that a body describing a circular path is under the action of a "centrifugal force."

The Conical Pendulum, or governor-ball.—Fig. 83. If a material point of mass $= M = G \div g$, suspended on a cord of length $= l$, is to maintain a uniform circular motion in a horizontal plane, with a given radius r, under the action of gravity and the cord, required the velocity c to be given it. At B we have the body free. The only forces acting are G and the cord-tension P. The sum of their normal components, i.e., ΣN, must $= Mc^2 \div r$, i.e., $P \sin \alpha = Mc^2 \div r$; but, since Σ (vert. comps.) $= 0$, $P \cos \alpha = G$. Hence $G \tan \alpha = Gc^2 \div gr$; $\therefore c = \sqrt{gr \tan \alpha}$. Let $u =$ number of revolutions per unit of time, then $u = c \div 2\pi r = \sqrt{g} \div 2\pi \sqrt{h}$; i.e., is inversely proportional to the square root of the vertical projection of the length of cord. The time of executing one revolution is $= 1 \div u$.

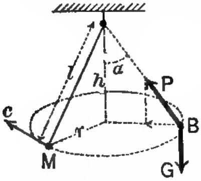

FIG. 83.

Elevation of the outer rail on railroad curves (considerations of traction disregarded).—Consider a single car as a material point, and *free*, having a given velocity $= c$. P is the rail-pressure against the wheels. So long as the car follows the track the resultant R of P and G must point toward the centre of curvature and have a value $= Mc^2 \div r$. But $R = G \tan \alpha$, whence $\tan \alpha = c^2 \div gr$. If therefore the ties are placed at this angle α with the horizontal, the pressure will come upon the tread and not on the flanges of the wheels; in other words, the car will not leave the track. (This is really the same problem as the preceding.)

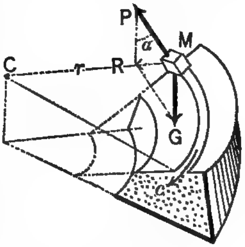

FIG. 84.

Apparent weight of a body at the equator.—This is less than the true weight or attraction of the earth, on account of the uniform circular motion of the body with the earth in its diurnal rotation. If the body hangs from a spring-balance, whose indication is G lbs. (apparent weight), while the true attraction is G' lbs., we have $G' - G = Mc^2 \div r$. For M we may use $G \div g$ (apparent values); for r about 20,000,000 ft.; for c, 25,000 miles in 24 hrs., reduced to feet per second. It results from this that G is $< G'$ by $\frac{1}{289}G'$ nearly, and (since $17^2 = 289$) hence if the earth revolved on its axis seventeen times as fast as at present, G would $= 0$, i.e., bodies would apparently have no weight, the earth's attraction on them being just equal to the necessary centripetal or deviating force necessary to keep the body in its orbit.

Centripetal force at any latitude.—If the earth were a homogeneous liquid, and at rest, its form would be spherical; but when revolving uniformly about the polar diameter, its form of relative equilibrium (i.e., no motion of the particles relatively to each other) is nearly ellipsoidal, the polar diameter being an axis of symmetry.

Lines of attraction on bodies at its surface do not intersect in a common point, and the centripetal force requisite to keep a suspended body in its orbit (a small circle of the ellipsoid), at any latitude β is the resultant, N, of the attraction or true weight G' directed (nearly) toward the centre, and of G the tension of the string. Fig. 85. $G =$ the apparent weight, indicated by a spring-balance and MA is its line of action (plumb-line) normal to the ocean surface. Evidently the apparent weight, and consequently g, are less than the true values, since N must be perpendicular to the polar axis, while the true values themselves, varying inversely as the square of MC, *decrease* toward the equator, hence the apparent values decrease still more rapidly as the latitude diminishes. The apparent g for any latitude β, at h ft. above sea-level, is (Chwolson, 1901), for foot-second units,*

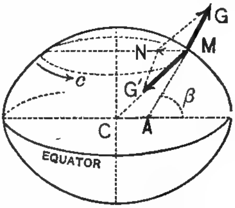

FIG. 85.

$$g = 32.1723 - 0.083315 \cos 2\beta - 0.000003h.$$

(The value 32.2 is accurate enough for practical purposes.) Since the earth's axis is really not at rest, but moving about the sun, and also about the centre of gravity of the moon and earth, the form of the ocean surface is periodically varied, i.e., the phenomena of the tides are produced.

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* At the equator, $g = 32.09$ at sea-level but decreases to 32.06 at an elevation of 10,000 ft. above the sea.

77. Cycloidal Pendulum.—This consists of a material point at the extremity of an imponderable, flexible, and inextensible cord of length $= l$, confined to the arc of a cycloid in a vertical plane by the cycloidal evolutes shown in Fig. 86. Let the oscillation begin (from rest) at A, a height $= h$ above O the vertex. On reaching any lower point, as B (height $= z$ above O), the point has acquired some velocity v, which is at this instant increasing at some rate $= p_t$. Now consider the point free, Fig. 87; the forces acting are P the cord-tension, normal to path, and G the weight, at an angle φ with the path. From § 74, eq. (5), $\Sigma T = Mp_t$ gives

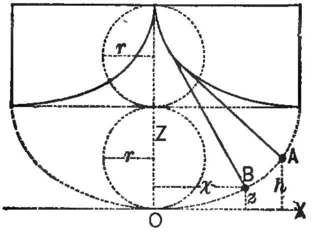

FIG. 86.

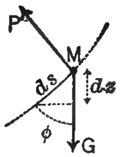

FIG. 87.

$$G \cos \varphi + P \cos 90^\circ = (G \div g)p_t; \; \therefore p_t = g \cos \varphi$$

Hence (eq. (7). § 74), $vdv = p_t ds$ gives

$vdv = g \cos \varphi ds$; but $ds \cos \varphi = - dz$; $\therefore vdv = - gdz$.

Summing between A and B, we have

$$\Big[\tfrac{1}{2}v^2\Big]_0^v = - g\int_h^z dz; \quad \text{or} \quad v^2 = 2g(h - z);$$

the same as if it had fallen freely from rest through the height $h - z$. (*This result evidently applies to any form of path when, besides the weight G, there is but one other force, and that always normal to the path.*)*

From $\Sigma N = Mv^2 \div r_1$, we have $P - G \sin \varphi = Mv^2 \div r_1$, whence P, the cord-tension at any point, may be found (here $r_1 =$ the radius of curvature at any point = length of straight portion of the cord).

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* Compare with lower part of p. 83.

To find the time of passing from A to O, a half-oscillation, substitute the above value of v^2 in $v = ds \div dt$, putting $ds^2 = dx^2 + dz^2$, and we have $dt^2 = (dx^2 + dz^2) \div [2g(h - z)]$. To find dx in terms of dz, differentiate the equation of the curve, which in this position is

$$x = r \text{ ver. sin.}^{-1} (z \div r) + \sqrt{2rz - z^2};$$

whence

$$dx = \frac{r dz}{\sqrt{2rz - z^2}} + \frac{(r - z)dz}{\sqrt{2rz - z^2}} = \frac{(2r - z)dz}{\sqrt{2rz - z^2}};$$

$$\therefore dx^2 = \left[\frac{2r}{z} - 1\right]dz^2$$

(r = radius of the generating circle). Substituting, we have

$$dt = \sqrt{\frac{r}{g}} \cdot \frac{(-dz)}{\sqrt{hz - z^2}};$$

$$\therefore \Big[_A^O t = \sqrt{\frac{r}{g}} \int_0^h \frac{dz}{\sqrt{hz - z^2}} = \sqrt{\frac{r}{g}} \Big[_0^h \text{ver. sin.}^{-1} \frac{z}{\frac{1}{2}h} = \pi\sqrt{\frac{r}{g}}.$$

Hence the whole oscillation occupies a time $= \pi\sqrt{l \div g}$ (since $l = 4r$). This is independent of h, i.e., the oscillations are *isochronal*. This might have been proved by showing that p_t is proportional to OB *measured along the curve;* i.e., that the motion is *harmonic*. (§ 59, Prob. 2.)

78. Simple Circular Pendulum.—If the material point oscillates in the arc of a circle, Fig. 88, proceeding as in the preceding problem, we have finally, after integration by series, as the time of a full oscillation, in one direction,*

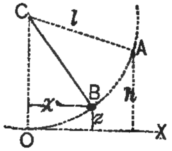

FIG. 88.

$$\left(\pi\sqrt{\frac{l}{g}}\right)\left[1 + \frac{1}{8} \cdot \frac{h}{l} + \frac{9}{256} \cdot \frac{h^2}{l^2} + \frac{225}{18432} \cdot \frac{h^3}{l^3} + \dots\right]$$

Hence for a small h the time is nearly $\pi\sqrt{l \div g}$, and the oscillations nearly isochronal. (For the Compound Pendulum, see § 117.)

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* See p. 651 of Coxe's translation of Weisbach's *Mechanics*.

[**Note.**—While the simple pendulum is purely ideal, the conception is a very useful one. A sphere of lead an inch in diameter and suspended by a silk thread, or very fine wire, more than 2 ft. in length, makes a close approximation to a simple pendulum; the length l being measured from the point of suspension to the middle of the sphere (strictly it should be a little greater). The length of the simple pendulum beating seconds (small amplitude) is about 3.26 ft.; (see p. 120)].

79. Change in the Velocity Square.—From eq. (7), § 74, we have $\frac{1}{2}(v^2 - c^2) = \int p_t ds$. But, from similar triangles, du being the projection of any ds of the path upon the resultant force R at that instant, $Rdu = Tds$ (or, Prin. of Virt. Vels. § 62, $Rdu = Tds + N \times 0$). T and N are the tangential and normal components of R. Fig. 89. Hence, finally,

$$\tfrac{1}{2}Mv^2 - \tfrac{1}{2}Mc^2 = \int Rdu, \quad . \quad . \quad . \quad . \quad . \quad (a)$$

for all elements of the curve between any two points. In general R is different in amount and direction for each ds of the path, but du is the distance through which R acts, **in its own direction**, while the body describes any ds; Rdu is called the **work done** by R when ds is described by the body. The above equation is read: *The difference between the initial and final kinetic energy of a body = the work done by the resultant force in that portion of the path.*

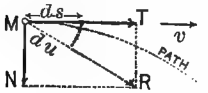

Fig. 89.

(These phrases will be further spoken of in Chap. VI.)

Application of equation (*a*) *to a planet in its orbit about the sun.*—Fig. 90. Here the only force at any instant is the attraction of the sun $R = C \div u^2$ (see Prob. 3, § 59), where C is a constant and u the variable radius vector. As u diminishes, v increases, therefore dv and du have contrary signs; hence equation (*a*) gives (c being the velocity at some initial point O)

$$\frac{1}{2}Mv_1^2 - \frac{1}{2}Mc^2 = -C\int_{u_0}^{u_1}\frac{du}{u^2} = C\left[\frac{1}{u_1} - \frac{1}{u_0}\right]; \quad (b)$$

$\therefore\ v_1 = \sqrt{c^2 + \frac{2C}{M}\left[\frac{1}{u_1} - \frac{1}{u_0}\right]}$, which is independent of the direction of the initial velocity c.

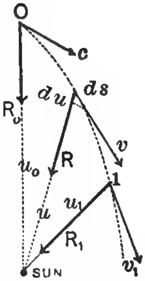

Fig. 90.

Application of eq. (*a*) *to a projectile in vacuo.*—G, the body's weight, is the only force acting, and therefore $= R$, while $M = G \div g$. Therefore equation (*a*) gives

$$\frac{G}{g}\cdot\frac{v_1^2-c^2}{2}=G\int_0^{y_1}dy=Gy_1;$$

$\therefore\ v_1=\sqrt{c^2+2gy_1}$, which is independent of the angle, α, of projection.

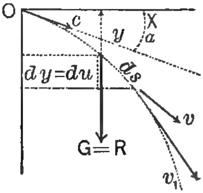

FIG. 91.

Application of equation (*a*) to a body sliding, *without friction*, on a fixed curved guide in a vertical plane; initial velocity $= c$ at O.—Since there is some pressure at each point between the body and the guide, to consider the body *free* in space, we must consider the guide removed and that the body describes the given curve as a result of the action of the two forces, its weight G, and the pressure P, of the guide against the body. G is constant, while P varies from point to point, though always (since there is no friction) *normal to curve.* At any point, R being the resultant of G and P, project ds upon R, thus obtaining du; on G, thus obtaining dy; on P, thus obtaining zero. But by the *principle of virtual velocities* (see § 62) we have $Rdu = Gdy + P \times \text{zero}^{*} = Gdy$, which substituted in eq. (*a*) gives

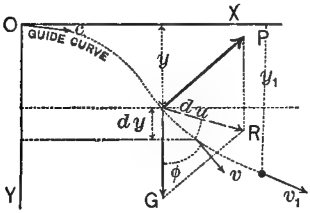

FIG 92.

$$\frac{G}{g}\frac{1}{2}(v_1^2-c^2)=\int_0^{y_1}Gdy=G\int_0^{y_1}dy=Gy_1;\quad\therefore v_1=\sqrt{c^2+2gy_1}$$

and therefore depends only on the *vertical distance* fallen through and the initial velocity, i.e., is *independent of the form of the guide.*

As to the value of P, the mutual pressure between the guide and body at any point, since ΣN must equal $Mv^2 \div r$, r being the variable radius of curvature, we have, as in § 77,

$$P - G\sin\varphi = Mv^2 \div r;\quad \therefore\ P = G[\sin\varphi + (v^2 \div gr)].$$

As, in general, φ and r are different from point to point of

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[*] It is quite essential that the guide be *fixed*, as well as smooth, in order that this projection be zero; since if the guide were in motion, the force P, although ⅂ to the guide, would not be ⅂ to the ds or element of the path of the body, for that path would then be different from the curve of the guide.

the path, P is not constant. Should the curve at the point in question be *convex* upward (instead of concave upward as in Fig. 92) we must write $G \sin \phi - P = Mv^2 \div r$; etc.

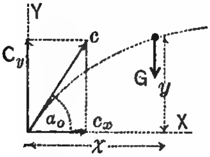

FIG. 93.

80. Projectiles in Vacuo.—A ball is projected into the air (whose resistance is neglected, hence the phrase *in vacuo*) at an angle $= \alpha_0$ with the horizontal;* required its path; assuming it confined to a vertical plane. Resolve the motion into independent horizontal (X) and vertical (Y) motions, G, the weight, the only force acting, being correspondingly replaced by its horizontal component = zero, and its vertical component $= -G$. Similarly the initial velocity along $X = c_x = c \cos \alpha_0$, along $Y, = c_y = c \sin \alpha_0$. The X acceleration $= p_x = 0 \div M = 0$, i.e., the X motion is uniform, the velocity v_x remains $= c_x = c \cos \alpha_0$ at all points, hence, reckoning the time from O, at the end of any time t we have

$$x = c(\cos \alpha_0)t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the Y motion, $p_y = (-G) \div M = -g$, i.e., it is uniformly retarded, the initial velocity being $c_y = c \sin \alpha_0$; hence, after any time t, the Y velocity will be (see § 56) $v_y = c \sin \alpha_0 - gt$, while the distance

$$y = c(\sin \alpha_0)t - \tfrac{1}{2}gt^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Between (1) and (2) we may eliminate t, and obtain as the equation of the trajectory or path

$$y = x \tan \alpha_0 - \frac{gx^2}{2c^2 \cos^2 \alpha_0}.$$

For brevity put $c^2 = 2gh$, h being the ideal *height due to the velocity* c, i.e., $c^2 \div 2g$ (see § 53; if the ball were directed vertically upward, a height $h = c^2 \div 2g$ would be actually attained, α_0 being $= 90°$), and we have

$$y = x \tan \alpha_0 - \frac{x^2}{4h \cos^2 \alpha_0}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is easily shown to be the equation of a parabola, with its axis vertical.

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* And with a velocity of c ft. per sec.

The horizontal range.—Fig. 94. Putting $y = 0$ in equation (3), we obtain

$$x\left[\tan \alpha_0 - \frac{x^2}{4h \cos^2 \alpha_0}\right] = 0,$$

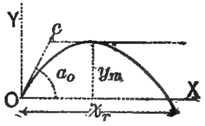

FIG. 94.

which is satisfied both by $x = 0$ (i.e., at the origin), and by $x = 4h \cos \alpha_0 \sin \alpha_0$. Hence the horizontal range for a given c and α_0 is $x_r = 4h \cos \alpha_0 \sin \alpha_0 = 2h \sin 2\alpha_0$.

For $\alpha_0 = 45°$ this is a maximum (c remaining the same), being then $= 2h$. Also, since $\sin 2\alpha_0 = \sin (180° - 2\alpha_0) = \sin 2(90° - \alpha_0)$, therefore any two complementary angles of projection give the same horizontal range.

Greatest height of ascent; that is, the value of y maximum, $= y_m$.—Fig. 94. Differentiate (3), obtaining

$$\frac{dy}{dx} = \tan \alpha_0 - \frac{x}{2h \cos^2 \alpha_0},$$

which, put $= 0$, gives $x = 2h \sin \alpha_0 \cos \alpha_0$, and this value of x in (3) gives $y_m = h \sin^2 \alpha_0$.

(Let the student obtain this more simply by considering the Y motion separately.)

81. Actual Path of Projectiles.—Small jets of water, so long as they remain unbroken, give close approximations to parabolic paths, as also any small dense object, e.g., a ball of metal, having a moderate initial velocity. The course of a cannon-ball, however, with a velocity of 1200 to 1400 feet per second is much affected by the resistance of the air, the descending branch of the curve being much steeper than the ascending; see Fig. 96*a*. The equation of this curve has not yet been determined, but only the expression for the slope (i.e., $dy : dx$) at any point. See Professor Bartlett's Mechanics, § 151 (in which the body is a sphere having no motion of rotation). Swift rotation about an axis, as well as an unsymmetrical form with reference to the direction of motion, alters the trajectory still further, and may deviate it from a vertical plane. The presence of wind would occasion increased irregularity. See Johnson's Encyclopædia, article "Gunnery." (See p. 823.)

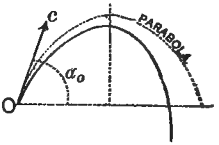

FIG. 96*a*.

82. Special Problem (imaginary; from Weisbach's Mechanics. *The equations are not homogeneous*).—Suppose a material point, mass $= M$, to start from the point O, Fig. 97, with a velocity $= 9$ feet per second along the $-\ Y$ axis, being subjected thereafter to a constant attractive X force, of a value $X = 12M$, and to a variable Y force increasing with the time (in seconds, reckoned from O), viz., $Y = 8Mt$. Required the path, etc. For the X motion we have $p_x = X \div M = 12$, and hence

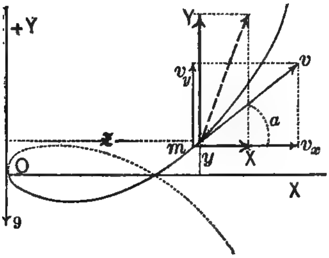

FIG. 97.

$$\int_0^{v_x} dv_x = \int_0^t p_x dt = 12\int_0^t dt; \text{ i.e., } v_x = 12t;$$

$$\text{and } \int_0^x dx = \int_0^t v_x dt; \text{ i.e., } x = 12\int_0^t tdt = 6t^2. \quad . \quad (1)$$

For the Y motion $p_y = Y \div M = 8t$, $\therefore \int_{-9}^{v_y} dv_y = 8\int_0^t tdt$;

$$\text{i.e., } v_y + 9 = 4t^2, \quad \text{and} \quad \int_0^y dy = \int_0^t v_y dt;$$

$$\therefore\ y = 4\int_0^t t^2 dt - 9\int_0^t dt, \quad \text{or} \quad y = \tfrac{4}{3}t^3 - 9t. \quad . \quad . \quad (2)$$

Eliminate t between (1) and (2), and we have, as the *equation of the path*,

$$y = \pm\frac{4}{3}\left(\frac{x}{6}\right)^{\frac{3}{2}} \mp 9\left(\frac{x}{6}\right)^{\frac{1}{2}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which indicates a curve of the third order.

The velocity at any point is (see § 74, eq. (1))

$$v = \sqrt{v_x^2 + v_y^2} = 4t^2 + 9. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The length of curve measured from O will be (since $v = ds \div dt$)

$$s = \int_0^s ds = \int_0^t vdt = 4\int_0^t t^2 dt + 9\int_0^t dt = \tfrac{4}{3}t^3 + 9t. \quad (5)$$

The *slope*, tan α, at any point $= v_y \div v_x = (4t^2 - 9) \div 12t$,

$$\text{and } \therefore \frac{d \tan \alpha}{dt} = \frac{4t^2 + 9}{12t^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The radius of curvature at any point (§ 74, eq. (6)), substituting $v_x = 12t$, also from (4) and (6), is

$$r = v^3 \div \left[v_x^2 \frac{d \tan \alpha}{dt} \right] = \frac{1}{12}[4t^2 + 9]^{\frac{3}{2}}, \quad . \quad . \quad (7)$$

and the *normal acceleration* $= v^2 \div r$ (eq. (4), § 74), becomes from (4) and (7) $p_n = 12$ (ft. per square second), a *constant.* Hence the centripetal or deviating force at any point, i.e., the ΣN of the forces X and Y, is *the same at all points*, and $= Mv^2 \div r = 12M$.

From equation (3) it is evident that the curve is symmetrical about the axis X. Negative values of t and s would apply to points on the dotted portion in Fig. 97, since the body may be considered as having started at any point whatever, so long as all the variables have their proper values for that point.

(Let the student determine how the conditions of this motion could be approximated to experimentally.)

83. Relative and Absolute Velocities.—Fig. 98. Let M be a material point having a uniform motion of velocity v_2 along a straight groove cut in the deck of a steamer, which itself has a uniform motion of translation, of velocity v_1, over the bed of a river. In one second M advances a distance v_2 along the groove, which simultaneously has moved a distance $v_1 = AB$ with the vessel. The absolute path of M during the second is evidently w (the diagonal formed on v_1 and v_2), which may therefore be called the *absolute velocity* of the body (considering the bed of the river as fixed); while v_2 is its *relative velocity*, i.e., relative to the vessel. If the motion of the vessel is not one of translation, the construction still holds good for an instant of time, but v_1 is then the velocity of that point of the deck over which M is passing at this instant, and v_2 is M's velocity relatively to that point alone.

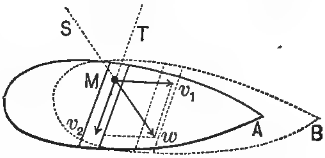

FIG. 98.

Conversely, if M be moving over the deck with a given absolute velocity $= w$, v_1 being that of the vessel, the relative velocity v_2 may be found by resolving w into two components, one of which shall be v_1; the other will be v_2.

If w is the absolute velocity and direction of the *wind*, the vane on the mast-head will be parallel to MT, i.e., to v_2 the relative velocity; while if the vessel be rolling and the mast-head therefore describing a sinuous path, the direction of the vane varies periodically.

Evidently the effect of the wind on the sails, if any, will depend on v_2 the relative, and not directly on w the absolute, velocity. Similarly, if w is the velocity of a jet of water, and v_1 that of a water-wheel channel, which the water is to enter without sudden deviation, or impact, the channel-partition should be made tangent to v_2 and not to w.

Again, the *aberration of light* of the stars depends on the same construction; v_1 is the absolute velocity of a locality of the earth's surface (being practically equal to that of the centre); w is the absolute direction and velocity of the light from a certain star. To see the star, a telescope must be directed along MT, i.e., parallel to v_2 the relative velocity; just as in the case of the moving vessel, the groove must have the direction MT, if the moving material point, having an absolute velocity w, is to pass down the groove without touching its sides. Since the velocity of light $= 192,000$ miles per second $= w$, and that of the earth in its orbit $= 19$ miles per second $= v_1$, the angle of aberration SMT, Fig. 98, will not exceed 20 seconds of arc; while it is zero when w and v_1 are parallel.

Returning to the wind and sail-boat, it will be seen from Fig. 98 that when $v_1 =$ or even $> w$, it is still possible for v_2 to be of such an amount and direction as to give, on a sail properly placed, a small wind-pressure, having a small fore-and aft component, which in the case of an ice-boat may exceed the small fore-and-aft resistance of such a craft, and thus v_1 will be still further increased; i.e., an ice-boat may sometimes travel faster than the wind which drives it. This has often been proved experimentally on the Hudson River. (See p. 819.)

84. Examples.—1. A platform-car on a straight level track carries a vertical *smooth* pole loosely encircled by an iron ring weighing 30 lbs., and is part of a train having a uniform northward motion with velocity of 20 miles/hour. The ring, at first fastened at the top of pole, 10 ft. above floor, is set free. Find its absolute velocity just before striking the floor and the distance the car has progressed during the fall of ring.

Solution.—The X-motion (horizontal) of the ring is that of the car and has a constant velocity $c_x = 20 \times 5280 \div 3600 = 29.34$ ft./sec. Its Y-motion (i.e., along pole) has initial velocity $=0$ and a constant downward acceleration $p_y=g$ (since the only force acting on ring is vertical and is its *own weight*). Hence from § 56 the time of the 10-ft. fall $=\sqrt{2\times 10\div 32.2}=0.788$ sec. and the Y-velocity generated at end of that time is $v_y=gt=25.4$ ft./sec. This is now combined with the simultaneous X-velocity of ring, i.e., 29.34, to give $v, =\sqrt{c_x^2+v_y^2}, =38.8$ ft./sec., for the required final absolute velocity of ring, which is therefore at this instant moving obliquely northward and downward at an angle of 40° 52′ with the horizontal (since $v_y\div c_x=25.4\div 29.34=0.8655=\tan 40° 52'$).

Example 2.—Pole and car, etc., as in example 1, but the train now has a *uniformly accelerated motion*, gaining 25 velocity-units (ft. per sec.) in each 5 secs. of time. The ring begins to drop when the train already has a velocity of 6 ft./sec. Find the final absolute velocity of ring; also the final pressure of pole on ring.

Solution.—The Y-motion of ring is the same as before, since the pressure on the ring from the pole (smooth vertical sides) must be horizontal and hence does not affect the Y-motion. Hence the time of descent is, as before, 0.788 sec. During this time the velocity of the train has increased to a value of $v_x=6+(25\div 5)0.788=9.94$ ft./sec., which is the velocity of the X-motion of the ring at the final instant, whence its final absolute velocity, $v, =\sqrt{(9.94)^2+(25.4)^2}, =27.3$ ft./sec., directed obliquely downward and northward at an angle of 68° 38′ with horizontal ($9.94\div 27.3=0.3642=\cos 68° 38'$) The pressure of the pole on ring is constant and $=P_x=Mp_x=(30\div 32.2)\times 5=4.65$ lbs.

Example 3.—Conical pendulum, Fig. 83, p. 78. Given $G=8$ lbs. and $l=2$ ft., at what angle α will the cord finally place itself with the vertical if a steady rotation is kept up at the rate of 50 revs./min.; and what will then be the tension in the cord?

Solution.—With the ft., lb., and sec. as units we have $u=0.8333$ revs./sec., $G=8$, $l=2$, $\alpha=?$ Hence from $u^2=g\div(4\pi^2h)$, we find $h=1.174$ ft. and $\cos\alpha, =h\div l, =0.5873$; $\therefore\ \alpha=54° 0'$. As for the tension in cord, $P=G\div\cos\alpha=13.62$ lbs.

[*Note.*—In this example, if the assigned value of u, or of the cord-length l, had been so small as to make $lu^2>g\div(4\pi^2)$, we should have obtained for $\cos\alpha$ a value >1.00; which is, of course, impossible for a cosine. That is, the value assigned for u must be $>\sqrt{g}\div(2\pi\sqrt{l})$, in order that the cord may depart at all from its original vertical position.]

Example 3.—Compute the length l of a simple pendulum which is to oscillate 4500 times in an hour. Amplitude small; 5°

Solution.—For small oscillations we have, from p. 81, $t=\pi\sqrt{l\div g}$ as the time of one oscillation; that is, for the foot and second as units,

$$3600\div 4500=\pi\sqrt{l\div 32.2};\quad \text{and therefore}\quad l=2.089 \text{ ft.}$$

Example 4.—A leaden ball weighing ½ ounce, and of diameter 0.53 in., is allowed to slide down the inside of a fixed and rigid hemispherical bowl, of perfectly smooth internal surface and with its upper edge in a horizontal plane. Its radius is 18 in. The ball is to start from rest at upper edge. Find the time occupied by the ball in reaching the lowest point, and the pressure under it as it passes that point; also the pressure in passing the 45° point.

Solution.—Regarding the ball as a material point we note that its motion is practically that of a simple pendulum with $l=[18-\frac{1}{2}(0.53)]$ in., $=1.478$ ft., for which (see Fig. 88, p. 81) the ratio $h\div l=1.00$. Hence (§ 78, p. 81) the time of a *half* oscillation (applicable here) will be (ft. and sec.),

$$\frac{\pi}{2}\sqrt{\frac{1.478}{32.2}}\left[1+\frac{1}{8}+\frac{9}{256}+\frac{225}{18432}\right]=\frac{\pi}{2}(0.2143)[1.1724]=0.395\text{ sec.}$$

(For a small amplitude this would be only $\frac{1}{2}\pi(.02143)[1.00]=0.337$ sec.)

At the bottom the velocity will be (p. 83), $v=\sqrt{2gl}$, whence $v^2\div gl=2$; and the pressure (see foot p. 83, with $\sin\phi=1.0$, is $P=\frac{1}{2}(1+2)=1.5$ ounces.

As the ball passes the 45° point its velocity is $v'=\sqrt{2g\times0.707l}$; i.e., $v'^2\div gl=1.414$, while $\sin 45°=0.707$; whence, for the pressure, P',

$$P'=\tfrac{1}{2}[0.707+1.414]=1.06\text{ ounces.}$$

Example 5.—A body at latitude 41° weighs apparently (i.e., by spring balance) 10 lbs.; what is the amount and direction of its real weight? (Fig. 85.) That is, we have given $G=10$ lbs. and angle $\beta=41°$; and desire the value of force G' and of the angle ϕ which it makes with MA (plumb line). (This angle, ϕ, $=$that at vertex G of the parallelogram in Fig. 85).

Solution.—At the equator the earth's radius is $r=20{,}920{,}000$ ft. and the velocity of objects at the surface is $c=1521$ ft./sec. The radius of the small circle at M is $r'=r\cos 41°=15{,}780{,}000$ ft., and hence the velocity of the 10-lb. body at M is c', $=(r'\div r)c$, $=1148$ ft./sec. Therefore the resultant N, $=Mc^2\div r'$, $=[(10\div32.2)(1148)^2]\div15{,}780{,}000=0.0259$ lbs.

Call the projection of N on GM prolonged, T, and its projection on a ⊥ to GM, S; then T, $=N\cos\beta$, $=0.01954$ lbs., and S, $=N\sin\beta$, $=0.01699$ lbs. We have also $\tan\phi=S\div[G+T]=0.0016957$; hence $\phi=0°\ 5'\ 48''$. Then G', $=(G+T)\sec\phi$, $=10.01955$ lbs.

[By a somewhat more refined process we obtain 10.01964 lbs. (Du Bois).]

Example 6.—A small compact jet of water (see Fig. 94, p. 85) issues obliquely from a nozzle. It strikes the horizontal plane of nozzle at 6 ft. from the latter, and its highest point is 26.4 in. above that plane. Find c, the velocity at nozzle, and the angle of projection α_0.

Solution.—From p. 85 (foot and second units) we have $4h\cos\alpha_0\sin\alpha_0=6$ ft., and $h\sin^2\alpha_0=2.2$ ft.; whence, by division $(\tan\alpha_0\div4)=(2.2\div6)$, or $\tan\alpha_0=1.4666$; and therefore $\alpha_0=55°\ 43'$. α_0 being now known we find from $h\sin^2\alpha_0=2.2$ that $h=3.22$ ft. But h simply stands for the expression $c^2\div2g$, and hence, finally, we obtain for the velocity of the jet where it leaves the nozzle $c=14.4$ ft. per sec.

Example 7.—If in Fig. 98, the absolute velocity of the air-particles (wind) is $w=10$ miles/hour and directly from the northwest, the boat's velocity being $=12$ miles/hour toward the east, in what direction and with what velocity does the wind appear to come, to a man on the boat?

Ans. From a direction 34° 52′ east of north, and at 8.62 mi./hr.

Example 8.—If to a passenger on board a boat going eastward at 15 miles/hour, the wind appears to come from the northeast and to have a velocity 10 miles/hour, what is the true or "absolute" velocity of the wind, and what is its true direction (angle with north and south line)?

Ans. 10.63 mi./hr., and from a point 48° 16′ west of north.

CHAPTER IV.

MOMENT OF INERTIA.

[NOTE.—For the propriety of this term and its use in Mechanics, see §§ 114, 216, and 229; for the present we deal only with the geometrical nature of these two kinds of quantity.]

85. Plane Figures.—Just as in dealing with the centre of gravity of a plane figure (§ 23), we had occasion to sum the series $\int z dF$, z being the distance of any element of area, dF, from an axis; so in subsequent chapters it will be necessary to know the value of the series $\int z^2 dF$ for plane figures of various shapes referred to various axes. This summation $\int z^2 dF$ of the products arising from multiplying each elementary area of the figure by the *square* of its distance from an axis is called the **moment of inertia** *of the plane figure with respect to the axis in question;* its symbol will be I. If the axis is perpendicular to the plane of the figure, it may be named the *polar mom. of inertia* (§ 94); if the axis lies in the plane, the *rectangular* mom. of inertia (§§ 90–93). Since the I of a plane figure evidently consists of *four dimensions of length*, it may always be resolved into two factors, thus $I = Fk^2$, in which $F =$ total area of the figure, while $k = \sqrt{I \div F}$, is called the **radius of gyration,** because if all the elements of area were situated at the *same* radial distance, k, from the axis, the moment of inertia would still be the same, viz.,

$$I = \int k^2 dF = k^2 \int dF = Fk^2.$$

For example, if the moment of inertia of a certain plane figure about a specified axis is 248 biquadratic inches (i.e., four-dimension inches; or in.^{4}), while its area is 12 sq. in. (or in.^{2}), the corresponding radius of gyration is

$$k = \sqrt{248 \div 12} = 4.55 \text{ in.}$$

86. Rigid Bodies.—Similarly, in dealing with the rotary motion of a rigid body, we shall need the sum of the series $\int \rho^2 dM$, meaning the summation of the products arising from multiplying the mass dM of each elementary volume dV of a rigid body by the square of its distance from a specified axis. This will be called the *moment of inertia of the body with respect to the particular axis mentioned* (often indicated by a subscript), and will be denoted by I. As before, it can often be conveniently written Mk^2, in which M is the whole mass, and k its "radius of gyration" for the axis used, k being $= \sqrt{I \div M}$. If the body is *homogeneous*, the heaviness, γ, of all its particles will be the same, and we may write

$$I = \int \rho^2 dM = (\gamma \div g)\int \rho^2 dV = (\gamma \div g)Vk^2.$$

87. If the body is a homogeneous plate of an *infinitely small thickness* $= \tau$, and of area $= F$, we have $I = (\gamma \div g)\int \rho^2 dV$ $= (\gamma \div g)\tau\int \rho^2 dF$; i.e., $= (\gamma \div g) \times$ *thickness* $\times$ *mom. inertia of the plane figure.*

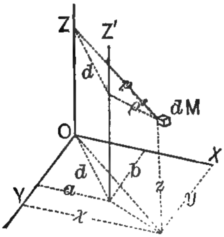

FIG. 99.

88. Two Parallel Axes. Reduction Formula.*—Fig. 99. Let Z and Z' be two parallel axes. Then I_z $= \int \rho^2 dM$, and $I_{z'} = \int \rho'^2 dM$. But d being the distance between the axes, so that $a^2 + b^2 = d^2$, we have $\rho'^2 = (x - a)^2 + (y - b)^2$ $= (x^2 + y^2) + d^2 - 2ax - 2by$, and $\therefore$

$$I_{z'} = \int \rho^2 dM + d^2 \int dM - 2a\int x dM - 2b\int y dM. \quad . \quad (1)$$

But $\int \rho^2 dM = I_z$, $\int dM = M$, and from the theory of the centre of gravity (see § 23, eq. (1), knowing that $dM = \gamma dV \div g$, and $\therefore$ that $[\int \gamma dV] \div g = M$) we have $\int x dM$ $= M\bar{x}$ and $\int y dM = M\bar{y}$; hence (1) becomes

$$I_{z'} = I_z + M(d^2 - 2a\bar{x} - 2b\bar{y}), \quad . \quad . \quad . \quad . \quad (2)$$

in which a and b are the x and y of the axis Z'; $\bar{x}$ and $\bar{y}$ refer to the centre of gravity of the body. If Z is a gravity-axis (call it g), both $\bar{x}$ and $\bar{y} = 0$, and (2) becomes

$$I_{z'} = I_g + Md^2. \ . \ . \ . \quad \text{or} \quad k_{z'}^2 = k_g^2 + d^2. \quad . \quad . \quad . \quad (3)$$

It is therefore evident that the mom. of inertia about a gravity-axis is smaller than about any other *parallel* axis.

Eq. (3) includes the particular case of a *plane figure*, by writing area instead of mass, i.e., when Z (now g) is a gravity-axis,

---

* The particle of mass $= dM$, shown in Fig. 99, is typical of the *vast number* of particles which form the rigid body. That is, a, b, and d are *constants*, but x, y, z, ρ, and ρ' are *variables*.

$$I_{z'} = I_g + Fd^2. \qquad \ldots \ldots \qquad (4)$$

89. Other Reduction Formulæ; for Plane Figures.—(The axes here mentioned lie in the plane of the figure.) For *two sets of rectangular axes*, having the *same* origin, the following holds good. Fig. 100. Since

$$I_X = \int y^2 dF, \quad \text{and} \quad I_Y = \int x^2 dF,$$

we have $$I_X + I_Y = \int (x^2 + y^2) dF.$$

Similarly, $$I_U + I_V = \int (v^2 + u^2) dF.$$

But since the x and y of any dF have the same hypothenuse as the u and v, we have $v^2 + u^2 = x^2 + y^2$; $\therefore\ I_X + I_Y = I_U + I_V$.

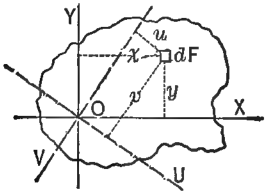

FIG. 100.

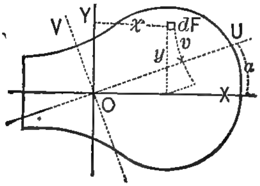

FIG. 100*a*.

Let X be an axis of symmetry; then, given I_X and I_Y (O is anywhere on X), required I_U, *U being an axis through O and making any angle α with X.* See Fig. 100*a*.

$$I_U = \int v^2 dF = \int (y \cos \alpha - x \sin \alpha)^2 dF; \text{ i.e.,}$$
$$I_U = \cos^2 \alpha \int y^2 dF - 2 \sin \alpha \cos \alpha \int xy dF + \sin^2 \alpha \int x^2 dF.$$

But since the area is symmetrical about X, in summing up the products $xydF$, for every term $x(+y)dF$, there is also a term $x(-y)dF$ to cancel it; which gives $\int xydF = 0$. Hence

$$I_U = \cos^2 \alpha I_X + \sin^2 \alpha I_Y.$$

The student may easily prove that if two distances a and b be set off from O on X and Y respectively, made inversely proportional to $\sqrt{I_X}$ and $\sqrt{I_Y}$, and an ellipse described on a and b as semi-axes; then the moments of inertia of the figure about any axes through O are inversely proportional to the squares of the corresponding semi-diameters of this ellipse; called therefore the *Ellipse of Inertia.* It follows therefore that the moments of inertia about *all gravity-axes* of a circle, or a regular polygon, are equal; since their ellipse of inertia must be a circle. Even if the plane figure is not symmetrical, an "ellipse of inertia" can be located at any point, and has the properties already mentioned; its axes are called the *principal axes* for that point.

90. The Rectangle.—*First, about its base.* Fig. 101. Since all points of a strip parallel to the base have the same co-ordinate, z, we may take the area of such a strip for $dF = bdz$;

$$\therefore I_B = \int_0^h z^2dF = b\int_0^h z^2dz$$

$$= \tfrac{1}{3}b\Big[_0^h z^3 = \tfrac{1}{3}bh^3; \text{ and } k = h \div \sqrt{3}.$$

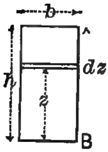

FIG. 101.

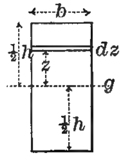

FIG. 102.

Secondly, about a *gravity-axis parallel to base.*

$$dF = bdz \;\therefore I_g = \int z^2dF = b\int_{-\frac{1}{2}h}^{+\frac{1}{2}h} z^2dz = \tfrac{1}{12}bh^3.$$

Hence the radius of gyration $= k = h \div \sqrt{12}$.

Thirdly, about *any other axis in its plane.* Use the results already obtained in connection with the reduction-formulæ of §§ 88, 89.

90a. The Triangle.—*First,* about an axis through the vertex and parallel to the base; i.e., I_V in Fig. 103. Here the length of the strip is variable; call it y. From similar triangles

$$y = (b \div h)z;$$

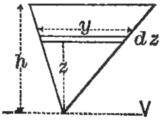

FIG. 103.

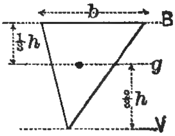

FIG. 104.

$$\therefore I_V = \int z^2dF = \int z^2ydz = (b \div h)\int_0^h z^3dz = \tfrac{1}{4}bh^3.$$

Secondly, about g, a gravity-axis parallel to the base. Fig 104. From § 88, eq. (4), we have, since $F = \tfrac{1}{2}bh$ and

$$d = \tfrac{2}{3}h,\; I_g = I_V - Fd^2 = \tfrac{1}{4}bh^3 - \tfrac{1}{2}bh \,.\, \tfrac{4}{9}h^2 = \tfrac{1}{36}bh^3.$$

$K\sqrt{\dfrac{\frac{1}{36}bh^3}{\frac{1}{2}bh}} = \sqrt{\frac{1}{2}h^2}$

Thirdly, Fig. 104, *about the base*; $I_B = ?$ From § 88, eq. (4), $I_B = I_g + Fd^2$, with $d = \frac{1}{3}h$; hence

$$I_B = \tfrac{1}{36}bh^3 + \tfrac{1}{2}bh \cdot \tfrac{1}{9}h^2 = \tfrac{1}{12}bh^3.$$

91. The Circle.—About any diameter, as g, Fig. 105. Polar co-ordinates, $I_g = \int z^2 dF$. Here we take $dF =$ area of an elementary rectangle $= \rho d\varphi \,.\, d\rho$, while $z = \rho \sin \varphi$.

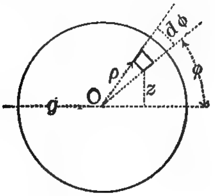

FIG. 105.

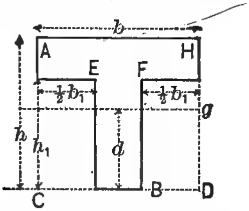

FIG. 106.

$$I_g = \int\int (\rho \sin \varphi)^2 \rho d\varphi d\rho = \int_0^{2\pi}\left[\sin^2 \varphi d\varphi \int_0^r \rho^3 d\rho\right]$$

$$= \frac{r^4}{4}\int_0^{2\pi} \sin^2 \varphi d\varphi = \frac{r^4}{4}\int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2\varphi)d\varphi$$

$$= \frac{r^4}{4}\int_0^{2\pi}\left[\frac{1}{2}d\varphi - \frac{1}{4} \,.\, \cos 2\varphi d(2\varphi)\right]$$

$$= \frac{1}{4}r^4\Big[_0^{2\pi}\left(\frac{1}{2}\varphi - \frac{1}{4}\sin 2\varphi\right)$$

$$= \frac{1}{4}r^4\left[\left(\frac{2\pi}{2} - 0\right) - (0 - 0)\right]. \quad \text{i.e., } I_g = \frac{1}{4}\pi r^4.$$

Hence the radius of gyration $= \frac{1}{2}r$.

92. Compound Plane Figures.—Since $I = \int z^2 dF$ is an infinite series, it may be considered as made up of separate groups or subordinate series, combined by algebraic addition, corresponding to the subdivision of the compound figure into component figures, each subordinate series being the moment of inertia of one of these component figures; but these separate moments *must all be referred to the same axis.* It is convenient to remember that the (*rectangular*) I of a plane figure remains unchanged if we conceive some or all of its elements shifted any distance parallel to the axis of reference. E.g., in Fig. 106, the sum of the I_B of the rectangle *CE*, and that of *FD* is = to the I_B of the imaginary rectangle formed by shifting one of them parallel to B, until it touches the other; i.e., I_B of $CE + I_B$ of $FD = \frac{1}{3}b_1h_1^3$ (§ 90). Hence the I_B of the $\top$ shape in Fig. 106 will be $= I_B$ of rectangle $AD - I_B$ of rect. $CE - I_B$ of rect. FD.

That is, $\quad I_B$ of $\top = \frac{1}{3}[bh^3 - b_1h_1^3]$. . . . (§ 90). . . (1)

About the gravity-axis, g, Fig. 106. To find the distance d from the base to the centre of gravity, we may make use of eq. (3) of § 23, writing areas instead of volumes, or, experimentally, having cut the given shape out of sheet-metal or card-board, we may balance it on a knife-edge. Supposing d to be known by some such method, we have, from eq. (4) of § 88, since the area $F = bh - b_1h_1$, $I_g = I_B - Fd^2$;

$$\text{i.e., } I_g = \tfrac{1}{3}[bh^3 - b_1h_1^3] - (bh - b_1h_1)d^2. \quad . \quad . \quad (2)$$

The double-$\top$ (*or* $\mathtt{I}$), and the *box* forms of Fig. 106*a*, *if symmetrical about the gravity-axis* g, have moments of inertia alike in form. Here the gravity-axis (parallel to base) of the compound figure is also a gravity axis (parallel to base) of each of the two component rectangles, of dimensions b and h, b_1 and h_1, respectively.

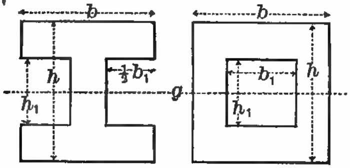

FIG. 106*a*.

Hence by algebraic addition we have (§ 90), for either compound figure,

$$I_g = \tfrac{1}{12}[bh^3 - b_1h_1^3]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(If there is no axis of symmetry parallel to the base we must proceed as in dealing with the $\top$-form.) Similarly for the ring, Fig. 107, or space between two *concentric* circumferences, we have, about any diameter or g (§ 91),

$$I_g = \tfrac{1}{4}\pi(r_1^4 - r_2^4). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

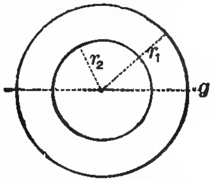

FIG. 107.

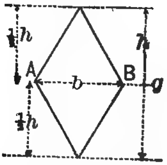

FIG. 108.

The rhombus about a gravity-axis, g, perpendicular to a diagonal, Fig. 108.—This axis divides the figure into two equal triangles, *symmetrically placed*, hence the I_g of the rhombus equals double the moment of inertia of one triangle about its base; hence (§ 90a)

$$I_g = 2 \cdot \tfrac{1}{12} b (\tfrac{1}{2}h)^3 = \tfrac{1}{48} bh^3. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(The result is the same, if either vertex, or both, be shifted any distance parallel to AB.)

For practice, the student may derive results for the *trapezoid*; for the forms in Fig. 106, when the inner corners are rounded into equal quadrants of circles; for the double-T, when the lower flanges are shorter than the upper; for the regular polygons, etc. (See table in the Cambria Steel Co.'s hand-book.)

93. If the plane figure be bounded, wholly or partially, by curves, it may be subdivided into an infinite number of strips, and the moments of inertia of these (referred to the desired axis) added by integration, *if the equations of the curves are known;* if not, Simpson's Rule,* for a finite even number of strips, of equal width, may be employed for an approximate result. If these strips are parallel to the axis, the I of any one strip = its length × its width × square of distance from axis; while if perpendicular to, and *terminating in*, the axis, its $I = \frac{1}{3}$ its width × cube of its length (see § 90).

A graphic method of determining the moment of inertia of any irregular figure will be given in a subsequent chapter.*

94. Polar Moment of Inertia of Plane Figures (§ 85).—Since the axis is now perpendicular to the plane of the figure, intersecting it in a point, O, the distances of the elements of area will all *radiate* from this point, and would better be denoted by ρ instead of z; hence, Fig. 109, $\int \rho^2 dF$ is the polar moment of inertia of any plane figure about a specified point O; this may be denoted by I_p. But $\rho^2 = x^2 + y^2$, for each dF; hence

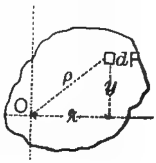

FIG. 109.

$$I_p = \int (x^2 + y^2) dF = \int x^2 dF + \int y^2 dF = I_Y + I_X.$$

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* See pp. 13, 79, 80, and 81 of the author's "*Notes and Examples in Mechanics*," and p. 454 of this book.

i.e., *the polar moment of inertia about any given point in the plane equals the sum of the rectangular moments of inertia about any two axes of the plane figure, which intersect at right angles in the given point.* We have therefore for the *circle about its centre*

$$I_p = \tfrac{1}{4}\pi r^4 + \tfrac{1}{4}\pi r^4 = \tfrac{1}{2}\pi r^4;$$

For a *ring* of radii r_1 and r_2,

$$I_p = \tfrac{1}{2}\pi(r_1^4 - r_2^4);$$

For the *rectangle about its centre*,

$$I_p = \tfrac{1}{12}bh^3 + \tfrac{1}{12}hb^3 = \tfrac{1}{12}bh(b^2 + h^2);$$

For the *square*, this reduces to

$$I_p = \tfrac{1}{6}b^4.$$

(See §§ 90 and 91.)

95. Slender, Prismatic, Homogeneous Rod.—Returning to the moment of inertia of rigid bodies, or solids, we begin with that of a *material line*, as it might be called, about an axis through its extremity making some angle α with the rod. Let l = length of the rod, F its cross-section (very small, the result being strictly true only when $F = 0$). Subdivide the rod into an infinite number of small prisms, each having F as a base, and an altitude $= ds$. Let γ = the heaviness of the material; then the mass of an elementary prism, or dM, $= (\gamma \div g)Fds$, while its distance from the axis Z is $\rho = s \sin \alpha$. Hence the moment of inertia of the rod with respect to Z as an axis is

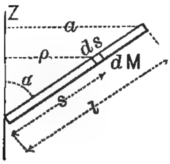

FIG. 110.

$$I_Z = \int \rho^2 dM = (\gamma \div g)F \sin^2 \alpha \int_0^l s^2 ds = \tfrac{1}{3}\cancel{(\gamma \div g)F}\overset{\checkmark}{l^3} \sin^2 \alpha.$$

But $\gamma Fl \div g$ = mass of rod and $l \sin \alpha = a$, the distance of the further extremity from the axis; hence $I_Z = \tfrac{1}{3}Ma^2$ and the *radius of gyration*, or k, is found by writing $\tfrac{1}{3}Ma^2 = Mk^2$; $\therefore k^2 = \tfrac{1}{3}a^2$, or $k = \sqrt{\tfrac{1}{3}}a$ (see § 86). If $\alpha = 90°$, $a = l$.

96. Thin Plates. Axis in the Plate.—Let the plates be homogeneous and of small constant thickness $= \tau$. If the surface of the plate be $= F$, and its heaviness γ, then its mass $= \gamma F\tau \div g$. From § 87 we have for the plate, about any axis,

$I = (\gamma \div g)\tau \times$ *mom. of inertia of the plane figure formed by the shape of the plate.* . . . . (1)

Rectangular plate. Gravity-axis parallel to base.—Dimensions b and h. From eq. (1) and § 90 we have

$$I_g = (\gamma \div g)\tau \,.\, \tfrac{1}{12}bh^3 = (\gamma bh\tau \div g)\tfrac{1}{12}h^2 = \tfrac{1}{12}Mh^2; \ \therefore k^2 = \tfrac{1}{12}h^2.$$

Similarly, if *the base is the axis*, $I_B = \frac{1}{3}Mh^2$, $\therefore k^2 = \frac{1}{3}h^2$.

Triangular plate. Axis through vertex parallel to base.—From eq. (1) and § 90*a*, dimensions being b and h,

$$I_V = (\gamma \div g)\tau\tfrac{1}{4}bh^3 = (\gamma\tfrac{1}{2}bh\tau \div g)\tfrac{1}{2}h^2 = \tfrac{1}{2}Mh^2; \ \therefore k^2 = \tfrac{1}{2}h^2.$$

Circular plate, with any diameter as axis.—From eq. (1) and § 91 we have

$$I_g = (\gamma \div g)\tau\tfrac{1}{4}\pi r^4 = (\gamma\pi r^2\tau \div g)\tfrac{1}{4}r^2 = \tfrac{1}{4}Mr^2; \ \therefore k^2 = \tfrac{1}{4}r^2.$$

97. Plates or Right Prisms of any Thickness (or Altitude). Axis Perpendicular to Surface (or Base).—As before, the solid is homogeneous, i.e., of constant heaviness γ; let the altitude $= h$. Consider an elementary prism, Fig. 111, whose length is parallel to the axis of reference Z. Its altitude $= h =$ that of the whole solid; its base $= dF =$ an element of F the area of the base of solid; and each point of it has the same ρ. Hence we may take its mass, $= \gamma h dF \div g$, as the dM in summing the series

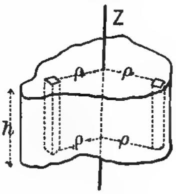

FIG. 111.

$$I_Z = \int\rho^2 dM;$$
$$\therefore I_Z = (\gamma h \div g)\int\rho^2 dF$$
$$= (\gamma h \div g) \times \text{polar mom. of inertia of base.} \quad . \ . \ . \quad (2)$$

By the use of eq. (2) and the results in § 94 we obtain the following:

Circular plate, or *right circular cylinder*, about the geometrical axis. $r =$ radius, $h =$ altitude.

$$I_g = (\gamma h \div g)\tfrac{1}{2}\pi r^4 = (\gamma h\pi r^2 \div g)\tfrac{1}{2}r^2 = \tfrac{1}{2}Mr^2; \ \therefore k^2 = \tfrac{1}{2}r^2.$$

Right parallelopiped or *rectangular plate.*—Fig. 112,

$$I_g = (\gamma h \div g)\tfrac{1}{12}bb_1(b_1^2 + b^2) = M\tfrac{1}{12}d^2; \ \therefore k^2 = \tfrac{1}{12}d^2.$$

For a *hollow cylinder*, about its geometric axis,

$$I_g = (\gamma h \div g)\tfrac{1}{2}\pi(r_1^4 - r_2^4) = \tfrac{1}{2}M(r_2^2 + r_1^2);\ \therefore k^2 = \tfrac{1}{2}(r_2^2 + r_1^2).$$

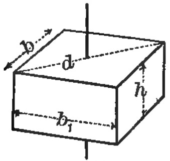

FIG. 112.

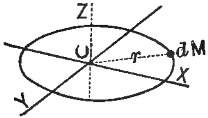

FIG. 113.

98. Circular Wire.—Fig. 113 (perspective). Let Z be a gravity-axis perpendicular to the plane of the wire; X and Y lie in this plane, intersecting at right angles in the centre O. The wire is homogeneous and of constant (small) cross-section. Since, referred to Z, each dM has the same $\rho = r$, we have $I_Z = \int r^2 dM = Mr^2$. Now I_X must equal I_Y, and (§ 94) their sum $= I_Z$,

$$\therefore I_X, \text{ or } I_Y, = \tfrac{1}{2}Mr^2, \qquad \text{and} \qquad k_X^2, \text{ or } k_Y^2 = \tfrac{1}{2}r^2.$$

99. Homogeneous Solid Cylinder, *about a diameter of its base.*—Fig. 114. $I_X = ?$ Divide the cylinder into an infinite number of laminæ, or thin plates, parallel to the base. Each is some distance z from X, of thickness dz, and of radius r (constant). In each draw a gravity-axis (of its own) parallel to X. We may now obtain the I_X of the whole cylinder by adding the I_X's of all the laminæ. The I_g of *any one lamina* (§ 96, circular plate) $=$ its mass $\times \frac{1}{4}r^2$; hence its I_X (eq. (3), § 88) $=$ its $I_g +$ (its mass) $\times z^2$. Hence for the whole cylinder

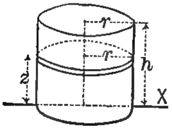

FIG. 114.

$$I_X = \int_0^h [(\gamma dz \pi r^2 \div g)(\tfrac{1}{4}r^2 + z^2)]$$

$$= (\pi r^2 \gamma \div g)\left[\tfrac{1}{4}r^2\int_0^h dz + \int_0^h z^2 dz\right];$$

$$\text{i.e., } I_X = (\pi r^2 h\gamma \div g)(\tfrac{1}{4}r^2 + \tfrac{1}{3}h^2) = M(\tfrac{1}{4}r^2 + \tfrac{1}{3}h^2).$$

100. Let the student prove (1) that if Fig. 114 represent **any right prism,** and k_F denote the radius of gyration of any one lamina, referred to its gravity-axis parallel to X, then the I_X of whole prism $= M(k_F^2 + \frac{1}{3}h^2)$; and (2) that the moment of inertia of the cylinder about a gravity-axis parallel to the base is $= M(\frac{1}{4}r^2 + \frac{1}{12}h^2)$.

101. Homogeneous Right Cone.—Fig. 115. *First*, about an axis V, *through the vertex and parallel to* the base. As before, divide into laminæ parallel to the base. Each is a circular thin plate, but its radius, x, is not $= r$, but, from proportion, is $x = (r \div h)z$.

The I of any lamina referred to its own gravity-axis parallel to V is (§ 96) = (its mass) $\times \frac{1}{4}x^2$, and its I_V (eq. (3), § 88) is $\therefore =$ its mass $\times \frac{1}{4}x^2 +$ its mass $\times z^2$.

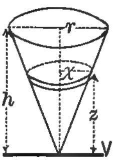

FIG. 115.

Hence for the whole cone,

$$I_V = \int_0^h (\pi x^2 dz\gamma \div g)[\tfrac{1}{4}x^2 + z^2]$$

$$= \frac{\gamma\pi r^2}{gh^2}\left[\frac{1}{4}\cdot\frac{r^2}{h^2} + 1\right]\int_0^h z^4 dz = M\frac{3}{20}[r^2 + 4h^2].$$

Secondly, about a gravity-axis parallel to the base.—From eq. (3), § 88, with $d = \frac{3}{4}h$ (see Prob. 7, § 26), and the result just obtained, we have $I = M\frac{3}{20}[r^2 + \frac{1}{4}h^2]$.

Thirdly, about its geometric axis, Z.—Fig. 116. Since the axis is perpendicular to each circular lamina through the centre, its I_Z (§ 97) is

$$= \text{its mass} \times \tfrac{1}{2}(\text{rad.})^2 = (\gamma\pi x^2 dz \div g)\tfrac{1}{2}x^2.$$

Now $x = (r \div h)z$, and hence for the whole cone

$$I_Z = \tfrac{1}{2}(\gamma\pi r^4 \div gh^4)\int_0^h z^4 dz = (\tfrac{1}{3}\pi r^2 h\gamma \div g)\tfrac{3}{10}r^2 = M\tfrac{3}{10}r^2.$$

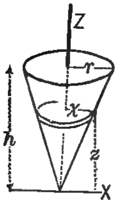

FIG. 116.

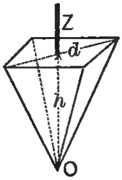

FIG. 117.

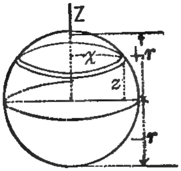

FIG. 118.

102. Homogeneous Right Pyramid of Rectangular Base.—*About its geometrical axis.* Proceeding as in the last paragraph, we derive $I_Z = M\frac{1}{20}d^2$, in which d is the diagonal of the base.

103. Homogeneous Sphere.—About any diameter. Fig. 118. $I_Z = ?$ Divide into laminæ perpendicular to Z. By § 97, and noting that $x^2 = r^2 - z^2$, we have finally, for the whole sphere,

$$I_Z = (\gamma\pi \div 2g)\Big[_{-r}^{+r}(r^4z - \tfrac{2}{3}r^2z^3 + \tfrac{1}{5}z^5) = \tfrac{8}{15}\gamma\pi r^5 \div g$$
$$= (\tfrac{4}{3}\pi r^3\gamma \div g)\tfrac{2}{5}r^2 = M\tfrac{2}{5}r^2;\ \therefore k_z^2 = \tfrac{2}{5}r^2.$$

For a *segment*, of one or two bases, put proper limits for z in the foregoing, instead of $+r$ and $-r$.

104. Other Cases.—*Parabolic plate*, Fig. 119, homogeneous and of (any) constant thickness, about an axis through O, the middle of the chord, and perpendicular to the plate. This is

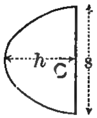

FIG. 119.

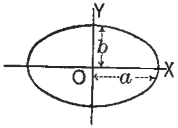

FIG. 120.

$$I = M\tfrac{1}{5}(\tfrac{1}{4}s^2 + \tfrac{8}{7}h^2).$$

The area of the segment is $= \frac{2}{3}hs$.

For an *elliptic plate*, Fig. 120, homogeneous and of any constant thickness, semi-axes a and b, we have about an axis through O, normal to surface $I_O = M\frac{1}{4}[a^2 + b^2]$; while for a very small constant thickness

$$I_X = M\tfrac{1}{4}b^2, \quad \text{and} \quad I_Y = M\tfrac{1}{4}a^2.$$

The area of the ellipse $= \pi ab$.

Considering Figs. 119 and 120 as *plane figures*, let the student determine their polar and rectangular moments of inertia about various axes.

For numerous other cases Kent's *Mechanical Engineers' Pocket-Book* may be consulted; also Trautwine's *Civil Engineers' Pocket-Book*.

105. Numerical Substitution.—The *moments of inertia of plane figures* involve dimensions of length alone, and will be utilized in the problems involving flexure and torsion of beams, where the inch is the most convenient linear unit. E.g., the polar moment of inertia of a circle of two inches radius about its centre is $\frac{1}{2}\pi r^4 = 25.13 +$ *biquadratic*, or *four-dimension*, inches, as it may be called. Since this quantity contains four dimensions of length, the use of the foot instead of the inch would diminish its numerical value in the ratio of the fourth power of twelve to unity.

The *moment of inertia of a rigid body, or solid*, however, $= Mk^2 = (G \div g)k^2$, in which G, the weight, is expressed in units of *force*, g involves both time and space (length), while k^2 involves length (two dimensions). Hence in any homogeneous formula in which the I of a solid occurs, we must be careful to employ units consistently; e.g., if in substituting $G \div g$ for M (as will always be done numerically) we put $g = 32.2$, we should use the *second* as unit of time, and the *foot* as linear unit.

106. Example.—Required the moment of inertia, about the axis of rotation, of a pulley consisting of a rim, four parallelopipedical arms, and a cylindrical hub which may be considered solid, being filled by a portion of the shaft. Fig. 121. Call the weight of the hub G, its radius r; similarly, for the rim, G_2, r_1 and r_2; the weight of one arm being $= G_1$. The total I will be the sum of the I's of the component parts, *referred to the same axis*, viz.: Those of the hub and rim will be $(G \div g)\frac{1}{2}r^2$ and $(G_2 \div g)\frac{1}{2}(r_1^2 + r_2^2)$, respectively (§ 97), while if the arms are *not very thick* compared with their length, we have for them (§§ 95 and 88)

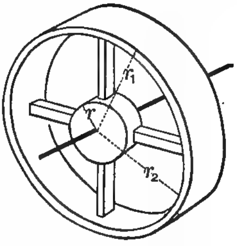

FIG. 121.

$$4(G_1 \div g)[\tfrac{1}{3}(r_2 - r)^2 - \tfrac{1}{4}(r_2 - r)^2 + [r + \tfrac{1}{2}(r_2 - r)]^2],$$

i.e.,
$$4(G_1 \div g)[\tfrac{1}{3}(r_2 - r)^2 + rr_2] \quad . \quad . \quad . \quad . \quad (4)$$

as an approximation (obtained by reduction from the axis at the extremity of an arm to a parallel gravity-axis, then to the required axis, then multiplying by four). In most fly-wheels, the rim is proportionally so heavy, besides being the farthest removed from the axis of rotation, that the moment of inertia of the other parts is only a small part of the whole.

Numerically let us have given $r = 4$, $r_2 = 36$, and $r_1 = 37$ inches; the respective weights being $G_2 = 500$ lbs. for the rim, $G_1 = 48$ lbs. for each arm, and $G = 120$ lbs. for the hub. The quantity g will be retained as a mere symbol. Using the foot-pound-second system of units we then have for the moment of inertia of the *hub* $(120 \div g)\frac{1}{2}[\frac{1}{3}]^2 = \quad 6.66 \div g$; for that of the *four arms* [by substitution in eq. (4) above]

$$4(48 \div g)\left[\frac{1}{3}\left(\frac{36}{12} - \frac{4}{12}\right)^2 + \frac{4}{12} \cdot \frac{36}{12}\right] = 647.2 \div g;$$

while for the *rim* we obtain $(500 \div g)\frac{1}{2}\left[\left(\frac{37}{12}\right)^2 + \left(\frac{36}{12}\right)^2\right] \quad = 4627.0 \div g.$

These results are seen to be approximately in the ratio of the numbers 1, 100, and 700; showing that the neglect of the hub and arms in computing the moment of inertia would give a result about $\frac{1}{8}$ too small.

Adding, we find for the total moment of inertia of the body about the axis of rotation the quantity $I = 5280.8 \div g$, for the units foot and pound. The unit of time is still involved in the quantity g.

We are now ready to compute the square of the corresponding *radius of gyration*, viz., k^2, by dividing I by the whole mass M, $= 812 \div g$ (see § 86); whence

$$k^2 = \left(\frac{5280}{g}\right) \div \left(\frac{812}{g}\right) = 6.50 \text{ sq. ft.},$$

and therefore k itself $= 2.55$ ft.

This is seen to be a little less than the 3.04 ft. value for k which would be implied in the approximate assumption that the moment of inertia is the same as if the whole mass were concentrated at the mid-point of the thickness of the rim, which assumption would be very nearly true if the masses of the hub and arms could be neglected.

107. Ellipsoid of Inertia.—The moments of inertia about all axes passing through any given point of any rigid body whatever may be proved to be inversely proportional to the squares of the diameters which they intercept in an imaginary ellipsoid, whose centre is the given point, and whose position in the body depends on the distribution of its mass and the location of the given point. The three axes which contain the three principal diameters of the ellipsoid are called the *Principal Axes* of the body for the given point. This is called the **ellipsoid of inertia.** (Compare § 89.) Hence the moments of inertia of any homogeneous regular polyedron about all gravity-axes are equal, since then the ellipsoid becomes a sphere. It can also be proved that for any rigid body, if the co-ordinate axes X, Y, and Z, are taken coincident with the three principal axes at any point, we shall have

$$\int xy\,dM = 0; \quad \int yz\,dM = 0; \quad \text{and} \quad \int zx\,dM = 0.$$

Note.—These three summations are called the "*products of inertia,*" and will occur in § 114 of this book.

CHAPTER V.

KINETICS OF A RIGID BODY.

108. General Method.—Among the possible* motions of a rigid body the most important for practical purposes (and fortunately the most simple to treat) are: a *motion of translation*, in which the particles move in parallel right lines with equal accelerations and velocities at any given instant; and *rotation about a fixed axis*, in which the particles describe circles in parallel planes with velocities and accelerations proportional (at any given instant) to their distances from the axis. Other motions will be mentioned later. To determine relations, or equations, between the elements of the motion, the mass and form of the body, and the forces acting (which do not necessarily form an unbalanced system), the most direct method to be employed is that of two *equivalent systems* of forces (§ 15), one consisting of the actual forces acting on the body, *considered free*, the other imaginary, consisting of the infinite number of forces which, applied to the separate material points composing the body, would account for their individual motions, as if they were an assemblage of particles without mutual actions or coherence. If the body were at rest, then considered *free*, and the forces referred to three co-ordinate axes, they would constitute a balanced system, for which the six summations ΣX, ΣY, ΣZ, $\Sigma(\text{mom.})_X$, $\Sigma(\text{mom.})_Y$, and $\Sigma(\text{mom.})_Z$, would each $= 0$; but in most cases of motion some or all of these sums are equal (at any given instant), not to zero, but to the corresponding summation of the imaginary equivalent system, i.e., to expressions involving the masses of the particles (or material points), their distribution in the body, and the elements of the motion. That is, we obtain six equations by putting the ΣX of the actual system equal to the ΣX of the imaginary, and so on; for a definite instant of time (since some of the quantities may be variable).

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* Motions of such character that the particles of the body do not change their relative positions. In other words, the body remains *rigid*.

108a. The "Imaginary System."—In conceiving the imaginary equivalent system in § 108, applied to the material points or particles (supposed destitute of mutual action, and not exposed to gravitation), which make up the rigid body, we employ the simplest system of forces that is capable, by the Mechanics of a Material Point, of producing the motion, which the particles actually have. If now the mutual actions, coherence, etc., were suddenly re-established, there would evidently be no change in the motion of the assemblage of particles; that is, in what is now a rigid body again, hence the imaginary system is equivalent to the actual system.

In applying this logic to the motion of translation of a rigid body (see § 109 and Fig. 122,) we reason as follows:

If the particles or elementary masses did not cohere together, being altogether without mutual action and not subjected to gravitation, their actual rectilinear motion in parallel lines, each having at a given instant the same velocity and also the *same acceleration*, p, as any other, could be maintained only by the application, to *each* particle, of a force having a value$=$*its mass* $\times p$, directed in the line of motion. In this way system (II.) is conceived to be formed and is evidently composed of parallel forces all pointing one way, whose resultant must be equal to their sum, viz. $\int dM \times p$. But since at this instant p is common to the motion of all the particles, this sum can be written $p\int dM$, $=$the whole mass $M \times p$.

If now the mutual coherence of contiguous particles were suddenly to be restored, system (II.) still acting, the motion of the assemblage of particles *would not be affected* (precisely as the falling motion *in vacuo* of two wooden blocks in contact is just the same whether they are glued together or not) and consequently we argue that the imaginary system (II.) is the equivalent of whatever system of forces the body is actually subjected to, viz. system (I.), (in which the body's own weight belongs) producing the actual motion.

Since the resultant of system (II.) is a single force, $=Mp$, parallel to the direction of the acceleration, and in a line passing through the center of gravity of the body, it follows that the resultant of the actual system is the same.

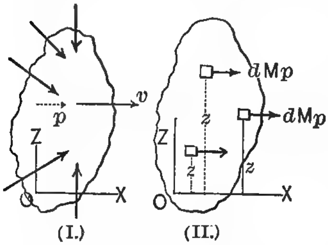

FIG. 122.

109. Translation.—Fig. 122. At a given instant all the particles have the same velocity $= v$, in parallel right lines (parallel to the axis X, say), and the same acceleration p. Required the ΣX of the acting forces, shown at (I.). (II.) shows the imaginary equivalent system, consisting of a force $=$ mass $\times$ acc. $= dMp$ applied parallel to X to each particle, since such a force would be necessary (from eq. (IV.) § 55) to account for the accelerated rectilinear motion of the particle, independently of the others. Putting $(\Sigma X)_I = (\Sigma X)_{II}$, we have

$$(\Sigma X)_I = \int p dM = p \int dM = Mp. \quad . \quad . \quad . \quad \text{(V.)}$$

It is evident that the resultant of system (II.) must be parallel to X; hence* that of (I.), which $= (\Sigma X)_I$ and may be denoted by R, must also be parallel to X; let $a =$ perpendicular distance from R to the plane YX; a will be parallel to Z. Now put $[\Sigma(\text{mom.})_Y]_I = [\Sigma(\text{mom.}_Y)]_{II}$, ($Y$ is an axis perpendicular to paper through O) and we have $-Ra = -\int dMpz$ $= -p\int dMz = -pM\bar{z}$ (§ 88), i.e., $a = \bar{z}$. A similar result may be proved as regards $\bar{y}$. Hence, *if a rigid body has a motion of translation, the resultant force must act in a line through the centre of gravity* (here more properly called the centre of mass), *and parallel to the direction of motion.* Or, practically, in dealing with a rigid body having a motion of translation, we may consider it concentrated at its centre of mass. If the velocity of translation is uniform, $R = M \times 0$ $= 0$, i.e., the forces are balanced.

109a. Example.—The symmetrical rigid body in Fig. 122*a* weighs ($G=$) 4 tons, and touches a smooth horizontal floor at the two points O and B, symmetrically situated. Its center of gravity, C, is 6 ft. above the floor; and it is required to find the effect of applying a horizontal force of $P=1$ ton, pointing to the right and 4 ft. below the level of the center of gravity C. Evidently a motion of translation will ensue from left to right, with some acceleration p, unless the body should begin to overturn about O or B as a pivot. The latter would be proved to be the case if either reaction, V_0 or V, of the floor against the body at O and B, is found to be negative as the result of an analysis which assumes translation to occur. The actual forces acting on the body are only four, viz.: G and P, and the unknown vertical reactions V and V_0. A special device (very convenient for the present case) will now be used as a means of solution. The resultant of the "equivalent system," II, in this case of translation (see Fig. 122), is $R, = Mp,$ lbs., acting through the center of gravity in a line parallel to that of the motion and in the direction of the acceleration, and hence is also the resultant of the actual system (just described). If, therefore, we annex to the actual system *its anti-resultant* (which is a force, R', of the same value, Mp, as R, and in same line, but pointing in the *opposite direction*) we thereby form a system under which the body would be in equilibrium; which would justify our writing $\Sigma X = 0$, $\Sigma Z = 0$, Σ(moms.) $= 0$, etc. (R' is called the "reversed inertia force" and is, of course, fictitious). With this system, then, in view, putting $\Sigma X = 0$ we obtain $P - R' = 0$; i.e., $R', = Mp, = 1$ ton; whence the acceleration $p = 1 \div (G \div g)$ $= 1 \div (4 \div 32.2) = 8.05$ ft./sec.^{2}

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* The forces of system (I.) *cannot form a couple;* since those of system (II.) do not reduce to a couple, *all pointing one way.*

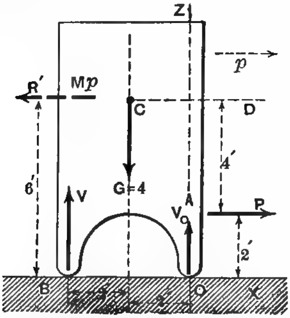

FIG. 122*a*.

By Σ(moms. about point A) $= 0$ we find $R' \times 4' + G \times 2' - V \times 4' = 0$ or $V = 3.0$ tons; and, by $\Sigma Z = 0$, $V + V_0 - G = 0$, or $V_0 = 4 - 3.0 = 1.0$ tons.

Since neither V nor V_0 is found to be negative the body does not tend to overturn but moves parallel to itself (i.e., translation) with a uniformly accelerated motion, the value of the acceleration being $p = 8.05$ ft./sec.^{2}; so that at the end of the first second the body would be 4.025 ft. from the start (no initial velocity); at the end of the second second, 16.1 ft.

If P were zero, or if P were applied horizontally through the center of gravity, V and V_0 would each be one half of G, i.e. 2 tons. It appears, therefore, that the effect of the eccentric application of P (viz. 4 ft. below the center of gravity C) is to increase V by 1.0 ton and diminish V_0 by an equal amount. If P acted 4 ft. *above* C, V and V_0 would change places in this respect. For V_0 to be just zero, P (in its present position) would need to have a value of 2 tons, and the body would be on the point of overturning toward the left. Or, again, with $P = 1$ ton, its line of application would have to be 8 ft. below or above C, for one of the reactions to be just zero. In fact, in the fictitious equilibrated system which includes R', since $P = R'$ (in this simple case) they form a couple; and hence the three forces G, V, and V_0 are equivalent to a couple of equal and opposite moment (viz. 4 ft.-tons in Fig. 122*a*).

From the above it is seen that in the case of the last car of a railroad train, when it has an accelerated motion (just leaving a station), the pressures under the front and rear trucks will be slightly different from their values when the motion is uniform or zero, if the pull in the coupling does not pass through the center of gravity of the car.

✓

110. Rotation about a Fixed Axis.—First, as to the elements of space and time involved. Fig. 123. Let O be the axis of rotation (perpendicular to paper), OY a fixed line of reference, and OA a convenient line of the rotating body, passing through the axis and perpendicular to it, accompanying the body in its angular motion, which is the same as that of OA. Just as in linear motion we dealt with linear space (s), linear velocity (v), and linear acceleration (p), so here we distinguish at any instant;

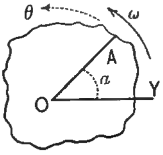

FIG. 123.

α, the *angular space* between OY and OA, (radians; or degrees, or revolutions);

$\omega = \frac{d\alpha}{dt}$, the *angular velocity*, or rate at which α is changing, (such as radians per sec., or revolutions per minute, etc.); and

$\theta = \frac{d\omega}{dt} = \frac{d^2\alpha}{dt^2}$, the *angular acceleration*, or rate at which ω is changing (radians per sec. per sec., e.g.)

These are all in angular measure and may be + or −, according to their direction against or with the hands of a watch. $d\alpha$ is a small increment of α, while $d^2\alpha$ is the difference between two $d\alpha$'s described in two consecutive small and *equal* time-intervals, each $= dt$.

(Let the student interpret the following cases: (1) at a certain instant ω is +, and θ −; (2) ω is −, and θ +; (3) α is −, ω and θ both +; (4) α +, ω and θ both −.) ? For rotary motion we have therefore, *in general*,

$$\omega = \frac{d\alpha}{dt}; \quad . \quad . \quad . \quad . \quad . \quad \text{(VI.)} \qquad \theta = \frac{d\omega}{dt} = \frac{d^2\alpha}{dt^2}; \quad . \quad . \quad \text{(VII.)}$$

$$\text{and} \quad \therefore \text{(by elimination)} \quad \omega d\omega = \theta d\alpha; \quad . \quad . \quad . \quad . \quad . \quad \text{(VIII.)}$$

corresponding to eqs. (I.), (II.), and (III.) in § 50, for rectilinear motion.

Hence, for *uniform rotary motion*, ω being constant and $\theta = 0$, we have $\alpha = \omega t$, t being reckoned from the instant when $\alpha = 0$.

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* See pp. 132, 133, of the "*Notes*," etc, for further illustration.

For *uniformly accelerated rotary motion* θ is constant, and if ω_0 denote the *initial* angular velocity (when α and $t=0$) we may derive, as in § 56, denoting the constant θ by θ_1,

$$\omega = \omega_0 + \theta_1 t; \quad . \ . \ (1) \qquad \alpha = \omega_0 t + \tfrac{1}{2}\theta_1 t^2; \ . \ . \ (2)$$

$$\alpha = \frac{\omega^2 - \omega_0^2}{2\theta_1}; \quad . \ . \ (3) \quad \text{and} \quad \alpha = \tfrac{1}{2}(\omega_0 + \omega)t. \ . \ . \ (4)$$

If in any problem in rotary motion θ, ω, and α have been determined for any instant, the corresponding *linear* values for any point of the body whose radial distance from the axis is ρ, will be $s = \alpha\rho$ (= distance described by the point measured along its circular path from its initial position), $v = \omega\rho$ = its velocity, and $p_t = \theta\rho$ its tangential acceleration, at the instant in question, if α, ω and θ, *are expressed in radians.*

Example.—(1) What value of ω, the angular velocity, is implied in the statement that a pulley is revolving at the rate of 100 revolutions per minute if the radian is unit angle?

100 revolutions per minute is at the rate of $2\pi \times 100 = 628.32$ radian units of angular space per minute $= 10.472$ per second. $\therefore$ $\omega = 628.32$ radians per minute or 10.472 radians per second.

(2) A grindstone whose initial speed of rotation is 90 revolutions per minute is brought to rest in 30 seconds, the angular retardation (or negative angular acceleration) being constant; required the angular acceleration, θ_1, and the angular space α described. Use the second and radian as units.

$\omega_0 = 2\pi\frac{90}{60} = 9.4248$ radians per second; $\therefore$ from eq. (1)

$$\theta_1 = \frac{\omega - \omega_0}{t} = -9.424 \div 30 = -0.3141 \text{ radians per sec. per sec.}$$

The angular space, from eq. (2) is

$$\alpha = \omega_0 t + \tfrac{1}{2}\theta_1 t^2 = 30 \times 9.42 - \tfrac{1}{2}(0.314)900 = 141.3$$

radians; that is, the stone has made 22.4 revolutions in coming to rest and a point 2 ft. from the axis has described a distance $s = \alpha\rho = 141.3 \times 2 = 282.6$ ft. in its circular path.

111. Rotation. Preliminary Problems. Axis Fixed.—For clearness in subsequent matter we now consider the following problem. Fig. 124 shows a rigid homogeneous right cylinder A of weight $G=200$ lbs. and radius $r=2$ ft., mounted on a horizontal axle and concentric with the same. The center of gravity of the cylinder is in the axis of rotation (Z). The axle carries a light and concentric drum, of 10 in. radius, from which a light *inextensible* cord may unwind as the attached weight B descends, thus imparting an accelerated rotary motion to the cylinder. The weights and masses of the drum, cord, and axle, and all friction, will be neglected; and the two journals will be considered as one. The cylinder being originally at rest we wish to determine its motion as produced by a constant downward pull or tension of 80 lbs. in the vertical cord. (The necessary weight, G', of the body to be used at B*, to secure this 80 lbs. tension in the cord, will be found later.) During this motion the real system of forces (system (I)) acting on the body A consists of the weight 200 lbs., always acting through Z, the *fixed* axis of rotation; the downward pull of 80 lbs. at 10 in. from the axis; the vertical component V of the reaction of the bearing; and the horizontal component (if any), H. At (II), Fig. 124, is shown an imaginary *equivalent system* capable of producing the same motion in the particles, each of mass $=dM$, if they were independent. Since each particle is moving in a circle of some radius ρ with some linear (tangential) acceleration p_t at any instant, the cylinder having at that same instant some angular velocity ω and some angular acceleration θ, we have $v=\omega\rho$ and $p_t=\theta\rho$. (ω and θ in *radians*.)

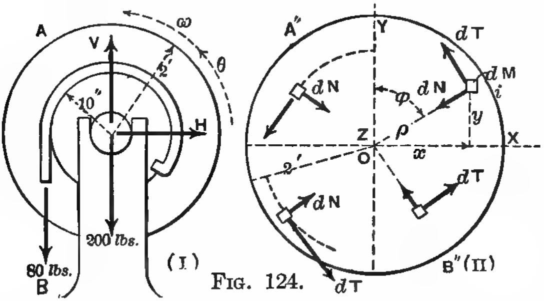

FIG. 124.

This circular motion of each particle could be produced (see eq. (5), p. 76) by a tangential force dT lbs., $=dMp_t$, $=\theta dM\rho$, accompanied by a normal force dN lbs., $=dMv^2\div\rho$, $=\omega^2 dM\rho$. Our equivalent system, then, in (II), consists of a dT and a dN of proper value applied to each particle of body A at a given instant. Axes X and Y are shown in Fig. 124, axis Z, the axis of rotation, being $\perp$ to the paper through origin O. Let us now, for any instant of the motion, equate Σ (moms.)$_z$ of the actual system, (I), to Σ (moms.)$_z$ in system (II); using the integral sign to denote a summation which extends over all the particles of body A (for this instant; the integral might therefore be called an *instantaneous* integral). This gives, if we note that each of the normal forces dN of system (II) has no moment about axis Z, and that θ is common to all the particles at this instant, (with ft.-lb.-sec. units),

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* The body B is not shown in the figure.

$$+80\times 10/12 = +\int dT\,.\,\rho = +\theta\int dM\rho^2 = +\theta I_z. \qquad (1)$$

The summation (instantaneous) $\int dM\rho^2$ is seen to be the quantity called "*moment of inertia*," about axis Z, of the body A and remains constant, since the ρ's do not change in value as the motion proceeds. For a solid homogeneous cylinder $I_z=\frac{1}{2}Mr^2$ (p. 99), and hence

$$800=6\theta[200\div 32.2](2)^2;\quad \text{i.e.,}\quad \theta=5.37 \text{ rads./sec.}^2$$

That is, θ is *constant* and the rotary motion of the cylinder is *uniformly accelerated.*

(**N. B.**—From eq. (1) we note that, in general, in order to obtain the angular acceleration, θ, of the rotary motion of a rigid body about a fixed axis Z we have only to treat the body as a "free body" and write Σ (moms.) *about axis of rotation = angul. accel.* × *mom. of inertia about Z.*)

112. Further Results in Preceding Problem.—As to the *necessary weight, G', of body B* (suspended on the cord and causing the motion of both bodies), in order to produce the 80 lbs. tension in the cord, we note that body B has the same motion (only in a right line), as a point in the circumference of the drum, where the acceleration is $p'=\theta\times\frac{10}{12}$ $=4.48$ ft./sec.^{2} That is, the motion of B will be *uniformly accelerated,* with an acceleration of 4.48 ft./sec.^{2} Hence the weight of B must not only produce the 80 lbs. tension in the cord but also accelerate the mass of B, $(M'=G'\div g)$ with an acceleration of 4.48 ft./sec.^{2} I.e., we have $G'=80+(G'\div g)p'$; which is nothing more than saying that the net accelerating force, $G'-80$, = mass × accel.; whence we find, on solving, $G'=92.9$ lbs. for the weight of the body to be used at B.

For example, in the first 3 sec. of time, starting from rest, B will descend a distance (see p. 54), $s_3=\frac{1}{2}p'(3)^2=20.16$ ft. and will have acquired a (linear) velocity of $v_3=p'\times 3=13.44$ ft./sec.; while body A will have turned through an angle of $\alpha_3=\frac{1}{2}\theta(3)^2=24.2$ radians, (or 3.85 revolutions) and will possess an angular velocity of $\omega_3=\theta\times 3=16.1$ rads./sec. or, $(16.1\div 2\pi=)$, 2.57 revs./sec.

Reaction of the bearing; (two journals considered as one). To find the two components H and V of this reaction, we again have recourse to the two equivalent systems of Fig. 124, acting on body A. (N.B.—The *upward* 80 lbs. and the force G' do not belong to system (I), since they act on body B.) During the motion, the coordinates x and y of each particle (of mass $=dM$) are continually changing, as also the angle ϕ

between the ρ of the particle and axis Y (but not ρ itself). At any given instant we note that $x=\rho\sin\phi$ and $y=\rho\cos\phi$, for each particle. Let us now put ΣY of system (I) equal to ΣY of system (II). This gives us

$$V-200-80=\int dT\sin\phi-\int dN\cos\phi \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As before, these integrals are "*instantaneous integrals,*" being extended over all the particles of the body at a *given instant* of time, [so that in general the value of each may change with the progress of the motion.

Substituting for dT and dN, etc., this may be written

$$V-200-80=\theta\int dM\rho\sin\phi-\omega^2\int dM\rho\cos\phi \quad . \quad . \quad . \quad (3)$$

or,
$$V-200-80=\theta\int dMx-\omega^2\int dMy, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Note that the value of θ, and also of ω, at this single instant are common to all the particles and have been factored out, as shown.

But the summation of $\int dMx$ is nothing more than $M\bar{x}$; where M is the mass of the whole cylinder A $(=200\div g)$ [see p. 18, eq. (1)], and $\bar{x}$ is the x coordinate of its center of gravity; and, similarly, $\int dMy=M\bar{y}$. We may therefore write

$$V-200-80=\theta M\bar{x}-\omega^2 M\bar{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But in the present case, since the center of gravity of body A is in the axis of rotation at all times, we have both $\bar{x}$ and $\bar{y}=$ zero at all times; and hence finally $V-200-80=0$; or $V=280$ lbs.

As to the horizontal component, H, of the bearing reaction, we place ΣX of system (I) equal to ΣX of system (II) and obtain

$$H=-\int dT\cos\phi-\int dN\sin\phi=-\theta\int dM\rho\cos\phi-\omega^2\int dM\rho\sin\phi, \quad (6)$$

i.e.,
$$H=-\theta\int dMy-\omega^2\int dMx,=-\theta M\bar{y}-\omega^2 M\bar{x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

But since $\bar{x}$ and $\bar{y}$ are zero at all times, H must be zero, from (7); and we therefore conclude that in this case the reaction of the bearing is purely vertical at all times and is V, $=280$ lbs.

113. Centre of Percussion of a Rod suspended from one End.—Fig. 126. The rod is initially at rest (see (I.) in figure), is straight, homogeneous, and of constant (small) cross-section. Neglect its weight. A horizontal force or pressure, P, due to a blow (and varying in amount during the blow), now acts upon it from the left, perpendicularly to the axis, Z, of suspension. An accelerated rotary motion begins about the fixed axis Z. (II.) shows the rod *free*, at a certain instant, with the reactions X_0 and Y_0 put in at O_0. (III.) shows an imaginary system which would produce the same effect at this instant, and consisting of a $dT=dM\theta\rho$, and a $dN=\omega^2 dM\rho$ applied to each dM, the rod being composed of an infinite number of dM's, each at some distance ρ from the axis. Considering that *the rotation has just begun*, ω, the

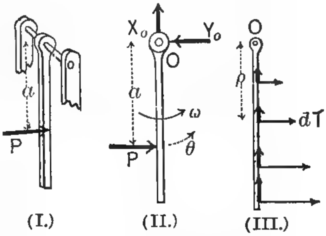

Fig. 126.

angular velocity is as yet small, and will be neglected. Required Y_0 the horizontal reaction of the support at O in terms of P. By putting $\Sigma Y_{II} = \Sigma Y_{III}$, we have

$$P - Y_0 = \int dT = \theta \int \rho dM = \theta M\bar{\rho}.$$

$\therefore\ \boldsymbol{Y_0 = P - \theta M\bar{\rho}}$; $\bar{\rho}$ is the distance of the centre of **gravity** from the axis (N.B. $\int \rho dM = M\bar{\rho}$ is only true when all the ρ's are parallel to each other). But the value of the angular acceleration θ at this instant depends on P and a, for Σ (mom.)$_Z$ in (II.) = Σ (mom.)$_Z$ in (III.), whence $Pa = \theta \int \rho^2 dM = \theta I_Z$, where I_Z is the *moment of inertia* of the rod about Z, and from § 95 $= \frac{1}{3}Ml^2$. Now $\bar{\rho} = \frac{1}{2}l$; hence, finally,

$$Y_0 = P\left[1 - \frac{3}{2} \cdot \frac{a}{l}\right].$$

If now Y_0 is to $= 0$, i.e., if there is to be *no shock between the rod and axis*, we need only apply P at a point whose distance $a = \frac{2}{3}l$ from the axis; for then $Y_0 = 0$. This point is called the **centre of percussion** for the given rod and axis. It and the point of suspension O are interchangeable (see § 118). (Lay a pencil on a table; tap it at a point distant one third of the length from one end; it will *begin to rotate* about a vertical axis through the farther end. Tap it at one end; it will begin to rotate about a vertical axis through the point first mentioned. Such an axis of rotation is called an *axis of instantaneous rotation*, and is different for each point of impact—just as the point of contact of a wheel and rail is the one point of the wheel which is momentarily at rest, and about which, therefore, all the others are turning *for the instant.* Tap the pencil at its centre of gravity, and a motion of translation begins; see § 109.)

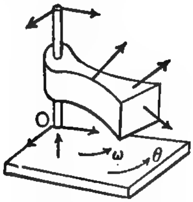

FIG. 127.

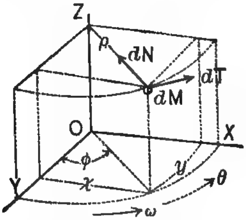

FIG. 128.

114. Rotation. Axis Fixed. General Formulæ.—Considering now a rigid body of any shape whatever, let Fig. 127 indicate the system of forces acting *at any given instant*, Z being the fixed axis of rotation, ω and θ the angular velocity and angular acceleration, *at the given instant.* X and Y are two axes, at right angles to each other and to Z, fixed in space. At this instant each dM of the body has a definite x, y, and φ (see Fig. 128), which will change, and also a ρ, and z, which will not change, as the motion progresses, and is pursuing a circular path with a velocity $= \omega\rho$ and a tangential acceleration $= \theta\rho$. Hence, if to each dM of the body (see Fig. 128) we imagine a tangential force $dT = dM\theta\rho$ and a normal force $= dM(\omega\rho)^2 \div \rho = \omega^2 dM\rho$ to be applied (eq. (5), § 74), and these alone, we have a system comprising an infinite number of forces, all parallel to XY, and equivalent to the actual system in Fig. 127. Let ΣX, etc., represent the sums (six) for Fig. 127, whatever they may be in any particular case, while for 128 we shall write the corresponding sums in detail. Noting that

$$\int dN \cos\varphi = \omega^2\int dM\rho\cos\varphi = \omega^2\int dMy = \omega^2 M\bar{y}, (\S\, 88);$$

that $\int dN \sin\varphi = \omega^2\int dM\rho\sin\varphi = \omega^2\int dMx = \omega^2 M\bar{x}$;

and similarly, that $\int dT \cos\varphi = \theta\int dM\rho\cos\varphi = \theta M\bar{y}$, and $\int dT\sin\varphi = \theta M\bar{x}$; while in the moment sums (the moment of $dT\cos\varphi$ about Y, for example, being $-\, dT\cos\varphi \,.\, z = -\,\theta dM\rho\,(\cos\varphi)z = -\theta dMyz$, the sum of the moms.$_Y$ of all the $(dT\cos\varphi)$'s $= -\,\theta\int dMyz$)

$$\int dT\cos\varphi z = \theta\int dMyz, \int dN\sin\varphi z = \omega^2\int dMxz, \text{ etc.,}$$

we have, since the systems are equivalent,

$$\Sigma X = +\,\theta M\bar{y} - \omega^2 M\bar{x}; \quad . \quad . \quad . \quad . \quad (\text{IX.})$$

$$\Sigma Y = -\,\theta M\bar{x} - \omega^2 M\bar{y}; \quad . \quad . \quad . \quad . \quad (\text{X.})$$

$$\Sigma Z = \quad 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XI.})$$

$$\Sigma \text{ moms.}_X = -\,\theta\int dMxz - \omega^2\int dMyz; \quad . \quad (\text{XII.})$$

$$\Sigma \text{ moms.}_Y = -\,\theta\int dMyz + \omega^2\int dMxz; \quad . \quad (\text{XIII.})$$

$$\Sigma \text{ moms.}_Z = \quad \theta\int dM\rho^2 = \theta I_Z. \quad . \quad . \quad . \quad (\text{XIV.})$$

These hold good for any instant. As the motion proceeds $\bar{x}$ and $\bar{y}$ change, as also the sums $\int dMxz$ and $\int dMyz$. If the body, however, is homogeneous, and *symmetrical about the plane* XY, $\int dMxz$ and $\int dMyz$ would always $=$ zero; since the z of any dM does not change, and for every term $dMy(+z)$, there would be a term $dMy(-z)$ to cancel it; similarly for $\int dMxz$. The eq. (XIV.), Σ (moms. about axis of rotat.) $= \int dT\rho = \theta \int dM\rho^2 =$ (*angular accel.*) $\times$ (*mom. of inertia of body about axis of rotat.*), shows how the sum $\int dM\rho^2$ arises in problems of this chapter. That a force $dT = dM\theta\rho$ should be necessary to account for the acceleration (tangential) $\theta\rho$ of the mass dM, is due to the so-called *inertia* of the mass (§ 54), and its moment $dT\rho$, or $\theta dM\rho^2$, might, with some reason, be called the *moment of inertia* of the dM, and $\int \theta dM\rho^2 = \theta \int dM\rho^2$ that of the whole body. But custom has restricted the name to the sum $\int dM\rho^2$, which, being without the θ, has no term to suggest the idea of inertia. For want of a better the name is still retained, and is generally denoted by I. (See §§ 86, etc.)

115. Example of the Preceding.—A homogeneous right parallelopiped is mounted on a vertical axle (no friction), as in figure. O is at its centre of gravity, *hence both* $\bar{x}$ *and* $\bar{y}$ *are zero.* Let its heaviness be γ, its dimensions h, $b_{\prime\prime}$ and b (see § 97). XY is a plane of symmetry, hence both $\int dMxz$ and $\int dMyz$ are zero at all times (see above). The tension P in the (inextensible) cord is caused by the hanging weight P_1 (but is not $= P_1$, unless the rotation is uniform). The figure shows both rigid bodies *free.* P_1 will have a motion of translation; the parallelopiped, one of rotation about a fixed axis. No masses are considered except $P_1 \div g$, and $bhb_1\gamma \div g$. The $I_Z = Mk_Z^2$ of the latter $=$ its mass $\times \frac{1}{12}(b_1^2 + b^2)$, § 97. At any instant, the cord being taut, if $p =$ linear acceleration of P_1, we have

$$p = \theta a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{eq. } (a)$$

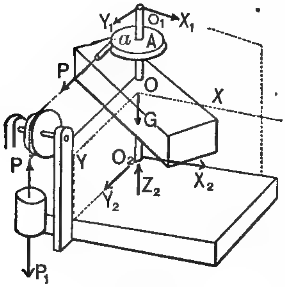

FIG. 129.

From (XIV.), $\quad Pa = \theta I_Z; \quad \therefore P = \theta I_Z \div a. \quad . \quad . \quad . \quad (1)$

For the free mass $P_1 \div g$ we have (§ 109) $P_1 - P =$ mass $\times$ acc.,

$$= (P_1 \div g)p = (P_1 \div g)\theta a\,; \quad \therefore P = P_1(1 - \theta a \div g). \quad (2)$$

Equate these two values of P and solve for θ, whence

$$\theta = \frac{P_1 a}{Mk_z^2 + (P_1 \div g)a^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

All the terms here are *constant*, hence θ is constant; therefore the rotary motion is *uniformly accelerated*, as also the translation of P_1. The formulæ of § 56, and (1), (2), (3), and (4) of § 110, are applicable. The tension P is also constant; see eq. (1). As for the five unknown reactions (components) at O_1 and O_2, the bearings, we shall find that they too are constant; for

from (IX.) we have $X_1 + X_2 = 0$; (4)
from (X.) we have $P + Y_1 + Y_2 = 0$; (5)
from (XI.) we have $Z_2 - G = 0$; (6)
from (XII.) we have $P . \overline{AO} + Y_1 . \overline{O_1O} - Y_2 . \overline{O_2O} = 0$; (7)
from (XIII.) we have $-X_1 . O_1O + X_2 . \overline{O_2O} = 0$. (8)

Numerical substitution in the above problem.—Let the parallelopiped be of wrought-iron; let $P_1 = 48$ lbs.; $a = 6$ in. $= \frac{1}{2}$ ft.; $b = 3$ in. $= \frac{1}{4}$ ft. (see Fig. 112); $b_1 = 2$ ft. 3 in. $= \frac{9}{4}$ ft.; and $h = 4$ in. $= \frac{1}{3}$ ft. Also let $\overline{O_1O} = \overline{O_2O} = 18$ in. $= \frac{3}{2}$ ft., and $\overline{AO} = 3$ in. $= \frac{1}{4}$ ft. Selecting the *foot-pound second* system of units, in which $g = 32.2$, the linear dimensions must be used in feet, the heaviness, γ, of the iron must be used in *lbs. per cubic foot*, i.e., $\gamma = 480$ (see § 7), and all forces in lbs., times in seconds.

The weight of the iron will be $G = V\gamma = bb_1h\gamma = \frac{1}{4} . \frac{9}{4} . \frac{1}{3} \times 480 = 90$ lbs.; its mass $= 90 \div 32.2 = 2.79$; and its moment of inertia about $Z = I_z = Mk_z^2 = M\frac{1}{12}(b_1^2 + b^2) = 2.79 \times 0.426 = 1.191$. (That is, the *radius of gyration*, k_z, $= \sqrt{0.426} = 0.653$ ft.; or the moment of inertia, or any result depending solely upon it, is just the same as if the mass were concentrated in a thin shell, or a line, or a point, at a distance of 0.653 feet from the axis.) We can now compute the angular acceleration, θ, from eq. (3);

$$\theta = \frac{48 \times \frac{1}{2}}{1.191 + (48 \div 32.2) \times \frac{1}{4}} = \frac{24}{1.191 + 0.372} = 15.36$$

radians per sec. per sec. The linear acceleration of P_1 is $p = \theta a = 7.68$ feet per sec. per sec. for the uniformly accelerated translation.

Nothing has yet been said of the velocities and initial conditions of the motions; for what we have derived so far applies to any point of time. Suppose, then, that the angular velocity $\omega =$ zero when the time, $t = 0$; and correspondingly the velocity, $v = \omega a$, of translation of P_1, be also $= 0$ when $t = 0$. At the end of any time t, $\omega = \theta t$ (§§ 56 and 110) and $v = pt = \theta at$; also the angular space, $\alpha = \frac{1}{2}\theta t^2$, described by the parallelopiped during the time t, and the linear space $s = \frac{1}{2}pt^2 = \frac{1}{2}\theta at^2$, through which the weight P_1 has sunk vertically. For example, during the first second the parallelopiped has rotated through an angle $\alpha = \frac{1}{2}\theta t^2 = \frac{1}{2} \times 15.36 \times 1 = 7.68$ radians, i.e., $(7.68 \div 2\pi) = 1.22$ revolutions, while P_1 has sunk through $s = \frac{1}{2}\theta at^2 = 3.84$ ft., vertically.

The tension in the cord, from (2), is

$$P = 48(1 - 15.36 \times \tfrac{1}{2} \div g) = 48(1 - 0.24) = 36.48 \text{ lbs.}$$

The pressures at the bearings will be as follows, *at any instant:* from (4) and (8), X_1 and X_2 must individually be zero; from (6) $Z_2 = G = V\gamma = 90$ lbs.; while from (5) and (7), $Y_1 = -21.28$ lbs., and $Y_2 = -15.20$ lbs., and should point in a direction opposite to that in which they were assumed in Fig. 129 (see last lines of § 39).

117. The Compound Pendulum is any rigid body allowed to oscillate without friction under the action of gravity when mounted on a horizontal axis. Fig. 131 shows the body *free*, in any position during the progress of the oscillation. C is the centre of gravity; let $\overline{OC} = s$. From (XIV.), § 114, we have Σ (mom. about fixed axis)

$$= \text{angul. acc.} \times \text{mom. of inertia.}$$

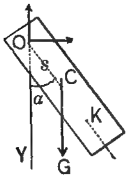

FIG. 131.

$$\therefore -Gs\sin\alpha = \theta I_0,$$

$$\text{and } \theta = -Gs\sin\alpha \div I_0 = -Mgs\sin\alpha \div Mk_0^2,$$

$$\text{i.e., } \theta = -gs\sin\alpha \div k_0^2 . \quad . \quad . \quad . \quad . \quad (1)$$

Hence θ is variable, proportional to sin α. Let us see what the length $l = \overline{OK}$, of a simple circular pendulum, must be, to have at this instant (i.e., for this value of α) the same angular acceleration as the rigid body. The linear (tangential) accelerations of K, the extremity of the required simple pendulum would be (§ 77) $p_t = -g\sin\alpha$, and hence its angular acceleration* would $= -g\sin\alpha \div l$. Writing this equal to θ in eq. (1), we obtain

$$l = k_0^2 \div s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But this is *independent of* α; therefore the length of the simple pendulum having an angular acceleration equal to that of the oscillating body is *the same in all positions of the latter*, and if the two begin to oscillate simultaneously from a position of rest at any given angle α_1 with the vertical, they will keep abreast of each other during the whole motion, and hence have the same duration of oscillation; which is ∴, *for small amplitudes* (§ 78),

---

* Most easily obtained by considering that if the body shrinks into a mere point at K, and thus becomes a simple pendulum, we have both k_0 and s equal to l; which in (1) gives $\theta = -g\sin\alpha \div l$.

$$t' = \pi\sqrt{l \div g} = \pi\sqrt{k_0^2 \div gs}, \quad . \quad . \quad . \quad . \quad (3)$$

K is called the *centre of oscillation* corresponding to the given *centre of suspension* O, and is identical with the *centre of percussion* (§ 113).

Example.—Required the time of oscillation of a cast-iron cylinder, whose diameter is 2 in. and length 10 in., if the axis of suspension is taken 4 in. above its centre. If we use 32.2 for g, all linear dimensions should be in feet and times in seconds. From § 100, we have

$$I_C = M(\tfrac{1}{4}r^2 + \tfrac{1}{12}h^2) = M(\tfrac{1}{4}\cdot\tfrac{1}{144} + \tfrac{1}{12}\cdot\tfrac{100}{144}) = M\tfrac{1}{144}\cdot\tfrac{103}{12}.$$

From eq. (3), § 88,

$$I_0 = I_C + Ms^2 = M[\tfrac{1}{144}\cdot\tfrac{103}{12} + \tfrac{1}{9}] = M \times 0.170;$$

$$\therefore k_0^2 = 0.170 \text{ sq. ft.}; \ \therefore t' = \pi\sqrt{0.170 \div (32.2 \times \tfrac{1}{3})} = 0.395 \text{ sec.}$$

118. The Centres of Oscillation and Suspension are Interchangeable.—(Strictly speaking, these centres are points in the line through the centre of gravity perpendicular to the axis of suspension.) Refer the centre of oscillation K to the centre of gravity, thus (Fig. 132, at (I.)):

$$s_1 = l - s = \frac{Mk_0^2}{Ms} - s = \frac{Mk_C^2 + Ms^2}{Ms} - s = \frac{k_C^2}{s}. \quad (1)$$

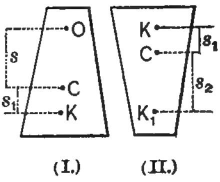

FIG. 132.

Now invert the body and suspend it at K; required CK_1, or s_2, to find the centre of oscillation corresponding to K as centre of suspension. By analogy from (1) we have $s_2 = k_C^2 \div s_1$; but from (1), $k_C^2 \div s_1 = s$ ∴ $s_2 = s$; in other words, K_1 is identical with O. Hence the proposition is proved.

Advantage may be taken of this to determine the length L of the theoretical simple pendulum vibrating seconds, and thus finally the acceleration of gravity from formula (3), § 117, viz., when $t' = 1.0$ and l (now $= L$) has been determined experimentally, we have

$$g \text{ (in ft. per sq. second)} = L \text{ (in ft.)} \times \pi^2. \quad . \quad . \quad (2)$$

This most accurate method of determining g at any locality requires the use of a bar of metal, furnished with a sliding weight for shifting the centre of gravity, and with two projecting blocks provided with knife-edges. These blocks can also be shifted and clamped. By suspending the bar by one knife-edge on a proper support, the duration of an oscillation is computed by counting the total number in as long a period of time as possible; it is then reversed and suspended on the other with like observations. By shifting the blocks between successive experiments, the duration of the oscillation in one position is made the same as in the other, i.e., the distance between the knife-edges is the length, l, of the simple pendulum vibrating in the computed time (if the knife-edges are not equidistant from the centre of gravity), and is carefully measured. The l and t' of eq. (3), § 117, being thus known, g may be computed. The length, in *feet*, of the simple pendulum vibrating seconds, at any latitude β, and at a height of h *ft.* above sea-level, is (Chwolson, 1902).

$$L = 3.25974 - 0.008441 \cos 2\beta - 0.0000003h.$$

119. Isochronal Axes of Suspension.—*In any compound pendulum, for any axis of suspension, there are always three others, parallel to it in the same gravity-plane, for which the oscillations are made in the same time as for the first.* For any assigned time of oscillation t', eq. (3), § 117, compute the corresponding distance $\overline{CO} = s$ of O from C;

i.e., from $$t'^2 = \pi^2 \frac{Mk_0^2}{Mgs} = \frac{\pi^2(Mk_C^2 + Ms^2}{Mgs},$$

we have $$s = (gt'^2 \div 2\pi^2) \pm \sqrt{(g^2t'^4 \div 4\pi^4) - k_C^2}. \quad . \quad . \quad (1)$$

Hence for a given t', there are two positions for the axis O parallel to any axis through C, in any gravity-plane, on both sides; i.e., *four parallel axes of suspension*, in any gravity-plane, giving equal times of vibration; for two of these axes we must reverse the body. E.g., if a slender, homogeneous, prismatic rod be marked off into thirds, the (small) vibrations will be of the same duration, if the centre of suspension is taken at either extremity, or at either point of division.

Example.—Required the positions of the axes of suspension, parallel to the base, of a right cone of brass, whose altitude is six inches, radius of base, 1.20 inches, and weight per cubic inch is 0.304 lbs., so that the time of oscillation may be a half-second. (N.B. For variety, use the inch-pound-second system of units, first consulting § 51.)

120. The Fly-Wheel in Fig. 133 at any instant experiences a pressure P' against its crank-pin from the connecting-rod and a resisting pressure P'' from the teeth of a spur-wheel with which it gears.* Its weight G acts through C (nearly), and there are pressures at the bearings, but these latter and G have no moments about the axis C (perpendicular to paper). The figure shows it *free*, P'' being assumed constant (in practice this depends on the resistances met by the machines which D drives, and the fluctuation of velocity of their moving parts). P', and therefore T its tangential component, are variable, depending on the effective steam-pressure on the piston at any instant, on the obliquity of the connecting-rod, and in high-speed engines on the masses and motions of the piston and connecting-rod. Let r = radius of crank-pin circle, and a the perpendicular from C on P''. From eq. (XIV.), § 114, we have

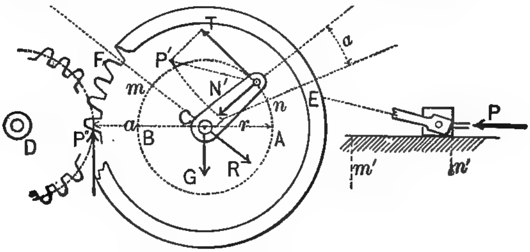

FIG. 133.

$$Tr - P''a = \theta I_C, \quad \therefore \theta = (Tr - P''a) \div I_C, \quad . \quad (1$$

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*Bearings at C not shown. P is the thrust in the piston-rod due to steam pressure on piston.

as the angular acceleration at any instant; substituting which in the general equation (VIII.), § 110, we obtain

$$I_C\omega d\omega = Trd\alpha - P''ad\alpha. \quad . \quad . \quad . \quad . \quad (2)$$

From (1) it is evident that if at any position of the crank-pin the variable Tr is equal to the constant $P''a$, θ is zero, and consequently the angular velocity ω is either a maximum or a minimum. Suppose this is known to be the case both at m and n; i.e., suppose T, which was zero at the dead-point A, has been gradually increasing, till at n, $Tr = P''a$; and thereafter increases still further, then begins to diminish, until at m Tr again $= P''a$, and continues to diminish toward the dead-point B. The angular velocity ω, whatever it may have been on passing the dead-point A, diminishes, since θ is negative, from A to n, where it is ω_n, a minimum; increases from n to m, where it reaches a maximum value, ω_m. n and m being known points, and supposing ω_n known, let us inquire what ω_m will be. From eq. (2) we have

$$I_C\int_{\omega_n}^{\omega_m}\omega d\omega = \int_n^m Trd\alpha - P''\int_n^m ad\alpha. \quad . \quad . \quad (3)$$

But $rd\alpha = ds =$ an element of the path of the crank-pin, and also the "virtual velocity" of the force T, and $ad\alpha = ds''$, an element of the path of a point in the pitch-circle of the fly-wheel, the small space through which P'' is overcome in dt. Hence (3) becomes

$$I_C\tfrac{1}{2}(\omega_m^2 - \omega_n^2) = \int_n^m Tds - P'' \times \text{linear arc } \overline{EF}. \quad (4)$$

To determine $\int_n^m Tds$ we might, by a knowledge of the varying steam-pressure, the varying obliquity of the connecting-rod, etc., determine T for a number of points equally spaced along the curve nm, and obtain an approximate value of this sum by Simpson's Rule; but a simpler method is possible by noting (see eq. (1), § 65) that each term Tds of this sum = the corresponding term Pdx in the series $\int_{n'}^{m'} Pdx$, in which $P =$ the effective steam-pressure on the piston in the cylinder at any instant, dx the small distance described by the piston while the crank-pin describes any ds, and n' and m' the positions of the piston (or of cross-head, as in Fig. 133) when the crank-pin is at n and m respectively. (4) may now be written

$$I_C\tfrac{1}{2}(\omega_m^2 - \omega_n^2) = \int_{n'}^{m'} Pdx - P'' \times \text{linear arc } \overline{EF}, \quad (5)$$

from which ω_m may be found as proposed. More generally, it is available, alone (or with other equations), to determine any one (or more, according to the number of equations) unknown quantity. This problem, in rotary motion, is analogous to that in § 59 (Prob. 4) for rectilinear motion. Friction and the inertia of piston and connecting-rod have been neglected. As to the time of describing the arc nm, from equations similar to (5), we may determine values of ω for points along nm, dividing it into an even number of equal parts, calling them ω_1, ω_2, etc., and then employ Simpson's Rule* for an approximate value of the sum $\Big[_n^m t = \int_n^m \frac{d\alpha}{\omega}$ (from eq. (VI.), § 110); e.g., with four parts, we would have

$$\Big[_n^m t = \frac{1}{12}(\text{angle } nCm, \text{in rads.})\left[\frac{1}{\omega_n} + \frac{4}{\omega_1} + \frac{2}{\omega_2} + \frac{4}{\omega_3} + \frac{1}{\omega_m}\right]. (6)$$

121. Numerical Example. Fly-Wheel.—(See Fig. 133 and the equations of § 120.) Suppose the engine is non-condensing and non-expansive (i.e., that P is constant), and that

$$P = 5500 \text{ lbs.}, \quad r = 6 \text{ in.} = \tfrac{1}{2} \text{ ft.}, \quad a = 2 \text{ ft.},$$

and also that the wheel is to make 120 revolutions per minute, i.e., that its *mean angular velocity* is to be

$$\omega' = \tfrac{120}{60} \times 2\pi, \text{ i.e., } \omega' = 4\pi \text{ “ radians” per sec.}$$

First, required the amount of the resistance P'' (constant) that there shall be no permanent change of speed, i.e., that the angular velocity shall have the same value at the end of a complete revolution as at the beginning. Since an equation of the form of eq. (5) holds good for any range of the motion, let

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[*] See p. 13 of the *"Notes and Examples in Mechanics."*

that range be a complete revolution, and we shall have zero as the left-hand member; $\int P dx = P \times 2$ ft. $= 5500$ lbs. $\times 2$ ft., or 11,000 foot-pounds (as it may be called); while P'' is unknown, and instead of lin. arc $\overline{EF}$ we have a whole circumference of 2 ft. radius, i.e., 4π ft.;

$$\therefore 0 = 11{,}000 - P'' \times 4 \times 3.1416; \quad \text{whence} \quad P'' = 875 \text{ lbs.}$$

Secondly, required the proper mass to be given to the fly-wheel of 2 ft. radius that in the forward stroke (i.e., while the crank-pin is describing its *upper* semicircle) the max. angular velocity ω_m shall exceed the minimum ω_n by only $\frac{1}{10}\omega'$, assuming (which is nearly true) that $\frac{1}{2}(\omega_m + \omega_n) = \omega'$. There being now three unknowns, we require three equations, which are, including eq. (5) of § 120, viz.:

$$Mk_C^2\tfrac{1}{2}(\omega_m + \omega_n)(\omega_m - \omega_n)$$
$$= \int_{n'}^{m'} P dx - P'' \times \text{linear arc } \overline{EF}; \quad (5)$$

$$\tfrac{1}{2}(\omega_m + \omega_n) = \omega' = 4\pi; \quad (7) \quad \text{and} \quad \omega_m - \omega_n = \tfrac{1}{10}\omega' = \tfrac{2}{5}\pi. \quad (8)$$

The points n and m are found most easily and with sufficient accuracy by a graphic process.* Laying off the dimensions to scale, by trial such positions of the crank-pin are found that T, the tangential component of the thrust P' produced in the connecting-rod by the steam-pressure P (which may be resolved into two components, along the connecting-rod and a normal to itself) is $=(a \div r)P''$, i.e., is $= 3500$ lbs. These points will be n and m (and two others on the lower semicircle). The positions of the piston n' and m', corresponding to n and m of the crank-pin, are also found graphically in an obvious manner. We thus determine the angle nCm to be 100°, so that linear arc $EF = \frac{100}{180}\pi \times 2$ ft. $= \frac{10}{9}\pi$ ft., while

$$\int_{n'}^{m'} P dx = 5500 \text{ lbs.} \times \int_{n'}^{m'} dx = 5500 \times \overline{n'm'} = 5500 \times 0.77 \text{ ft.},$$

$n'm'$ being scaled from the draft.

Now substitute from (7) and (8) in (5), and we have, with $k_C = 2$ ft. (which assumes that the mass of the fly-wheel is concentrated in the rim),

---

* See p. 85, "*Notes and Examples,*" etc.

$$(G \div g) \times 4 \times 4\pi \times \tfrac{2}{5}\pi = 5500 \times 0.77 - 875 \times \tfrac{10}{9}\pi,$$

which being solved for G (with $g = 32.2$; since we have used the foot and second), gives $G = 600.7$ lbs.

The points of max. and min. angular velocity on the back-stroke may be found similarly, and their values for the fly-wheel as now determined; they will differ but slightly from the ω_m and ω_n of the forward stroke. Professor Cotterill says that the rim of a fly-wheel should never have a max. velocity $>$ 80 ft. per sec.; and that if made in segments, not more than 40 to 50 feet per second. In the present example we have for the forward stroke, from eqs. (7) and (8), $\omega_m = 13.2$ (π-measure units) per second; i.e., the corresponding velocity of the wheel-rim is $v_m = \omega_m a = 26.4$ feet per second.

122. Angular Velocity Constant. Fixed Axis.—If ω is constant, the angular acceleration, θ, must be = zero at all times, which requires Σ (mom.) about the axis of rotation to be = 0 (eq. (XIV.), § 114). An instance of this occurs when the only forces acting are the reactions at the bearings on the axis, and the body's weight, parallel to or intersecting the axis; the values of these reactions are now to be determined for different forms of bodies, in various positions relatively to the axis. (The opposites and equals of these reactions, i.e., the forces with which the axis acts upon the bearings, are sometimes stated to be due to the "*centrifugal forces*," or "centrifugal action," of the revolving body.)

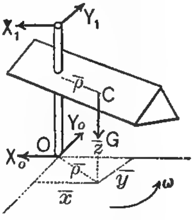

FIG. 134.

Take the axis of rotation for Z, then, with $\theta = 0$, the equations of § 114 reduce to

$$\Sigma X = -\omega^2 M\bar{x}; \quad . \quad . \quad . \quad . \quad (\text{IX}a.)$$

$$\Sigma Y = -\omega^2 M\bar{y}; \quad . \quad . \quad . \quad . \quad (\text{X}a.)$$

$$\Sigma Z = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XI}a.)$$

$$\Sigma \text{ moms.}_X = -\omega^2 \int dMyz; \quad . \quad . \quad . \quad (\text{XII}a.)$$

$$\Sigma \text{ moms.}_Y = +\omega^2 \int dMxz; \quad . \quad . \quad . \quad (\text{XIII}a.)$$

$$\Sigma \text{ moms.}_Z = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XIV}a.)$$

For greater convenience, let us suppose the axes X and Y (since their position is arbitrary so long as they are perpendicular to each other and to Z) *to revolve with the body* in its uniform rotation.

122a. *If a homogeneous body have a plane of symmetry and rotate uniformly about any axis Z perpendicular to that plane (intersecting it at O), then the acting forces are equivalent to a single force, $= \omega^2 M\bar{\rho}$, applied at O and acting in a gravity-line, but directed away from the centre of gravity.* It is evident that such a force $P = \omega^2 M\bar{\rho}$, applied as stated (see Fig. 135), will satisfy all six conditions expressed in the foregoing equations, taking X through the centre of gravity, so that $\bar{x} = \bar{\rho}$. For, from (IX*a*.), P must $= \omega^2 M\bar{\rho}$, while in each of the other summations the left-hand member will be zero, since P lies in the axis of X; and as their right-hand members will also be zero for the present body ($\bar{y} = 0$; and each of the sums $\int dMyz$ and $\int dMxz$ is zero, since for each term $dMy(+z)$ there is another $dMy(-z)$ to cancel it; and similarly, for $\int dMxz$), they also are satisfied; Q.E.D. Hence a single point of support at O will suffice to maintain the uniform motion of the body, and the pressure against it will be equal and opposite to P.*

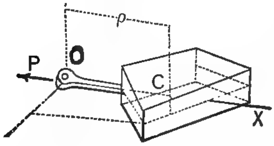

FIG. 135.

First Example.—Fig. 136. Supposing (for greater safety) that the uniform rotation of 210 revolutions per minute of each segment of a fly-wheel is maintained solely by the tension in the corresponding arm, P; required the value of P if the segment and arm together weigh $\frac{1}{30}$ of a ton, and the distance of their centre of gravity from the axis is $\bar{\rho} = 20$ in., i.e., $= \frac{5}{3}$ ft. With the *foot-ton-second* system of units, with $g = 32.2$, we have

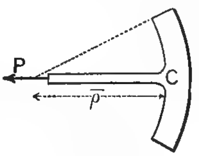

FIG. 136.

$$P = \omega^2 M\bar{\rho} = [\tfrac{210}{60} \times 2\pi]^2 \times [\tfrac{1}{30} \div 32.2] \times \tfrac{5}{3} = 0.83 \text{ tons},$$

or 1660 lbs.

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* That is, neglecting gravity. The body's weight, if considered, will take its place among the *actual* forces acting on the body.

Second Example.—Fig. 137. Suppose the uniform rotation of the same fly-wheel depends solely on the tension in the rim, required its amount. The figure shows the half-rim free, with the two equal tensions, P', put in at the surfaces exposed. Here it is assumed that the arms exert no tension on the rim. From § 122*a* we have $2P' = \omega^2 M\bar{\rho}$, where M is the mass of the half-rim, and $\bar{\rho}$ its gravity co-ordinate, which may be obtained approximately by § 26, Problem 1, considering the rim as a circular wire, viz., $\bar{\rho} = 2r \div \pi$.

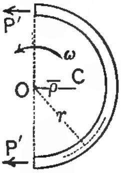

FIG. 137.

Let $M = (180 \text{ lbs.}) \div g$, with $r = 2$ ft. We have then

$$P' = \tfrac{1}{2}(22)^2(180 \div 32.2)(4 \div \pi) = 1718.0 \text{ lbs.}$$

(In reality neither the arms nor the rim sustain the tensions just computed; in treating the arms we have supposed no duty done by the rim, and *vice versâ*. The actual stresses are less, and depend on the yielding of the parts. Then, too, we have supposed the wheel to take no part in the transmission of motion by belting or gearing, which would cause a bending of the arms, and have neglected its weight.)

122b. *If a homogeneous body have a line of symmetry and rotate uniformly about an axis parallel to it (O being the foot of the perpendicular from the centre of gravity on the axis), then the acting forces are equivalent to a single force* $P = \omega^2 M\bar{\rho}$, *applied at O and acting in a gravity-line away from the centre of gravity.* Taking the axis X through the centre of gravity, Z being the axis of rotation, Fig. 138, while Z' is the line of symmetry, pass an auxiliary plane $Z'Y'$ parallel to ZY. Then the sum $\int dMxz$ may be written $\int dM(\bar{\rho} + x')z$ which $= \bar{\rho}\int dMz + \int dMx'z$. But $\int dMz = M\bar{z} = 0$, since $\bar{z} = 0$, and every term $dM(+ x')z$ is cancelled by a numerically

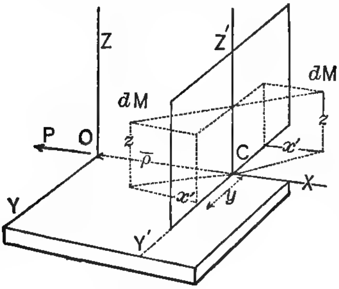

FIG. 138.

equal term $dM(-x')z$ of opposite sign. Hence $\int dMxz = 0$. Also $\int dMyz = 0$, since each positive product is annulled by an equal negative one (from symmetry about Z'). Since, also, $\bar{y} = 0$, all six conditions in § 122 are satisfied. Q. E. D.

If the homogeneous body is any solid of revolution *whose geometrical axis is parallel to the axis of rotation*, the foregoing is directly applicable.

122c. *If a homogeneous body revolve uniformly about any axis lying in a plane of symmetry, the acting forces are equivalent to a single force $P = \omega^2 M\bar{\rho}$, acting parallel to the gravity-line which is perpendicular to the axis (Z), and away from the centre of gravity, its distance from any origin O in the axis Z being $= [\int dMxz] \div M\bar{\rho}$ (the plane ZX being a gravity-plane).*—Fig. 139. From the position of the body we have $\bar{\rho} = \bar{x}$, and $\bar{y} = 0$; hence if a value $\omega^2 M\bar{\rho}$ be given to P and it be made to act through Z and parallel to X, and away from the centre of gravity, all the conditions of § 122 are satisfied except (XIIa.) and (XIIIa.). But symmetry about the plane XZ makes $\int dMyz = 0$, and satisfies (XIIa.), and by placing P at a distance $a = \int dMxz \div M\bar{\rho}$ from O along Z we satisfy (XIIIa.). Q. E. D.

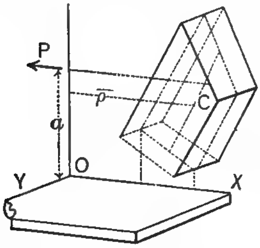

FIG. 139.

Example.—A *slender*, homogeneous, prismatic rod, of length $= l$, is to have a uniform motion, about a vertical axis passing through one extremity, maintained by a cord-connection with a fixed point in this axis. Fig. 140. Given ω, φ, l, ($\bar{\rho} = \frac{1}{2}l \cos\varphi$), and F the cross-section of the rod, let $s =$ the distance from O to any dM of the rod, dM being $= F\gamma ds \div g$. The x of any $dM = s \cos\varphi$; its $z = s \,.\, \sin\varphi$;

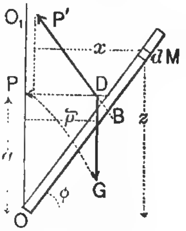

FIG. 140.

$$\therefore \int dMxz = (F\gamma \div g) \sin\varphi \cos\varphi \int_0^l s^2 ds$$

$$= \tfrac{1}{3}(F\gamma l \div g)l^2 \sin\varphi \cos\varphi = \tfrac{1}{3}Ml^2 \sin\varphi \cos\varphi.$$

Hence $a, = \int dMxz \div M\bar{\rho}$, is $= \frac{2}{3}l \sin \varphi$, and the line of action of $P\ (= \omega^2 M\bar{\rho} = \omega^2 (F\gamma l \div g) \frac{1}{2}l \cos \varphi)$ is therefore *higher up than the middle of the rod.* Find the intersection D of G and the horizontal drawn through Z at distance a from O. Determine P' by completing the parallelogram GP', attaching the cord so as to make it coincide with P'; for this will satisfy the condition of maintaining the motion, when once begun, viz., that the acting forces G, and the cord-tension P', shall be equivalent to a force $P = \omega^2 M\bar{\rho}$, applied horizontally through Z at a distance a from O.

123. Free Axes. Uniform Rotation.—Referring again to § 122 and Fig. 134, let us inquire under what circumstances the lateral forces, X_1, Y_1, X_0, Y_0, with which the bearings press the axis, to maintain the motion, are individually zero, i.e., *that the bearings are not needed, and may therefore be removed* (except a smooth horizontal plane to sustain the body's weight), leaving the motion undisturbed like that of a top "asleep." For this, not only must ΣX and ΣY both be zero, but also (since otherwise X_1 and X_0 might form a *couple*, or Y_1 and Y_0 similarly) $\Sigma\,(\text{moms.})_X$ and $\Sigma\,(\text{moms.})_Y$ must each = zero. The necessary peculiar distribution of the body's mass about the axis of rotation, then, must be as follows (see the equations of § 122):

First, $\bar{x}$ and $\bar{y}$ each = 0, i.e., *the axis must be a gravity-axis.*

Secondly, $\int dMyz = 0$, and $\int dMxz = 0$, the origin being anywhere on Z, the axis of rotation.

An axis (Z) (of a body) fulfilling these conditions is called a **Free Axis**, and since, if either one of the three *Principal Axes* for the centre of gravity (see § 107) be made an axis of rotation (the other two being taken for X and Y), the conditions $\bar{x} = 0$, $\bar{y} = 0$, $\int dMxz = 0$, and $\int dMyz = 0$, are all satisfied, *it follows that every rigid body has at least three free axes, which are the Principal Axes* of Inertia of the centre of gravity at right angles to each other.*

In the case of *homogeneous bodies* free axes can often be determined by inspection: e.g., any diameter of a sphere; any transverse diameter of a right circular cylinder through its centre of gravity, as well as its geometrical axis; the geometrical axis of any solid of revolution; etc.

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* See § 107. p. 104.

124. Rotation about an Axis which has a Motion of Translation. —*Take only the particular case where the moving axis is a gravity-axis.* At any instant, let the velocity and acceleration of axis Z be v and p; the angular velocity and acceleration about that axis, ω and θ. Then, since the actual motion of a dM in any dt is compounded of its motion of rotation about the gravity-axis and the motion of translation in common with that axis, we may, in forming the imaginary equivalent system in Fig. 141, consider each dM as subjected to the simultaneous action of $dP = dMp$ parallel to X, of the tangential $dT = dM\theta\rho$, and of the normal $dN = dM(\omega\rho)^2 \div \rho = \omega^2 dM\rho$. Take X in the direction of translation, Z (perpendicular to paper through O) is the moving gravity-axis; Y perpendicular to both. At any instant we shall have, then, the following conditions for the acting forces (remembering that $\rho \sin \varphi = y, \int dMy = M\bar{y} = 0$; etc.):

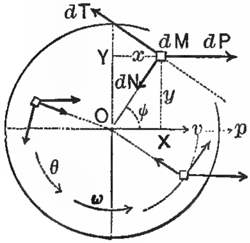

FIG. 141.

$$\Sigma X = \int dP - \int dT \sin \varphi - \int dN \cos \varphi = Mp; \quad . \quad (1)$$

$$\Sigma Y = \int dT \cos \varphi - \int dN \sin \varphi \qquad = 0; \quad . \quad . \quad (2)$$

$$\Sigma \text{ moms.}_Z = \int dT\rho - \int dPy = \theta \int dM\rho^2 = \theta I_Z = \theta M k_Z^2, \quad (3)$$

and three other equations not needed in the following example.

Example.—A homogeneous solid of revolution rolls (without slipping) down a rough inclined plane. Investigate the motion. Considering the body *free*, the acting forces are G (known) and N and P, the unknown normal and tangential components of the action of the plane on the roller. If slipping occurs, then P is the sliding friction due to the pressure N (§ 156); here, however, it is less by hypothesis (perfect rolling). At any instant the four unknowns are found by the equations

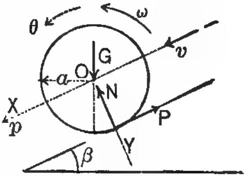

FIG. 142.

$$\Sigma X,\text{ i.e., } G\sin\beta - P, = (G \div g)p; \quad . \quad (1)$$

$$\Sigma Y,\text{ i.e., } G\cos\beta - N, = 0; \quad . \quad . \quad . \quad . \quad (2)$$

$$\Sigma\text{ moms.}_Z,\text{ i.e., } \qquad Pa, = \theta M k_Z^2; \quad . \quad . \quad (3)$$

while on account of the perfect rolling, $\theta a = p$. . . . (4)

Solving, we have, for the acceleration of translation,

$$p = g\sin\beta \div [1 + (k_Z^2 \div a^2)].$$

(If the body slid without friction, p would $= g\sin\beta$.) Hence for a cylinder (§ 97), k_Z^2 being $= \frac{1}{2}a^2$, we have $p = \frac{2}{3}g\sin\beta$; and for a sphere (§ 103) $p = \frac{5}{7}g\sin\beta$.

(If the plane is so steep or so smooth that both rolling and slipping occur, then θa no longer $= p$, but the ratio of P to N is known from experiments on sliding friction; hence there are still four equations.)

The motion of translation being thus found to be uniformly accelerated, we may use the equations of § 56.

Numerically, if a homogeneous solid sphere took 1.20 sec. to descend (from rest() 10 ft. along a rough inclined plane, with $\beta = 30°$, did any slipping occur, or was the motion perfect rolling? From p. 54 we have $s = \frac{1}{2}p_1t^2$, that is, $10 = \frac{1}{2} \cdot \frac{5}{7} \cdot g\sin 30° \cdot t^2$, for perfect rolling; from which we obtain $t = 1.32$ seconds, which is > 1.20 sec. Hence some slipping must have occurred. (The time of descent would have been only $t = \sqrt{2s \div \frac{1}{2}g} = 1.114$ sec., if the surfaces had been perfectly smooth; and the sphere would have had simply a motion of translation, the force P being zero).

N.B.—A hollow sphere would occupy a *longer time* than a solid one in descending the plane (if rough); since the ratio $k_z \div a$ is greater for the former.

125. Parallel-Rod of a Locomotive.—When the locomotive moves uniformly, each dM of the rod between the two (or three) driving-wheels rotates with uniform velocity about a centre of its own on the line BD, Fig. 143, and with a velocity v* and radius r common to all, and likewise has a horizontal *uniform* motion of translation. Hence if we inquire what are the reactions P

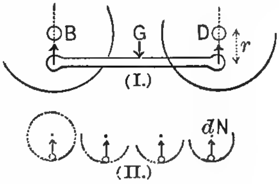

FIG. 143.

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* This velocity is that which the dM has *relatively to the frame of the locomotive*, in a circular path. E.g., if the locomotive (frame) has a velocity of 60 miles per hour and the radius r is one-third of the radius of the driver, then v is 20 miles per hour.

of its supports, as induced *solely by its weight and motion*, when in its lowest position (independently of any thrust along the rod), we put ΣY of (I.) $= \Sigma Y$ of (II.) (II. shows the imaginary equivalent system), and obtain

$$2P - G = \int dN = \int dMv^2 \div r = (v^2 \div r)\int dM = Mv^2 \div r.$$

Example.—Let the velocity of translation $= 50$ miles per hour, the radius* of the pins be 18 in. $= \frac{3}{2}$ ft., and $=$ *half that of the driving-wheels*, while the weight of the rod is 200 lbs. With $g = 32.2$, we must use the foot and second, and obtain

$$v = \tfrac{1}{2}[50 \times 5280 \div 3600] \text{ ft. per second} = 36.6;$$

while $\quad M = 200 \div 32.2 = 200 \times .0310 = 6.20;$

and finally $P = \frac{1}{2}[200 + 6.2(36.6)^2 \div \frac{3}{2}] = 2868.3$ lbs.,

or nearly $1\frac{1}{2}$ tons, *about thirty times that due to the weight alone.*

126. So far in this chapter the motion has been prescribed, and the necessary conditions determined, to be fulfilled by the acting forces at any instant. Problems of a converse nature, i.e., where the initial state of the body and the acting forces are given while the resulting motion is required, are of much greater complexity, but of rare occurrence in practice.

For further study in this direction the reader is referred to Routh's "*Rigid Dynamics*," Rankine's "*Applied Mechanics*," Schell's "*Theorie der Bewegung und der Kraefte*," and Worthington's "*Dynamics of Rotation*" (this last being a small but clearly written and practical book).

In Wood's "Analytical Mechanics" will be found the proof of "Euler's Equations," which are the basis of the treatment of the gyroscope in the book of that name by Gen. J. G. Barnard (Van Nostrand's *Science Series*, No. 90). The article on the gyroscope in Johnson's Cyclopædia is by Gen. Barnard. Perry's "*Spinning Tops*" is an interesting popular book.

The Brennan "Monorail Car" (model) is described in the *Engineering Record* for Aug. 31, 1907, p. 226, and depends for its stability (there being but one rail under the car) upon two gyroscope wheels revolving at 7000 revs. per min. in a vertical plane parallel to the rail. See also *McClure's Magazine* for Dec., 1907, p. 163.

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* Or, rather, the radius of the circular path of the pin-centre, whose velocity in this path is 25 miles per hour.

CHAPTER VI.

WORK, ENERGY, AND POWER.

127. Remark.—These quantities as defined and developed in this chapter, though compounded of the fundamental ideas of matter, force, space, and time, enter into theorems of such wide application and practical use as to more than justify their consideration as separate kinds of quantity.

128. Work in a Uniform Translation. Definition of Work.—Let Fig. 144 represent a rigid body having a motion of translation parallel to X, acted on by a system of forces P_1, P_2, R_3, and R_4, which remain constant.*

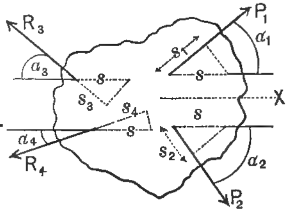

Fig. 144.

Let s be any distance described by the body during its motion; then ΣX must be zero (§ 109), i.e., noting that R_3 and R_4 have negative X components (the supplements of their angles with X are used),

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 - R_3 \cos \alpha_3 - R_4 \cos \alpha_4 = 0;$$

or, multiplying by s and transposing, we have (noting that $s \cos \alpha_1 = s_1$ the *projection* of s on P_1, that $s \cos \alpha_2 = s_2$, the *projection* of s on P_2, and so on),

$$P_1 s_1 + P_2 s_2 = R_3 s_3 + R_4 s_4. \quad . \quad . \quad . \quad . \quad (a)$$

The projections s_1, s_2, etc., may be called the *distances described in their respective directions* by the forces P_1, P_2, etc.; P_1 and P_2 having moved *forward*, since s_1 and s_2 fall *in front* of the initial position of their points of application; R_3 and R_4 *backward*, since s_3 and s_4 fall *behind* the initial positions in their case. (By forward and backward we refer to the direction of each force in turn.) The name **Work** is given to the *product of a force by the distance described in the direction of the force by the point of application.* If the force moves *forward* (see above), it is called a *working-force*, and is said *to do* the work (e.g:, P_1s_1) expressed by this product; while if *backward*, it is called a *resistance*, and is then said *to have the work* (e.g., R_3s_3), *done upon it*, in *overcoming it* through the distance mentioned (it might also be said to have done negative work).

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* Constant in *direction* as well as amount.

Eq. (a) above, then, proves the theorem that: *In a uniform translation, the working forces do an amount of work which is entirely applied to overcoming the resistances.*

129. Unit of Work.—Since the work of a force is a product of force by distance, it may logically be expressed as so many foot-pounds, inch-pounds, kilogram-meters, according to the system of units employed. The ordinary English unit is the foot-pound, or ft.-lb. It is of the same quality as the *moment* of a force.

130. Power.—Work as already defined does not depend on the time occupied, i.e., the work P_1s_1 is the same whether performed in a long or short time; but the element of time is of so great importance in all the applications of dynamics, as well as in such practical commercial matters as water-supply, consumption of fuel, fatigue of animals, etc., that the *rate of work* is a consideration both of interest and necessity.

Power is the rate at which work is done, and one of its units is one foot-pound per second in English practice; a larger one will be mentioned presently.

The *power exerted* by a *working force*, or *expended* upon a *resistance*, may be expressed symbolically as

$$L = P_1s_1 \div t, \quad \text{or} \quad R_3s_3 \div t,$$

in which t is the time occupied in doing the work P_1s_1 or R_3s_3 (see Fig. 144); or if v_1 is the component in the direction of the force P_1 of the velocity v of the body, we may also write

$$L = P_1v_1, \text{ ft.-lbs. per sec.} \quad . \quad . \quad . \quad . \quad (b)$$

131. *Example.*—Fig. 145, shows as a *free body* a sledge which is being drawn *uniformly* up a rough inclined plane by a cord parallel to the plane. Required the total power exerted (and expended), if the tension in the cord is $P_1 = 100$ lbs., the weight of sledge $R_3 = 160$ lbs., $\beta = 30°$, and the sledge moves 240 ft. each minute. N and R_4 are the normal and parallel (i.e., $R_4 =$ friction) components of the reaction of the plane on the sledge. From eq. (1), § 128, the work done while the sledge advances through $s = 240$ ft. may be obtained either from the working forces, which in this case are represented by P_1 alone, or from the resistances R_3 and R_4. Take the former method first. Projecting s upon P_1 we have $s_1 = s$.

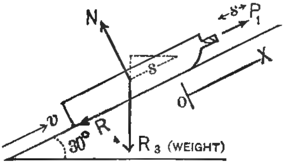

FIG. 145.

Hence $P_1 s_1$ or $100 \text{ lbs.} \times 240 \text{ ft.} = 24{,}000$ ft.-lbs.

of work done in 60 seconds. That is, the *power exerted by the working forces* is

$$L = P_1 s_1 \div t = 400 \text{ ft.-lbs. per second.}$$

As to the other method, we notice that R_3 and R_4 are resistances, since the projections $s_3 = s \sin \beta$, and $s_4 = s$, would fall back of their points of application in the initial position, while N is *neutral*, i.e., is neither a working force nor a resistance, since the projection of s upon it is zero.

From $\Sigma X = 0$ we have $-R_4 - R_3 \sin \beta + P_1 = 0,$

and from $\Sigma Y = 0$ (§ 109) $N - R_3 \cos \beta = 0;$

whence R, the friction $= 20$ lbs., and $N = 138.5$ lbs. Also, since $s_3 = s \sin \beta = 240 \times \frac{1}{2} = 120$ ft., and $s_4 = s, = 240$ ft., we have for the work done upon the resistances (i.e., in overcoming them) in 60 seconds

$$R_3 s_3 + R_4 s_4 = 160 \times 120 + 20 \times 240 = 24{,}000 \text{ ft.-lbs.,}$$

and the *power expended in overcoming resistances,*

$$L = 24{,}000 \div 60 = 400 \text{ ft.-lbs. per second,}$$

as already derived. Or, in words the power exerted by the tension in the cord is expended entirely in raising the weight a vertical height of 2 feet, and overcoming the friction through a distance of 4 feet along the plane, every second; *the motion being a uniform translation.*

132. Horse-Power.—As an average, a horse can exerts a tractive effort or pull of 100 lbs., at a uniform pace of 4 ft. per second, for ten hours a day without too great fatigue. This gives a power of 400 ft.-lbs. per second; but Boulton & Watt in rating their engines, and experimenting with the strong dray-horses of London, fixed upon 550 ft.-lbs. per second, or 33,000 ft.-lbs. per minute, as a convenient large unit of power. (The French horse-power, or *cheval-vapeur*, is slightly less than the English, being 75 kilogrammeters per second, or 32,550 ft.-lbs. per minute.) This value for the horse-power is in common use. In the example in § 131, then, the power of 400 ft.-lbs. per second exerted in raising the weight and overcoming friction may be expressed as ($400 \div 550 =$) $\frac{8}{11}$ of a horse-power. A man can work at a rate equal to about $\frac{1}{12}$ of a horse-power, with proper intervals for eating and sleeping.

133. Kinetic Energy. Retarded Translation.—In a retarded translation of a rigid body whose mass $= M$, suppose there are no working-forces, and that the resistances are constant and their resultant is R. (E.g., Fig. 146 shows such a case; a sledge, having an initial velocity c and sliding on a rough horizontal plane, is gradually retarded by the friction R.) R is parallel to the direction of translation (§ 109) and the acceleration is $p = -R \div M$; hence from $vdv = pds$ we have

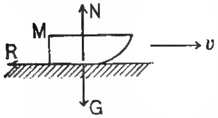

FIG. 146.

$$\int vdv = -(1 \div M)\int Rds. \quad . \quad . \quad . \quad . \quad (1)$$

But the projection of each ds of the motion upon R is $= ds$ itself; i.e. (§ 128), Rds is the *work done upon R*, in overcoming it through the small distance ds, and $\int Rds$ is the sum of all such amounts of work throughout any definite portion of the motion. Let the range of motion be between the points where the velocity $= c$, and where it $=$ zero (i.e., the mass has come to rest). With these limits in eq. (1), (0 and s' being the corresponding limits for s), we have

$$\frac{Mc^2}{2} = \int_0^{s'} Rds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

That is, *in giving up all its velocity c the body has been able to do the work $\int Rds$* (this, if R remains constant, reduces to Rs') or its equal $\frac{Mc^2}{2}$. If, then, by **energy** we designate the *ability to perform work,* we give the name **kinetic energy** of a moving body to *the product of its mass by half the square of its velocity* $\left(\frac{Mv^2}{2}\right)$; i.e., energy due to motion.

Example.—If the sledge in Fig. 146 has initially a velocity of $c = 10$ ft./sec. and its weight is $G = 322$ lbs. (so that its mass in the ft.-lb.-sec. system is $M = 10$) its initial kinetic energy is $Mc^2 \div 2 = 500$ ft.-lbs. If the friction or resistance, R, is constant and has a value of 20 lbs., we compute $s' = 25$ ft. (from $500 = Rs'$) as the distance the sledge will go in overcoming this resistance; i.e., in giving up all its kinetic energy. If the sledge goes 40 ft. we conclude the average resistance to have been only 12.5 lbs.; since $500 \div 40 = 12.5$. Now suppose R variable, say $= (20 + 4s)$ lbs., (s in ft.), and we have $500 = \int_0^{s'} [20 + 4s]ds = 20s' + 2s'^2$; $\therefore\ s' = 11.6$ ft.

134. Work and Kinetic Energy in any Translation.—Let P be the resultant of the working forces at any instant, R that of the resistances; they (§ 109) will both act in a **gravity-line*** parallel to the direction of translation. The acceleration at any instant is $p = (\Sigma X \div M) = (P - R) \div M$; hence from $vdv = pds$ we have

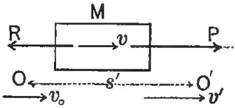

FIG. 147.

$$Mvdv = Pds - Rds. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating between any two points of the motion as O and O' where the velocities are v_0 and v', we have after transposition

$$\int_0^{s'} Pds = \int_0^{s'} Rds + \left[\frac{Mv'^2}{2} - \frac{Mv_0^2}{2}\right]. \quad . \quad . \quad (d)$$

But P being the resultant of P_1, P_2, etc., and R that of R_1, R_2, etc., we may prove, as in § 62, that if du_1, du_2, etc., be the respective projections of any ds upon P_1, P_2, etc., while dw_1, dw_2, etc., are those upon R_1, R_2, etc., then

$$Pds = P_1du_1 + P_2du_2 + \ldots \quad \text{and} \quad Rds = R_1dw_1 + R_2dw_2 \ldots ;$$

and (d) may be rewritten

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* That is, a line passing through the centre of gravity.

$$\int_0^{s'} P_1 du_1 + \int_0^{s'} P_2 du_2 + \dots$$
$$= \int_0^{s'} R_1 dw_1 + \int_0^{s'} R_2 dw_2 + \dots + \left[\frac{Mv'^2}{2} - \frac{Mv_0^2}{2}\right]; \quad (e)$$

or, in words: *In any translation, a portion of the work done by the working forces is applied in overcoming the resistances while the remainder equals the change in the kinetic energy of the body.*

It will be noted that the bracket in (e) depends only on the initial and final velocities, and not upon any intermediate values; hence, if the initial state is one of rest, and also the final, the total change in kinetic energy is zero, and the work of the working forces has been entirely expended in the work of overcoming the resistances; but at intermediate stages the former exceeds the work so far needed to overcome resistances, and this excess is said to be *stored* in the moving mass; and as the velocity gradually becomes zero, this stored energy becomes available for aiding the working forces (which of themselves are then insufficient) in overcoming the resistances, and is then said to be *restored.* (The function of a fly-wheel might be stated in similar terms, but as that involves rotary motion it will be deferred.)

Work applied in increasing the kinetic energy of a body is sometimes called "work of inertia," as also the work done by a moving body in overcoming resistances, and thereby losing speed.

135. Example of Steam-Hammer.—Let us apply eq. (e) to determine the velocity v' attained by a steam-hammer at the lower end of its stroke (the initial velocity being $= 0$), just before delivering its blow upon a forging, supposing that the steam-pressure P_2 at all stages of the downward stroke is given by an *indicator.* Fig. 148. Weight of moving mass is 322 lbs.; $\therefore M = 10$ (foot-pound-second system), $l = 1$ foot. The *working forces* at any instant are $P_1 = G = 322$ lbs.; P_2, which is variable, but whose values at the seven *equally spaced* *points a, b, c, d, e, f, g*, are 800, 900, 900, 800, 600, 500, 450 lbs., respectively. R_1 the exhaust-pressure (16 lbs. per sq. inch × 20 sq. inches piston-area) = 320 lbs., is the only resistance, and is constant. Hence from eq. (e), since here the projections du_1, etc., of any ds upon the respective forces are equal to each other and $= ds$,

$$P_1\int_0^l ds + \int_0^l P_2 ds = R_1\int_0^l ds + \frac{Mv'^2}{2}. \quad (1)$$

The term $\int P_2 ds$ can be obtained approximately* by Simpson's Rule, using the above values for six equal divisions, which gives

$$\tfrac{1}{18}[800 + 4(900 + 800 + 500) + 2(900 + 600) + 450]$$

= 725 ft.-lbs. of work. Hence, making all the substitutions, we have, since $\int_0^l ds = 1$ ft.,

$$322 \times 1 + 725 = 320 \times 1 + \tfrac{1}{2}Mv'^2; \ \therefore \ \tfrac{1}{2}Mv'^2 = 727 \text{ ft.-lbs.}$$

of energy to be expended on the forging. (Energy is evidently expressed in the same kind of unit as work.) We may then say that the forging receives a blow of 727 ft.-lbs. energy. The pressure actually felt at the surface of the hammer varies from instant to instant during the compression of the forging and the gradual stopping of the hammer, and depends on the readiness with which the hot metal yields.

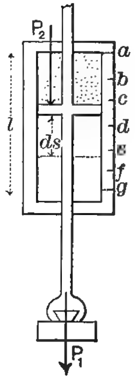

FIG. 148.

If the *mean resistance* encountered is R_m, and the depth of compression s'', we would have (neglecting the force of gravity, and noting that now the initial velocity is v', and the final zero), from eq. (c),

$$\tfrac{1}{2}Mv'^2 = R_m s''; \text{ i.e., } R_m = [727 \div s'' \text{ (ft.)}] \text{ lbs.}$$

E.g., if $s'' = \frac{1}{5}$ of an inch $= \frac{1}{60}$ of a foot, $R_m = 43620$ lbs., and the maximum value of R would probably be about double this near the end of the impact. If the anvil also sinks during the impact a distance s''', we must substitute $s''' + s''$ instead of s''; this will give a smaller value for R_m.

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* See p. 13 of "*Notes and Examples in Mechanics.*"

By mean value for R is meant [eq. (c)] that value, R_m, which satisfies the relation

$$R_m s' = \int_0^{s'} R ds.$$

This may be called more explicitly a *space-average*, to distinguish it from a *time-average*, which might appear in some problems, viz., a value R_{tm}, to satisfy the relation (t' being the duration of the impact)

$$R_{tm} t' = \int_0^{t'} R dt,$$

and is different from R_m.

From $\frac{1}{2}Mv'^2 = 727$ ft.-lbs., we have $v' = 12.06$ ft. per sec., whereas for a free fall it would have been $\sqrt{2 \times 32.2 \times 1} = 8.03$. (This example is virtually of the same kind as Prob. 4, § 59, differing chiefly in phraseology.)

136. Pile-Driving.*—The safe load to be placed upon a pile after the driving is finished is generally taken as a fraction (from $\frac{1}{6}$ to $\frac{1}{8}$) of the resistance of the earth to the passage of the pile as indicated by the effect of the last few blows of the ram, in accordance with the following approximate theory: Toward the end of the driving the resistance R encountered by the pile is nearly constant, and is assumed to be that met by the ram at the head of the pile; the distance s' through which the head of the pile sinks as an effect of the last blow is observed. If G, then, is the weight of the ram, $= Mg$, and h the height of free fall, the velocity due to h, on striking the pile, is $c = \sqrt{2gh}$ (§ 52), and we have, from eq. (c),

$$\tfrac{1}{2}Mc^2, \text{ i.e., } Gh, = \int_0^{s'} R ds = Rs' \quad . \quad . \quad (1)$$

(R being considered constant); hence $R = Gh \div s'$, and the *safe load* (for ordinary wooden piles),

$$P = \text{from } \tfrac{1}{6} \text{ to } \tfrac{1}{8} \text{ of } Gh \div s'. \quad . \quad . \quad . \quad . \quad (2)$$

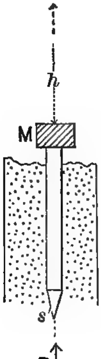

FIG. 149.

Maj. Sanders recommends $\frac{1}{8}$ from experiments made at Fort

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* See also p. 87 of the author's *Notes and Examples in Mechanics.*

Delaware in 1851; Molesworth, $\frac{1}{8}$; General Barnard, $\frac{1}{6}$, from extensive experiments made in Holland.

Of course from eq. (2), given P, we can compute s'.

(Owing to the uncertainty as to how much of the resistance R is due to friction of the soil on the sides of the pile, and how much to the inertia of the soil around the shoe, the more elaborate theories of Weisbach and Rankine seem of little practical account.)

137. *Example.*—In preparing the foundation of a bridge-pier it is desired that each pile (placing them 4 ft. apart) shall bear safely a load of 72 tons. If the ram weighs one ton, and falls 12 ft., what should be the effect of the last blow on each pile? Using the foot-ton-second system of units, and Molesworth factor $\frac{1}{8}$, eq. (2) gives

$$s' = \tfrac{1}{8}(1 \times 12 \div 72) = \tfrac{1}{48} \text{ of a foot} = \tfrac{1}{4} \text{ of an inch.}$$

That is, the pile should be driven until it sinks only $\frac{1}{4}$ inch under each of the last few blows.

138. Kinetic Energy Lost in Inelastic Direct Central Impact.—Referring to § 60, and using the same notation as there given, we find that if the united kinetic energy possessed by two inelastic bodies after their impact, viz., $\frac{1}{2}M_1C^2 + \frac{1}{2}M_2C^2$, C having the value $(M_1c_1 + M_2c_2) \div (M_1 + M_2)$, be deducted from the amount before impact, viz., $\frac{1}{2}M_1c_1^2 + \frac{1}{2}M_2c_2^2$, the *loss of kinetic energy during impact of two inelastic bodies* is *

$$W = \frac{\frac{1}{2}M_1M_2}{M_1 + M_2}(c_1 - c_2)^2. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

An equal amount of energy is also lost by partially elastic bodies during the first period of the impact, but is partly regained in the second. If the bodies were perfectly elastic, we would find it wholly regained and the resultant loss zero, from the equations of § 60; but this is not quite the reality, on account of internal vibrations.

The *kinetic energy still remaining in two inelastic bodies* after impact (they move together as one mass) is

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* See *Eng. News*, July, 1888, pp. 33 and 34.

$\frac{1}{2}(M_1 + M_2)C^2$, or, after inserting the value of $C = (M_1c_1 + M_2c_2) \div (M_1 + M_2)$, we have

$$W' = \frac{1}{2} \cdot \frac{[M_1c_1 + M_2c_2]^2}{M_1 + M_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

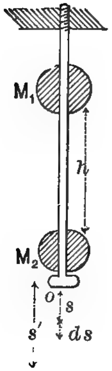

FIG. 150.

Example 1.—The weight $G_1 = M_1g$ falls freely through a height h, impinging upon a weight $G_2 = M_2g$, which was initially at rest. After their (*inelastic*) impact they move on together with the combined kinetic energy just given in (2), which, since c_1 and c_2, the velocities before impact, are respectively $\sqrt{2gh}$ and 0, may be reduced to a simpler form. This energy is soon absorbed in overcoming the flange-pressure R, which is proportional (so long as the elasticity of the rod is not impaired) to the elongation s, as with an ordinary spring. If from previous experiment it is known that a force R_0 produces an elongation s_0, then the variable $R = (R_0 \div s_0)s$. Neglecting the weight of the two bodies as a working force, we now have, from eq. (*d*),

$$0 = \frac{R_0}{s_0}\int_0^{s'} s ds + 0 - \frac{M_1^2gh}{M_1 + M_2};$$

$$\text{i.e.,} \frac{R_0}{s_0} \cdot \frac{s'^2}{2} = \frac{M_1^2gh}{M_1 + M_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When $s = s'$, i.e., when the masses are (momentarily) at rest in the lowest position, the flange-pressure or tensile stress in the rod is a maximum, $R' = (R_0 \div s_0)s'$, whence $s' = R's_0 \div R_0$; and (3) may be written

$$\cancel{\frac{R'}{2}s' = \frac{M_1^2gh}{M_1 + M_2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or

$$\cancel{\frac{R'^2s_0}{2R_0}} = \frac{M_1^2gh}{M_1 + M_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Eq. (3) gives the final elongation of the rod, and (5) the greatest tensile force upon it, provided the elasticity of the rod is not impaired. The form $\frac{1}{2}R's'$ in (4) may be looked upon as a direct integration of $\int_0^{s'} Rds$, viz., the mean resistance ($\frac{1}{2}R'$) multiplied by the whole distance (s') gives the work done in overcoming the variable R through the successive ds's.

If the elongation is considerable, the working-forces G_1 and G_2 cannot be neglected, and would appear in the term $+(G_1 + G_2)s'$ in the right-hand members of (3), (4), and (5). The upper end of the rod is firmly fixed, and the rod itself is of small mass compared with M_1 and M_2.

Example 2.—Two cars, Fig. 151, are connected by an elastic chain on a horizontal track. Velocities before impact (i.e., before the stretching of the chain begins, by means of which they are brought to a common velocity at the instant of greatest tension R', and elongation s' of the chain) are $c_1 = c_1$, and $c_2 = 0$.

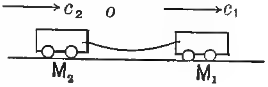

FIG. 151.

During the stretching, i.e., the first period of the impact, the kinetic energy lost by the masses has been expended in stretching the chain, i.e., in doing the work $\frac{1}{2}R's'$; hence we may write (the elasticity of the chain not being impaired) (see eq. (1))

$$\frac{1}{2}\frac{M_1M_2c_1^2}{M_1+M_2} = \frac{1}{2}R's' = \frac{R_0}{s_0}\cdot\frac{s'^2}{2} = \frac{R'^2s_0}{2R_0}, \quad . \quad . \quad (6)$$

in which the different symbols have the same meaning as in Example 1, in which the rod corresponds to the chain of this example.

In this case the mutual accommodation of velocities is due to the presence of the chain, whose *stretching* corresponds to the *compression* (of the parts in contact) in an ordinary impact.

In numerical substitution, 32.2 for g requires the use of the units foot and second for space and time, while the unit of force may be anything convenient.

139. Work and Energy in Rotary Motion. Axis Fixed.—The rigid body being considered free, let an axis through O perpendicular to the paper be the axis of rotation, and resolve all forces not intersecting the axis into components parallel and perpendicular to the axis, and the latter again into components tangent and normal to the circular path of the point

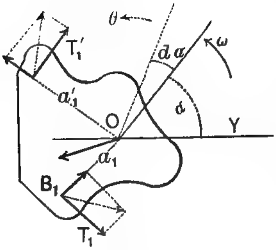

FIG. 152.

of application. These tangential components are evidently the only ones of the three sets mentioned which have moments about the axis, those having moments of the same sign as ω (the angular velocity at any instant) being called *working forces*, T_1, T_2, etc.; those of opposite sign, *resistances*, T_1', T_2', etc.; for when in time dt the point of application B_1, of T_1, describes the small arc $ds_1 = a_1 d\alpha$, whose projection on T_1 is $= ds_1$, this projection falls *ahead* (i.e., in direction of force) of the position of the point at the beginning of dt, while the reverse is true for T_1'.

From eq. (XIV.), § 114, we have for θ (angul. accel.)

$$\theta = \frac{(T_1a_1 + T_2a_2 + \dots) - (T_1'a_1' + T_2'a_2' + \dots)}{I}, \quad (1)$$

which substituted in $\omega d\omega = \theta d\alpha$ (from § 110) gives (remembering that $a_1 d\alpha = ds_1$, etc.), after integration and transposition,

$$\int_0^n T_1 ds_1 + \int_0^n T_2 ds_2 + \text{etc.}$$

$$= \int_0^n T_1' ds_1' + \int_0^n T_2' ds_2' + \text{etc.} + [\tfrac{1}{2}\omega_n^2 I - \tfrac{1}{2}\omega_0^2 I], \quad (2)$$

where 0 and n refer to any two (initial and final) positions of the rotating body. Eq. (4), § 120, is an example of this.

Now $\frac{1}{2}\omega_n^2 I = \frac{1}{2}\omega_n^2 \int dM\rho^2 = \int \frac{1}{2} dM(\omega_n\rho)^2$, which, since $\omega_n\rho$ is the actual velocity of any dM at this (final) instant, is nothing more than the sum of the amounts of kinetic energy possessed at this instant by all the particles of the body; a similar statement may be made for $\frac{1}{2}\omega_0^2 I$. (ω_0 and ω_n in *radians*.)

Eq. (2) therefore may be put into words as follows:

Between any two positions of a rigid body rotating about a fixed axis, the work done by the working forces is partly used in overcoming the resistances, and the remainder in changing the kinetic energy of the individual particles. If in any case this remainder is negative, the final kinetic energy is less than the initial, i.e., the work done by the working forces is less than that necessary to overcome the resistances through their respective spaces, and the deficiency is made up by the *restoring* of some of the initial kinetic energy of the rotating body. A moving fly-wheel, then, is a reservoir of kinetic energy.

Example.—A 668-lb pulley (see page 103) is found to have a radius of gyration of $\sqrt{7.91}$ ft., and a moment of inertia about its axis, Z, of $Mk^2 = (668 \div g)\,7.91$. Let us suppose it mounted on a short shaft of $(r_0 =)$ 2 in. radius (whose I_z may be neglected) supported in proper bearings. The pulley and shaft are in contact with nothing except the bearings, which offer a friction T_1', tangent to outer surface of shaft, of 120 lbs. If the pulley has an initial rotary speed of 300 revs./min., in how many turns, n, will it be brought to rest? Evidently $\omega_n = 0$, while ω_0, $= 2\pi 300 \div 60$, $= 31.41$ *rads./sec.* That is, the initial kinetic energy is $\frac{1}{2}\omega_0{}^2 Mk^2$, $= \frac{1}{2}(31.41)^2(668 \div 32.2)\,7.91$, $= 80{,}810$ ft.-lbs.: and the final, *zero*. $T_1' = 120$ lbs., *constant*, and the work done on T_1' is $T_1' \int_0^n ds_1' = 120 \,.\, n(2\pi) \,.\, \frac{1}{6} = 125.6n$ ft.-lbs. Hence from eq. (2) we have $0 = 125.6n + [0 - 80{,}810]$; i.e., $n = 643$ turns. *Ans.*

140. Work of Equivalent Systems the Same.—*If two plane systems of forces acting on a rigid body are equivalent* (§ 15*a*), *the aggregate work done by either of them during a given slight displacement or motion of the body parallel to their plane is the same.* By aggregate work is meant what has already been defined as the sum of the "virtual moments" (§§ 61 to 64), in any small displacement of the body, viz., the algebraic sum of the products, $\Sigma\,(Pdu)$, obtained by multiplying each force by the projection (du) of the displacement of (or small space described by) its point of application upon the force. (We here class resistances as negative working forces.)

Call the systems A and B; then, if all the forces of B were *reversed in direction* and applied to the body along with those of A, the compound system would be a *balanced system*, and hence we should have (§ 64), for a small motion parallel to the plane of the forces,

$$\Sigma(Pdu) = 0,\ \text{i.e.,}\ \Sigma(Pdu)\ \text{for}\ A - \Sigma(Pdu)\ \text{for}\ B = 0,$$

or

$$+\,\Sigma(Pdu)\ \text{for}\ A = +\,\Sigma(Pdu)\ \text{for}\ B.$$

But $+\,\Sigma\,(Pdu)$ for A is the aggregate work done by the forces of A during the given motion, and $+\,\Sigma(Pdu)$ for B is a similar quantity for the forces of B (not reversed) during the same small motion if B acted alone. Hence the theorem is proved, and could easily be extended to space of three dimensions.

141. Relation of Work and Kinetic Energy for any Extended Motion of a Rigid Body Parallel to a Plane.

—(If at any instant any of the forces acting are not parallel to the plane mentioned, their components lying in or parallel to that plane, will be used instead, since the other components obviously would be neither working forces nor resistances.) Fig. 153 shows an initial position, *o*, of the body; a final, *n*; and any intermediate, as *q*. The forces of the system acting may vary in any manner during the motion.

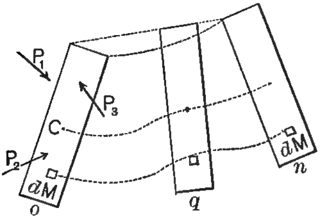

FIG. 153.

In this motion each *dM* describes a curve of its own with varying velocity v, tangential acceleration p_t, and radius of curvature r; hence in any position q, an imaginary system B (see Fig. 154), equivalent to the actual system A (at q in Fig. 153), would be formed by applying to each dM a tangential force $dT = dMp_t$, and a normal force $dN = dMv^2 \div r$. By an infinite number of consecutive small displacements, the body passes from o to n. In the small displacement of which q is the initial position, each dM describes a space ds, and dT does the work $dTds = dMvdv$, while dN does the work $dN \times 0 = 0$. Hence the total work done by B in the small displacement at q would be

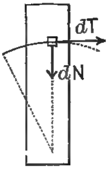

FIG. 154.

$$= dM'v'dv' + dM''v''dv'' + \text{etc.}, \quad . \quad . \quad . \quad (1)$$

including all the dM's of the body and their respective velocities at this instant.

But the work at q in Fig. 153 by the actual forces (i.e., of system A) during the same small displacement must (by § 140) be equal to that done by B, hence

$$P_1du_1 + P_2du_2 + \text{etc.} = dM'v'dv' + dM''v''dv'' + \text{etc.} \quad (q)$$

Now conceive an equation like (q) written out for each of the small consecutive displacements between positions o and n and corresponding terms to be added; this will give

$$\int_0^n P_1du_1 + \int_0^n P_2du_2 + \text{etc.}$$
$$= dM'\int_0^n v'dv' + dM''\int_0^n v''dv'' + \text{etc.}$$
$$= \tfrac{1}{2}dM'(v_n'^2 - v_0'^2) + \tfrac{1}{2}dM''(v_n''^2 - v_0''^2) + \text{etc.}$$

The second member may be rewritten so as to give, finally,

$$\int_0^n P_1du_1 + \int_0^n P_2du_2 + \text{etc.} = \Sigma(\tfrac{1}{2}dMv_n^2) - \Sigma(\tfrac{1}{2}dMv_0^2), \quad \text{(XV.)}$$

or, in words, *the work done by the acting forces (treating a resistance as a negative working force) between any two positions is equal to the gain (or loss) in the aggregate kinetic energy of the particles of the body between the two positions.* To avoid confusion, Σ has been used instead of the sign $\int$ in one member of (XV.), in which v_n is the final velocity of any dM (not the same for all necessarily) and v_0 the initial.

(The same method of proof can be extended to three dimensions.)

Since kinetic energy is always essentially positive, if an expression for it comes out negative as the solution of a problem, some impossible conditions have been imposed.

142. Work and Kinetic Energy in a Moving Machine.—Defining a *mechanism* or *machine* as a series of rigid bodies jointed or connected together, so that working-forces applied to one or more may be the means of overcoming resistances occurring anywhere in the system, and also of changing the amount of kinetic energy of the moving masses, let us for simplicity consider a machine the motions of whose parts are all parallel to a plane, and let all the forces acting on any one piece, considered free, at any instant be parallel to the same plane.

Now consider each piece of the machine, or of any series of its pieces, as a free body, and write out eq. (XV.) for it between any two positions (whatever initial and final positions are selected for the first piece, those of the others must be corresponding initial and corresponding final positions), and it will be found, on adding up corresponding members of these equations, that the terms involving those components of the mutual pressures (between the pieces considered) which are *normal* to the rubbing surfaces at any instant will cancel out, while their components tangential to the rubbing surfaces (i.e., *friction*, since if the surfaces are perfectly smooth there can be no tangential action) will appear in the algebraic addition as resistances multiplied by the distances rubbed through, *measured on the rubbing surfaces.* For example, Fig. 155, where one rotating piece both presses and rubs on another. Let the normal pressure between them at A be $R_2 = P_2$; it is a working force for the body of mass M'', but a resistance for M', hence the separate symbols for the numerically equal forces (action and reaction).

Similarly, the friction at A is $R_3 = P_3$; a resistance for M', a working-force for M''. (In some cases, of course, friction may be a resistance for both bodies.) For a small motion, A describes the small arc AA' about O' in dealing with M', but for M'' it describes the arc AA'' about O'', $A'A''$ being parallel to the surface of contact AD, while AB is perpendicular to $A'A''$. In Figs. 156 and 157 we see M' and M'' free, and their corresponding small rotations indicated. During these motions the kinetic energy (K. E.) of each mass has changed by amounts $d(\text{K. E.})_{M'}$ and $d(\text{K. E.})_{M''}$ respectively, and hence eq. (XV.) gives, for each free body in turn,

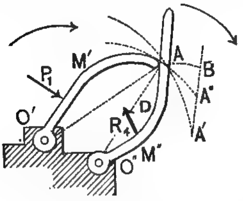

FIG. 155.

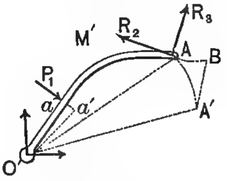

FIG. 156.

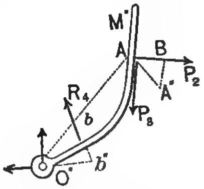

FIG. 157.

$$P_1\overline{aa'} - R_2\overline{AB} - R_3\overline{A'B} = d(\text{K. E.})_{M'} \quad . \quad (1)$$

$$-R_4\overline{bb''} + P_2\overline{AB} + P_3\overline{A''B} = d(\text{K. E.})_{M''}. \quad . \quad (2)$$

Now add (1) and (2), member to member, remembering that $P_2 = R_2$ and $P_3 = R_3 = F_3$ = friction, and we have

$$P_1\overline{aa'} - F_3\overline{A'A''} - R_4\overline{bb''} = d(\text{K. E.})_{M'} + d(\text{K. E.})_{M''}, \quad (3)$$

in which the mutual actions of M' and M'' do not appear, except the friction, *the work done in overcoming which, when the two bodies are thus considered collectively, is the product of the friction by the distance $A'A''$ of actual rubbing measured on the rubbing surface.* For any number of pieces, then, *considered free collectively*, the assertion made at the beginning of this article is true, since any finite motion consists of an infinite number of small motions to each one of which an equation like (3) is applicable.

Summing the corresponding terms of all such equations, we have

$$\int_0^n P_1 du_1 + \int_0^n P_2 du_2 + \text{etc.} = \Sigma(\text{K. E.})_n - \Sigma(\text{K. E.})_0 . \text{(XVI.)}$$

This is of the same form as (XV.), but instead of applying to a single rigid body, deals with any assemblage of rigid parts forming a machine, or any part of a machine (a similar proof will apply to three dimensions of space); but it must be remembered that it excludes all the *mutual* actions* of the pieces considered except friction, which is to be introduced in the manner just illustrated. A flexible inextensible cord may be considered as made up of a great number of short rigid bodies jointed without friction, and hence may form part of a machine without vitiating the truth of (XVI.).

$\Sigma(\text{K. E.})_n$ signifies the sum obtained by adding the amounts of kinetic energy ($\frac{1}{2}dMv_n^2$ for each elementary mass) possessed by all the particles of all the rigid bodies at their final positions; $\Sigma(\text{K. E.})_0$, a similar sum at their initial positions. For example, the K. E. of a rigid body having a motion of translation of velocity v, $= \frac{1}{2}v^2 \int dM = \frac{1}{2}Mv^2$; that of a rigid body having an angular velocity ω about a fixed axis Z, $= \frac{1}{2}\omega^2 I_Z$ (§ 139); while, if it has an angular velocity ω about a gravity-axis Z, which has a velocity v_Z of translation at right angles to itself, the (K. E.) at this instant may be proved to be (§ 143)

$$\tfrac{1}{2}Mv_Z^2 + \tfrac{1}{2}\omega^2 I_Z,$$

the sum of the amounts *due to the two motions separately.*

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* These *mutual actions* consist only of *actions by contact* (pressure, rub, etc.). No magnetic or electrical attractions or repulsions are here considered.

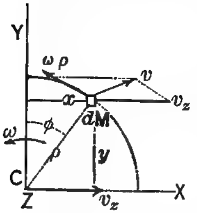

FIG. 158.

143. K. E. of Combined Rotation and Translation.—The last statement may be thus proved. Fig. 158. At a given instant the velocity of any dM is v, the diagonal formed on the velocity v_Z of translation, and the rotary velocity $\omega\rho$ relatively to the moving gravity-axis Z (perpendicular to paper) (see § 71),

$$\text{i.e., } v^2 = v_Z^2 + (\omega\rho)^2 - 2(\omega\rho)v_Z \cos \varphi;$$

hence we have K. E., at this instant,

$$= \int \tfrac{1}{2} dMv^2 = \tfrac{1}{2} v_Z^2 \int dM + \tfrac{1}{2}\omega^2 \int dM\rho^2 \; - \; \omega v_Z \int dM\rho \cos \varphi,$$

but $\rho \cos \varphi = y$, and $\int dMy = M\bar{y} = 0$, since Z is a gravity-axis,

$$\therefore \text{K. E.} = \tfrac{1}{2} M v_Z^2 + \tfrac{1}{2}\omega^2 I_Z. \quad \text{Q. E. D.}$$

It is interesting to notice that the K. E. due to rotation, viz., $\frac{1}{2}\omega^2 I_Z = \frac{1}{2}M(\omega k)^2$, is the same as if the whole mass were concentrated in a point, line, or thin shell, at a distance k, the radius of gyration, from the axis.

Example.—A solid homogeneous sphere of radius $r=6$ in. and weight $=322$ lbs. is rolling down an incline. At a certain instant the velocity of its centre is 10 ft. per sec. and hence, *if no slipping occurs*, its angular velocity about its centre is ω, $=v_Z \div r$, $=10 \div \frac{1}{2}$, $=20$ radians/sec. Consequently, at this instant (see § 103, p. 102) its *total kinetic energy* is $\frac{1}{2}(322 \div 32.2)[(10)^2 + (20)^2 \cdot \frac{2}{5}(\frac{1}{2})^2] = 700$ ft.-lbs.

144. Example of a Machine in Operation.—Fig. 159. Consider the four consecutive moving masses, M', M'', M''', and M^{iv} (being the piston; connecting-rod; fly-wheel, crank, drum, and chain; and weight on inclined plane) as *free*, collectively. Let us apply eq. (XVI.), the initial and final positions being taken when the crank-pin is at its dead-points o and n; i.e., we deal with the progress of the pieces made while the crank-pin describes its upper semicircle. Remembering that the mutual actions between any two of these four masses can be left out of account (except friction), the only forces to be put in are the actions of other bodies on each one of these four, and are shown in the figure. The only *mutual* friction considered will be at the crank-pin, and if this as an average $= F''$, the work done on it between o and $n = F''\pi r''$, where $r'' =$ radius of crank-pin. The work done by P_1 the effective steam-pressure (let it be constant) during this period is $= P_1 l'$; that done in overcoming F_1, the friction between piston and cylinder, $= F_1 l'$; that done *upon* the weight G'' of connecting-rod is cancelled by the work done *by it* in the descent following; the work done

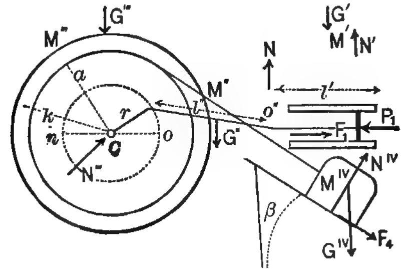

FIG. 159.

upon G^{iv}, $= G^{iv}\pi a \sin\beta$, where $a =$ radius of drum; that upon the friction F_4, $= F_4\pi a$. The pressures N, N', N^{iv}, and N''', and weights G' and G''', are neutral, i.e., do no work either positive or negative. Hence the left-hand member of (XVI.) becomes, between o and n,

$$P_1 l' - F_1 l' - F''\pi r'' - G^{iv}\pi a \sin\beta - F_4\pi a, \quad . \quad . \quad (1)$$

provided the respective distances are *actually described* by these forces, i.e., if the masses have sufficient initial kinetic energy to carry the crank-pin beyond the point of minimum velocity, with the aid of the working force P_1, whose effect is small up to that instant.

As for the total initial kinetic energy, i.e., $\Sigma(\text{K. E.})_0$, let us express it in terms of the velocity of crank-pin at o, viz., V_0. The $(\text{K.E.})_0$ of M' is nothing; that of M'', which at this instant is rotating about its right extremity (*fixed* for the instant) with angular velocity $\omega'' = V_0 \div l''$, is $\frac{1}{2}\omega''^2 I_0''$; that of M''' $= \frac{1}{2}\omega'''^2 I_C'''$, in which $\omega''' = V_0 \div r$; that of M^{iv} (translation) $= \frac{1}{2}M^{iv}v_0^{iv2}$, in which $v_0^{iv} = (a \div r)V_0$. $\Sigma(\text{K. E.})_n$ is expressed in a corresponding manner with V_n (final velocity of crank-pin) instead of V_0. Hence the right-hand member of (XVI.) will give (putting the radius of gyration of M'' about $O'' = k''$, and that of M''' about $C = k$)

$$\tfrac{1}{2}(V_n^2 - V_0^2)\left[M''\frac{k''^2}{l''^2} + M'''\frac{k^2}{r^2} + M^{iv}\frac{a^2}{r^2}\right]. \quad . \quad . \qquad (2)$$

By writing (1)=(2), we have an equation of condition, capable of solution for any one unknown quantity, to be satisfied for the extent of motion considered. It is understood that the chain is always taut, and that its weight and mass are neglected.

145. Numerical Case of the Foregoing.—(Foot-pound-second system of units for space, force, and time; this requires $g = 32.2$.)

Suppose the following data:

FEET.	LBS.	LBS.	MASS UNITS.
			(and ∴)
$l' = 2.0$	$P_1 = 6000$	$G' = 60$	$M' = 1.86$
$l'' = 4.0$	$F_1 = 200$	$G'' = 50$	$M'' = 1.55$
$a = 1.5$	F'' (av'ge) $= 400$	$G''' = 400$	$M''' = 12.4$
$r = 1.0$	$F_4 = 300$	$G^{iv} = 3220$	$M^{iv} = 100.0$
$k = 1.8$			
$k'' = 2.3$		Also let $V_0 = 4$ ft. per sec.; $\beta = 30°$	
$r'' = 0.1$			

Denote (1) by W and the large bracket in (2) by $\overline{M}$ (this by some is called the total mass "*reduced*" to the crank-pin). Putting (1) = (2) we have, solving for the unknown V_n,

$$V_n = \sqrt{\frac{2W}{\overline{M}} + V_0^2} . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

For above values,

$$W = 12.000 - 400 - 125.7 - 7590.0 - 1417.3$$
$$= 2467 \text{ foot-pounds};$$

while $\overline{M} = 0.5 + 40.3 + 225.0 = 265.8$ mass-units;

whence $V_n = \sqrt{18.56 + 16} = \sqrt{34.56} = 5.88$ ft. per second.

As to whether the crank-pin actually reaches the dead-point n, requires separate investigations to see whether V becomes zero or negative between o and n (a negative value is inadmissible, since a reversal of direction implies a different value for W), i.e., whether the proposed extent of motion is realized; and these are made by assigning some other intermediate position m, as a final one, and computing V_m, remembering that when m is not a dead-point the $(K.\ E.)_m$ of M' is not zero, and must be expressed in terms of V_m, and that the $(K.\ E.)_m$ of the connecting-rod M'' must be obtained from § 143.

146. Regulation of Machines.—As already illustrated in several examples (§ 121), a fly-wheel of sufficient mass and radius may prevent too great fluctuation of speed in a single stroke of an engine; but to prevent a permanent change, which must occur if the work of the working force or forces (such as the steam-pressure on a piston, or water-impulse in a turbine) exceeds for several successive strokes or revolutions the work required to overcome resistances (such as friction, gravity, resistance at the teeth of saws, etc., etc.) through their respective spaces, automatic governors are employed to diminish the working force, or the distance through which it acts per stroke, until the normal speed is restored; or *vice versâ*, if the speed slackens, as when new resistances are temporarily brought into play. Hence when several successive periods, strokes (or other cycle), are considered, the kinetic energy of the moving parts will disappear from eq. (XVI.), leaving it in this form:

work of working-forces = work done upon resistances.

147. Power of Motors.—In a mill where the same number of machines are run continuously at a constant speed proper for their work, turning out per hour the same number of barrels of flour, feet of lumber, or other commodity, the motor (e.g., a steam-engine, or turbine) works at a constant rate, i.e., develops a definite horse-power (H.P.), which is thus found in the case of *reciprocating steam-engines* (double-acting),

$$\text{H.P.} = \left\{\begin{array}{l}\text{total mean effective}\\ \text{steam-pressure on}\\ \text{piston in lbs.}\end{array}\right\} \times \left\{\begin{array}{l}\text{distance in feet}\\ \text{travelled by pis-}\\ \text{ton per second.}\end{array}\right\} \div 550,$$

i.e., the work (in ft.-lbs) done per second by the working force divided by 550 (see § 132). The total effective pressure at any instant is the excess of the forward over the back-pressure, and by its mean value (since steam is usually used expansively) is meant such a value P' as, multiplied by the length of stroke l, shall give

$$P'l = \int_0^l P dx,$$

where P is the variable effective pressure and dx an element of its path. If u is the number of strokes per second, we may also write (*foot-pound-second system*).

$$\text{H.P.} = P'lu \div 550 = \left[\int_0^l P dx\right] u \div 550. \quad \text{(XVII.)}$$

Very often the number of revolutions *per minute*, m, of the crank is given, and then

$$\text{H.P.} = P' \text{ (lbs.)} \times 2l \text{ (feet)} \times m \div 33{,}000.$$

If $F =$ area of piston we may also write $P' = Fp'$, where p' is the mean effective steam-pressure per unit of area. Evidently, to obtain P' in lbs., we multiply F in sq. in. by p' in lbs. per sq. in., or F in sq. ft. by p' in lbs. per sq. foot; the former is customary. p' in practice is obtained by measurements and computations from "indicator-cards" (see § 152, p. 159, where the value of P' is found by Simpson's Rule); or $P'l$, i.e., $\int_0^l P dx$, may be computed theoretically as in § 59, Problem 4.

The power as thus found is expended in overcoming the friction of all moving parts (which is sometimes a large item), and the resistances peculiar to the kind of work done by the machines. The work periodically *stored* in the increased kinetic energy of the moving masses is *restored* as they periodically resume their minimum velocities.

Example.—In the steady running of a (reciprocating) steam-engine operating a mill, the value of the mean total effective pressure on the piston is $P' = 16{,}800$ lbs. and the radius of the circle described by the crank-pin is 10 in. (so that the length of stroke is $l = 20$ inches). The fly-wheel turns at rate of 330 revs./min. Find the horse-power developed. Substituting, we find $\text{H.P.} = [16{,}800 \times 2 \times \frac{20}{12} \times 330] \div 33{,}000 = 560$ H.P.

148. Potential Energy.—There are other ways in which work or energy is stored and then restored, as follows:

First. In raising a weight G through a height h, an amount of work $= Gh$ is done *upon* G, as a *resistance*, and if at any subsequent time the weight is allowed to descend through the same vertical distance h (the form of path is of no account), G, now a *working force*, *does* the work Gh, and thus in aiding the motor repays, or restores, the Gh expended by the motor in raising it. If h is the vertical height through which the centre of gravity rises and sinks periodically in the motion of the machine, the force G may be left out of account in reckoning the expenditure of the motor's work, and the body when at its highest point is said to possess an amount Gh of **potential energy,** i.e., *energy of position*, since it is capable of doing the work Gh in sinking through its vertical range of motion.

Second. So far, all bodies considered have been by express stipulation *rigid*, i.e., incapable of changing shape. To see the effect of a lack of rigidity as affecting the principle of work and energy in machines, take the simple case in Fig. 160. A helical spring at a given instant is acted on at each end by a force P in an axial direction (they are equal, supposing the mass of the spring small). As the machine operates of which it is a member, it moves to a new consecutive position B, suffering a further elongation $d\lambda$ in its length (if P is increasing). P on the right, a working force, does the work Pdx'; how is this expended? P on the left has the work Pdx done upon it, and the mass is too small to absorb kinetic energy or to bring its weight into consideration. The remainder, $Pdx' - Pdx = Pd\lambda$, is expended in stretching the spring an additional amount $d\lambda$, and is capable of restoration if the spring retains its elasticity. Hence the work done in changing the form of bodies *if they are elastic* is said to be stored in the form of **potential energy.** That is, in the operation of machines, the name *potential energy* is also given to the energy stored and restored periodically in the changing and regaining of form of elastic bodies.

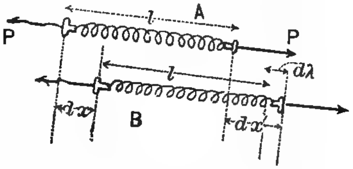

FIG. 160.

Example.—A given helical spring, when held stretched $s' = \frac{1}{2}$ ft. beyond its "natural" (or unstrained) length, exerts a pull of $R' = 1200$ lbs. at its two ends; and the "potential energy" residing in it is = *mean force* × *distance*, $= \frac{1}{2}R's'$, $= (\frac{1}{2})\ 1200\ (\frac{1}{2})$, $= 300$ ft.-lbs. If such a stretched spring be placed on a car of 644 lbs. weight on a level track and properly connected with a driving-wheel, which does not slip on the track, its recovery of natural length may be made the means of starting the car into motion and causing it to attain a final velocity of $v = 5.47$ ft./sec. (if no friction is met with); from $\frac{1}{2}(644 \div 32.2)v^2 = 300$.

149. Other Forms of Energy.—Numerous experiments with various kinds of apparatus have proved that for every 778 (about) ft.-lbs. of work spent in overcoming friction, one British unit of heat is produced (viz., the quantity of heat necessary to raise the temperature of one pound of water from 32° to 33° Fahrenheit); while from converse experiments, in which the amount of heat used in operating a steam-engine was all carefully estimated, the disappearance of a certain portion of it could only be accounted for by assuming that it had been converted into work at the same rate of (about) 778 ft.-lbs. of work to each unit of heat (or 427 kilogrammetres to each French unit of heat). This number 778 or 427, according to the system of units employed, is called the *Mechanical Equivalent of Heat*

Heat then is energy, and is supposed to be of the kinetic form due to the rapid motion or vibration of the molecules of a substance. A similar agitation among the molecules of the (hypothetical) ether diffused through space is supposed to produce the phenomena of light, electricity, and magnetism. Chemical action being also considered a method of transforming energy (its possible future occurrence as in the case of coal and oxygen being called potential energy), the well-known doctrine of the *Conservation of Energy*, in accordance with which energy is indestructible, and the doing of work is simply the conversion of one or more kinds of energy into equivalent amounts of others, is now an accepted hypothesis of physics.

Work consumed in friction, though practically lost, still remains in the universe as heat, electricity, or some other subtile form of energy.

150. Power Required for Individual Machines. Dynamometers of Transmission.—If a machine is driven by an endless belt from the main-shaft, A, Fig. 161, being the driving-pulley on the machine, the working force which drives the machine, in other words the "grip" with which the belt takes hold of the pulley tangentially, $= P - P'$, P and P' being the tensions in the "driving" and "following" sides of the belt respectively. The belt is supposed not to slip on the pulley. If v is the velocity of the pulley-circumference, the work expended on the machine per second, i.e., the *power*, is

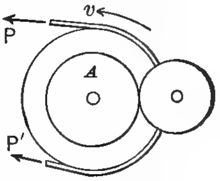

FIG. 161.

$$L=(P-P')v, \text{ ft.-lbs. per sec.} \quad . \quad . \quad . \quad . \quad (1)$$

To measure the force $(P - P')$, an apparatus called a *Dynamometer of Transmission* may be placed between the main shaft and the machine, and the belt made to pass through it in such a way as to measure the tensions P and P', or principally their difference, without meeting any resistance in so doing; that is, the power is *transmitted*, not absorbed, by the apparatus. One invention for this purpose (mentioned in the *Journal of the Franklin Institute* some years ago) is shown (*in principle*) in Fig. 162. A vertical plate carrying four pulleys and a scale-pan is first balanced on the pivot C. The belt being then adjusted, as shown, and the power turned on, a sufficient weight G is placed in the scale-pan to balance the plate again, for whose equilibrium we must have $Gb = Pa - P'a$, since the P and P' on the right *have action-lines passing through C*. The velocity of belt, v, is obtained by a simple counting device. Hence $(P - P')$ and v become known, and ∴ L from (1).

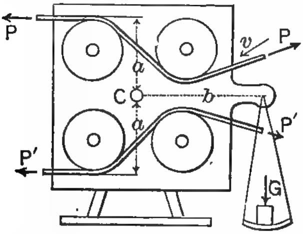

FIG. 162.

Many other forms of transmission-dynamometers are in use, some applicable whether the machine is driven by belting or gearing from the main shaft. Emerson's *Hydrodynamics* describes his own invention[*] on p. 283, and gives results of measurements with it; e.g., at Lowell, Mass., the power required to drive 112 looms, weaving 36-inch sheetings, No. 20 yarn, 60 threads to the inch, speed 130 picks to the minute, was found to be 16 H.P., i.e., $\frac{1}{7}$ H.P., to each loom (p. 335).

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[*] Prof. Flather's "*Dynamometers*" is a standard book (1907).

Example.—The endless belt connecting the pulley (running at $n=180$ revs./min., with a radius of $r=2$ ft.) of an engine shaft with that of a planing machine is led over the idle pulleys of the apparatus in Fig. 162, as there shown (engine pulley on left, and that of machine on right; but neither shown in figure). To balance the plate in position shown (with $a=2$ ft. and $b=4$ ft.) is found to require a weight $G=210$ lbs. We have, therefore, from $(P-P')\,a=Gb$, $P-P'=210\times4\div2=420$ lbs. as the net working force operating the machine; while the velocity of the belt is v, $=n\,2\pi r$, $=(180\div60)\,2\pi2=37.70$ ft./sec. Hence the power transmitted through the dynamometer of transmission is L, $=(P-P')v$, $=420\times37\;70)=15{,}834$ ft.-lbs. per sec., or 28 8 H P.

151. Dynamometers of Absorption.—These are so named since they furnish in themselves the resistance (friction or a weight) in the overcoming (or raising) of which the power is expended or absorbed. Of these the *Prony Friction Brake* is the most common, and is used for measuring the power developed by a given motor (e.g., a steam-engine or turbine) not absorbed in the friction of the motor itself. Fig. 163 shows one fitted to a vertical pulley driven by the motor. By tightening the bolt B, the velocity v of pulley-rim may be made constant at any desired value (within certain limits) by the consequent friction. v is measured by a counting apparatus, while the friction (or *tangential* components of action between pulley and brake), $=F$, becomes known by noting the weight G which must be placed in the scale pan to balance the arm between the checks; then with G' = weight of brake and b' = the horizontal distance of its center of gravity from the center of pulley, we have, for the equilibrium of the brake (moments about pulley center),

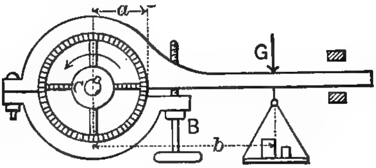

FIG. 163.

$$Fa=Gb+G'b' \; ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the work done on F per unit of time, or *power*, is

$$L=Fv, \text{ ft.-lbs. per sec.} \quad . \quad . \quad . \quad . \quad (2)$$

(In case the pulley is horizontal, a bell-crank must be interposed between the arm and the scale-pan.)

Example.—A vertical pulley of $a=2$ ft. radius and run by a turbine water-wheel, is gripped by a Prony brake, as in Fig. 163, with arm $b=4$ ft. 9 in. A load of $G=160$ lbs. is placed in the scale pan, the water turned on, and the bolt B screwed up until the friction F of pulley-rim on brake is just sufficient to lift the weight and hold the brake in equilibrium Weight of brake is $G'=40$ lbs., with centre of gravity $b'=1.5$ ft. on right of pulley centre. The speed to which the pulley now adjusts itself is at rate of 210 revs./min. The friction is $F=(Gb+Gb')\div a=(160\times 4.75+40\times 1.5)\div 2=410$ lbs.; the velocity of pulley-rim is $v=(210\div 60)\ 2\pi\times 2=44$ ft./sec.; hence the power developed is $Fv=410\times 44=18{,}040$ ft.-lbs. per sec.; or 32.8 H.P.

Note.—For an account of various modern designs of absorption and transmission dynamometers, the reader is referred to Prof. Flather's book, already mentioned in the foot-note on p. 157. This is a recent and a standard work. In the "*Notes and Examples in Mechanics*" by the present writer, brief descriptions are given (pp. 96 and 97) of the Appold and the Carpentier dynamometers of absorption, with the theory of the same; both of these are "automatic" or "self-adjusting."

It must be carefully noted by the student that in the *absorption* dynamometer, which for purposes of test temporarily takes the place of useful machines, the power is absorbed and *converted into heat*, necessitating cooling devices if the parts are to run smoothly and lubricants are to remain unaffected; whereas in the dynamometer of transmission the power simply passes through *without heating effect.*

152. The Indicator, used with steam and other fluid engines, is a special kind of dynamometer in which the automatic motion of a pencil describes a curve on paper whose ordinates are proportional to the fluid pressures exerted in the cylinder at successive points of the stroke. Thus, Fig. 164, the back-pressure being constant and $=P_b$, the ordinates P_0, P_1, etc., represent the effective pressures at equally spaced points of division. The mean effective pressure P' (see § 147) is, for this figure, by Simpson's Rule (six equal spaces),

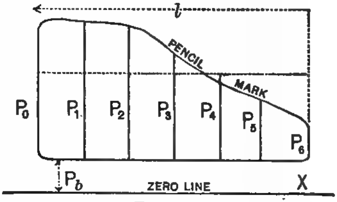

FIG. 164.

$$P'=\tfrac{1}{18}[P_0+4(P_1+P_3+P_5)+2(P_2+P_4)+P_6].$$

This gives a near approximation. The power is now found by § 147.

153. Mechanical Efficiency of a Steam or Vapor Engine (gas, petroleum, gasoline, or alcohol vapor, etc.). By the term "*mechanical efficiency*" is meant the ratio of the power obtained at the rim of the pulley or fly-wheel on the main shaft of the engine (where it would be connected with machinery to be operated or where in a test the resistance of brake-friction would be overcome) to the power exerted directly on the piston of the engine by the pressure of the fluid concerned. This latter item becomes known through the use of the "indicator" (see preceding paragraph) and is hence often called the "*indicated power;*" the power spent on friction provided by a Prony brake, for testing purposes, being called the "*brake-power.*"

Example.—If from indicator-cards the value of P', or total mean effective pressure on the piston of an engine, is found to be 12,000 lbs., the piston speed being at the (mean) rate of 6 ft. per sec., the "indicated power" of the engine is $=72,000$ ft.-lbs./sec. Now, when the engine is running under these same conditions of pressure and speed, if it is found by the use of a Prony friction brake that the power spent on brake friction consists of overcoming a friction of 6000 lbs. through 10 ft. each second, and that therefore the power obtained at the brake, or "brake power," is equal only to 60,000 ft.-lbs./sec., the *mechanical efficiency* of the engine (in this test) is $60,000 \div 72,000$, $=0.833$ or $83\frac{1}{3}$ per cent. In other words, $16\frac{2}{3}$ per cent of the power exerted by the fluid pressure on the piston, or "indicated power," is lost in the overcoming of the friction of the *engine itself*, i.e., among the moving parts situated between the piston and the rim of the test pulley.

153a. Efficiency of Power Transmission.—In transmitting power through a long line of shafting, or by ropes or belts, or water in pipes, or by electric current, the *efficiency* is the ratio of the power obtained at the receiving station to that put in at the sending station. For example,

Example.—An engine exerts power at the rate of (say) 600,000 ft.-lbs./sec., in running a "dynamo" at the sending or power station. The electric current so generated is conducted 60 miles through wires to a receiving station, where by operating an electric motor it enables a pulley to be run within a Prony brake from whose indications it is found that a power of 360,000 ft.-lbs./sec. is there obtainable. Hence the efficiency of transmission is $360,000 \div 600,000$, $=60$ per cent.

154. Boat-Rowing.—Fig. 166. During the stroke proper, let P = mean pressure on one oar-handle; hence the pressures on the foot-rest are $2P$, resistances. Let M = mass of boat and load, v_0 and v_n its velocities at beginning and end of stroke. P_1 = pressures between oar-blade and water. R = mean resistance of water to the boat's passage at this (mean) speed. These are the only (horizontal) forces to be considered as acting on the boat and two oars, considered free collectively. During the stroke the boat describes the space $s_1 = CD$, the oar-handle the space $s_2 = AB$, while the oar-blade slips backward through the small space (the "slip") $= s_1$ (average). Hence by eq. (XVI.), § 142,

$$2Ps_2 - 2Ps_3 - Rs_3 - 2P_1s_1 = \tfrac{1}{2}M(v_n^2 - v_0^2);$$

i.e., $2P(s_2-s_3)=2P\times\overline{AE}=2Ps=Rs_3+2P_1s_1+\tfrac{1}{2}M(v_n^2-v_0^2)$;

or, in words, the product of the oar-handle pressures into the distance described by them *measured on the boat*, i.e., the work done by these pressures *relatively to the boat*, is entirely accounted for in the work of slip and of liquid-resistance, and increasing the kinetic energy of the mass. (The useless work due to slip is inevitable in all paddle or screw propulsion, as well as a certain amount lost in machine-friction, not considered in the present problem.) During the "recover" the velocity decreases again to v_0. (See example on p. 97, *Notes*, etc.)

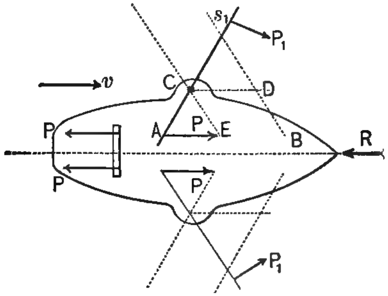

FIG. 166.

155. Examples.—1. What work is done* on a level track, in bringing up the velocity of a train weighing 200 tons, from zero to 30 miles per hour, if the total frictional resistance (at any velocity, say) is 10 lbs. per ton, and if the change of speed is accomplished in a length of 3000 feet?

(*Foot-ton-second system.*) 30 miles per hour = 44 ft. per sec. The mass

$$= 200 \div 32.2 = 6.2;$$

∴ the change in kinetic energy,

$$(= \tfrac{1}{2}Mv^2 - \tfrac{1}{2}M \times 0^2),$$
$$= \tfrac{1}{2}(6.2) \times 44^2 = 6001.6 \text{ ft.-tons.}$$

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* That is, what work is done by the pull, or tension, P, in the draw-bar between the locomotive and the "tender."

The work done in overcoming friction $=Fs$, i.e.,

$= 200 \times 10 \times 3000 = 6{,}000{,}000$ ft.-lbs. $= 3000$ ft.-tons;
$\therefore$ total work $= 6001.6 + 3000 = 9001.6$ ft.-tons.

(If the track were an up-grade, 1 in 100 say, the item of $200 \times 30 = 6000$ ft.-tons would be added.)

Example 2.—Required the rate of work, or power, in Example 1. The power is variable, depending on the velocity of the train at any instant. Assume the motion to be uniformly accelerated, then the working force is constant; call it P. The acceleration (§ 56) will be $p=v^2\div 2s=1936\div 6000=0.322$ ft. per sq. sec.; and since $P - F = Mp$, we have

$$P = 1 \text{ ton} + (200 \div 32.2) \times 0.322 = 3 \text{ tons},$$

which is $6000 \div 200 = 30$ lbs. per ton of train, of which 20 is due to its inertia, since when the speed becomes uniform the work of the engine is expended on friction alone.

Hence when the velocity is 44 ft. per sec., the engine is working at the rate of $Pv = 264{,}000$ ft.-lbs. per sec., i.e., at the rate of 480 H. P.;

At $\frac{1}{4}$ of 3000 ft. from the start, at the rate of 240 H. P., half as much;

At a uniform speed of 30 miles an hour the power would be simply $1 \times 44 = 44$ ft.-tons per sec. $= 160$ H. P.

Example 3.—The resistance offered by still water to the passage of a certain steamer at 10 knots an hour is 15,000 lbs. What power must be developed by its engines, at this uniform speed, considering no loss in "slip" nor in friction of machinery? *Ans.* 461 H. P.

Example 4.—Same as 3, except that the speed is to be 15 knots (i.e., nautical miles; each = 6086 feet) an hour, assuming that the resistances are as the square of the speed (approximately true). *Ans.* 1556 H. P.

Example 5.—Same as 3, except that 12% of the power is absorbed in the "slip" (i.e., in pushing aside and backwards the water acted on by the screw or paddle), and 8% in friction of machinery. *Ans.* 576 H. P.

Example 6.—In Example 3, if the crank-shaft makes 60 revolutions per minute, the crank-pin describing a circle of 18 inches radius, required the average* value of the tangential component of the thrust (or pull) of the connecting-rod against the crank-pin. *Ans.* 26890 lbs.

Example 7.—A solid sphere of cast-iron is *rolling* up an incline of 30°, and at a certain instant its centre has a velocity of 36 inches per second. Neglecting friction of all kinds, how much further will the ball mount the incline (see § 143)?

Ans. 0.390 ft.

Example 8.—In Fig. 163, with $b = 4$ ft. and $a = 16$ inches, it is found in one experiment that the friction which keeps the speed of the pulley at 120 revolutions per minute is balanced by a weight $G = 160$ lbs. Required the power thus measured. (With $b' = 0$.) *Ans.* 14.6 H. P.

Although in Examples 1 to 6 the steam cylinder is itself in motion, the work per stroke is still = mean effective steam-pressure on piston × length of stroke, for this is the final form to which the separate amounts of work done by, or upon, the two cylinder heads and the two sides of the piston will reduce, when added algebraically. See § 154.

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* By "average value" is meant such a value, T_m, as multiplied into the length of path described by the crank-pin per unit of time shall give the power exerted.

CHAPTER VII.

FRICTION.

156. Sliding Friction.—When the surfaces of contact of two bodies are perfectly smooth, the direction of the pressure or pair of forces between them is normal to these surfaces, i.e., to their tangent-plane; but when they are rough, and moving one on the other, the forces or actions between them incline away from the normal, each on the side opposite to the direction of the (relative) motion of the body on which it acts. Thus, Fig. 167, a block whose weight is G, is drawn on a rough horizontal table by a horizontal cord, the tension in which is P. On account of the roughness of one or both bodies the action of the table upon the block is a force P_1, inclined to the normal (which is vertical in this case) at an angle $= \varphi$ *away* from the direction of the relative velocity v. This angle φ is called the *angle of friction*, while the tangential component of P_1 is called the *friction* $= F$. The normal component N, which in this case is equal and opposite to G the weight of the body, is called the *normal pressure.*

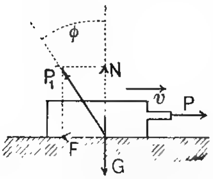

FIG. 167.

Obviously $F = N \tan \varphi$, and denoting $\tan \varphi$ by f, we have

$$F = fN. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

f is called the *coefficient of friction*, and may also be defined as the ratio of the friction F to the normal pressure N which produces it.

In Fig. 167, if the motion is accelerated (acc. $= p$), we have (eq. (IV.), § 55) $P - F = Mp$; if uniform, $P - F = 0$; from which equations (see also (1)) f may be computed. In the latter case f may be found to be different with different velocities (the surfaces retaining the same character of course), and then a uniformly accelerated motion is impossible unless $P - F$ were constant.

As for the lower block or table, forces the equals and opposites of N and F (or a single force equal and opposite to P_1) are comprised in the system of forces acting upon it.

As to whether F is a *working force* or a *resistance*, when either of the two bodies is considered free, depends on the circumstances of its motion. For example, in friction-gearing the tangential action between the two pulleys is a resistance for one, a working force for the other.

If the force P, Fig. 167, is just sufficient to start the body, or is just on the point of starting it (this will be called *impending motion*), F is called the *friction of rest.* If the body is at rest and P is not sufficient to start it, the tangential component will then be $<$ the friction of rest, viz., just $= P$. As P increases, this component continually equals it in value, and P_1 acquires a direction more and more inclined from the normal, until the instant of impending motion, when the tangential component $= fN =$ the *friction of rest.* When motion is once in progress, the friction, called then the *friction of motion*, $= fN$, in which f is not necessarily the same as in the friction of rest.

157. Variation of Friction and of the Coefficient of Friction, f.—Careful distinction must be made between the friction of *dry surfaces* and of those that are *lubricated;* though in the latter case as the supply of lubricant (oil, soap, graphite, etc.) is reduced from the extreme state of "flooding," the friction approximates in variation and magnitude to that of dry surfaces. Also, if the pressure is very great, the lubricant may be pressed out and the phenomena reduced to those of dry surfaces, which under great pressures "*seize,*" i.e., abrade, one another.

With **dry surfaces** the amount of friction, F (lbs.), depends on the nature of the materials and their initial roughness, being somewhat reduced as they become more polished, when a sliding motion has been long continued. With the surfaces in a given condition it is found (unless the pressure is very low) that increase of velocity diminishes the friction, as is unfortunately the case with railroad brakes, the friction between a brake-shoe and the rim of the car-wheel being least at the first application of the brakes, when the velocity of rubbing is greatest (see p. 168). The friction increases with the normal pressure N (the coefficient f, itself, increasing with N when N is large) and is somewhat smaller after motion begins than when motion is "impending" (friction of rest).

With **well lubricated surfaces,** however, the following may be said: The nature of the materials of the two bodies has but slight influence on the amount of friction, F, and when motion has begun, the friction is very much less than that of "impending motion." The friction is practically independent of the pressure when the lubrication is very copious (bearings "flooded") (resembling, therefore, "fluid friction;" see p. 695), the coefficient f being as small as 0.001 or under (Tower); but with more scanty lubrication conditions may approach those of dry surfaces. As to the effect of velocity (Goodman), the frictional resistance varies directly as the speed for low pressures. For high pressures, however, it is relatively great at low velocities, a minimum at about 100 ft./min., and afterwards increases approximately as the *square root* of the speed. A rise of temperature has a very important influence in diminishing the viscosity of the oil and enlarging the diameter of the bearing of a shaft more than that of the shaft itself.

In the problems of this chapter the coefficient f will be considered as constant; so that where it really varies (as when the velocity changes) an average value will be understood.

158. Experiments on Sliding Friction.—These may be made with simple apparatus. If a block of weight $= G$, Fig. 168, be placed on an inclined plane of uniformly rough surface, and the latter be gradually more and more inclined from the horizontal until the block *begins* to move, the value of β at this instant $= \varphi$, and $\tan \varphi = f =$ coefficient of friction of rest. For from $\Sigma X = 0$ we have F, i.e., fN, $= G \sin \beta$; from $\Sigma Y = 0$, $N = G \cos \beta$; whence $\tan \beta = f$, $\therefore \beta$ must $= \varphi$.

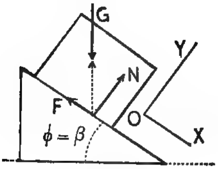

FIG. 168.

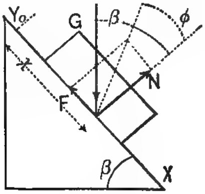

FIG. 169.

Suppose β so great that the motion is accelerated, the body starting from rest at o, Fig. 169. If it is found that the distance x varies as the square of the time, then (§ 56) the motion is uniformly accelerated (along the axis X). (Notice in the figure that G is no longer equal and opposite to P_1, the resultant of N and F, as in Fig. 168.) We have, then

$\Sigma Y = 0$, which gives $N - G \cos \beta = 0$;

$\Sigma X = Mp_1$, which gives $G \sin \beta - fN = (G \div g)p_1$;

while (from § 56) $p_1 = 2x \div t^2$. $x = \frac{1}{2} p t^2$

Hence, by elimination, x and the corresponding time t having been observed, we have for the coefficient of friction of motion

$$f = \tan \beta - \frac{2x}{gt^2 \cos \beta}$$

as an average (since the acceleration may not be uniform).

In view of § 157, it is evident that if a value β_m has been found experimentally for β such that the block, *once started by hand*, preserves a uniform motion down the plane, then, since $\tan \beta_m = f$ for friction of motion, β_m may be less than the β in Fig. 168, for friction of rest.

159. Another apparatus consists of a horizontal plane, a pulley, cord, and two weights, as shown in Fig. 59. The masses of the cord and pulley being small and hence neglected, the analysis of the problem when G is so large as to cause an accelerated motion is the same as in that example [(2) in § 57], except in Fig. 60, where the frictional resistance fN should be put in pointing toward the left. N still $= G_1$, and $\therefore$

$$S - fG_1 = (G_1 \div g)p; \quad \ldots \ldots \quad (1)$$

while for the other free body in Fig. 61 we have, as before,

$$G - S = (G \div g)p. \quad \ldots \ldots \quad (2)$$

From (1) and (2), S the cord-tension can be eliminated, and solving for p, writing it equal to $2s \div t^2$, s and t being the observed distance described (from rest) and corresponding time, we have finally for friction of motion (as an average)

$$f = \frac{G}{G_1} - \frac{G + G_1}{G_1} \cdot \frac{2s}{gt^2}. \quad \ldots \ldots \quad (3)$$

If G, Fig. 59, is made just sufficient to start the block, or sledge, G_1, we have for the friction of rest

$$f = \frac{G}{G_1}. \quad \ldots \ldots \ldots \quad (4)$$

160. Results of Experiments on Sliding Friction.—For accounts of recent experiments (and others) and deductions therefrom, the reader may consult the Engineer's "Pocket-books" of Kent and Trautwine; also Thurston's "*Friction* and *Lost Work*," Barr and Kimball's "*Machine Design*," and "*Lubrication and Lubricants*," by Archbutt and Deely. The following table gives a few values for the coefficient of friction, f, for slow motion, taken from the results obtained by Morin and others. Small changes in the condition of the surfaces may produce considerable variation in the value of f. Our knowledge is still quite incomplete in this respect.

TABLE FOR FRICTION OF SLOW MOTION.

No.	Surfaces.	Unguent.	Angle ϕ.	$f = \tan \phi$.
1	Wood on wood.	None.	14° to 26½°	0.25 to 0.50
2	Wood on wood.	Soap.	2° to 11½°	0.04 to 0.20
3	Metal on wood.	None.	26½° to 31½°	0.50 to 0.60
4	Metal on wood.	Water.	15° to 20°	0.25 to 0.35
5	Metal on wood.	Soap	11½°	0.20
6	Leather on metal.	None.	29½°	0.56
7	Leather on metal.	Greased.	13°	0.23
8	Leather on metal.	Water.	20°	0.36
9	Leather on metal.	Oil.	8½°	0.15
10	Metal on metal	None	8½° to 18°	0.15 to 0.30
11	Metal on metal	Water	18° (average)	0.30
12	Metal on metal	Oil		0.001 to 0.080

For the coefficient of *friction of rest*, the above values might be increased from 20 to 40 per cent., roughly speaking.

As showing the effect of velocity in diminishing the friction of dry surfaces, we may note that in the Galton-Westinghouse experiments with railroad brakes (cast-iron brake-shoes on steel-tired car-wheels), values for f were found as follows: When the velocity of rubbing was 10 miles per hour, $f=0.24$; for 20 miles per hour, $f=0.19$; for 30, 40, 50, and 60 miles per hour f was found to be 0.164, 0.14, 0.116, and 0.074, respectively. The foregoing values of f were obtained immediately on the application of the brake, but when the brake-shoe and wheel had been in contact some five seconds at a constant velocity, f was reduced some 20 or 30 per cent.; while for a contact of 15 seconds still further reduction ensued. The value of f for a "skidding" car-wheel (i.e., held fast by the brake pressure) sliding or "skidding" on the rail, was reduced from 0.25 for impending skidding, to 0.09 at a velocity of 7 miles per hour; and to 0.03 for 60 miles per hour. (See p. 190.)

That increasing the velocity of lubricated surfaces diminishes the coefficient of friction is well shown in the results obtained by Mr. Wellington, in 1884, with journals revolving at different speeds, viz.,

For velocity =	0.00	2.16	4.86	21.42	53.01	106 ft./min.
Coeff. f =	0.118	0.094	0.069	0.047	0.035	0.026

For a sledge on dry ground Morin found $f=0.66$. For stone on stone, see p. 555 of this book.

161. Cone of Friction.—Fig. 170. Let A and B be two rough blocks, of which B is immovable, and P the resultant of all the forces acting on A. except the pressure from B. B can furnish any required normal pressure N to balance $P \cos \beta$, but the limit of its tangential resistance is fN. So long then as β is $< \varphi$ the angle of friction, or in other words, so long as the line of action of P is *within the "cone of friction"*

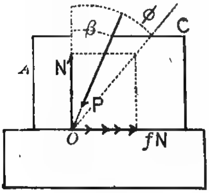

FIG. 170.

generated by revolving OC about ON, the block A will not slip on B, and the tangential resistance of B is simply $P \sin \beta$; but if β is $> \varphi$, this tangential resistance being only fN and $< P \sin \beta$, A will begin to slip, with an acceleration.

162. Problems in Sliding Friction.—In the following problems f is supposed known at points where rubbing occurs, or is impending. As to the pressure N to which the friction is due, it is generally to be considered unknown until determined by the conditions of the problem. Sometimes it may be an advantage to deal with the single unknown force P_1 (resultant of N and fN) acting in a line making the known angle φ with the normal (on the side *away* from the motion).

PROBLEM 1.—Required the value of the weight P, Fig. 171, the slightest addition to which will cause motion of the horizontal rod OB, resting on rough planes at 45°. The weight G of the rod may be applied at the middle. Consider the rod free; at each point of contact there is an unknown N and a friction due to it fN; the tension in the cord will be $= P$, since there is no acceleration and no friction at pulley. Notice the direction of the frictions, both opposing the impending motion. Take axes OX and OY as shown, and note the intersections, A and C, of the line of G with axes OX and OY. The student should not rush to the conclusion that, if G were transferred to A and there resolved into components along AO and AB, the value of N (and N_1) would be equal to that of one of these components, viz., mG (where m denotes sin 45°). Few problems in mechanics are so simple as to admit of an immediate mental solution, and guess-work should be carefully avoided.

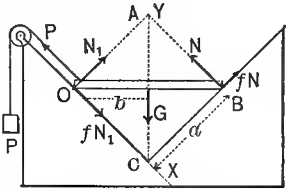

FIG. 171.

It will be found of advantage to take C as a center of moments. The line of P at O is considered as passing through point C, as also the lines of fN and fN_1. For equilibrium (impending slipping) we have, therefore,

$$\Sigma X = 0; \text{ i.e., } fN_1 + mG - N - P = 0; \quad . \quad . \quad . \quad (1)$$

$$\Sigma Y = 0; \text{ i.e., } N_1 + fN - mG = 0; \quad . \quad . \quad . \quad . \quad (2)$$

$$\Sigma(\text{moms.})_C = 0; \text{ i.e., } Na - N_1 a = 0. \quad . \quad . \quad . \quad (3)$$

The three unknowns P, N, and N_1, can now be found. From (3) we have $N=N_1$, which in (2) gives $N=\dfrac{mG}{1+f}$. Also from (1) we now find $P=2fN$; and hence finally

$$P=\frac{2mfG}{1+f}=\frac{f\sqrt{2}}{1+f}\cdot G.$$

PROBLEM 2.—Fig. 172. A rod, centre of gravity at middle, leans against a rough wall, and rests on an equally rough floor; how small may the angle α become before it slips? Let a = the half-length. The figure shows the rod free, and following the suggestion of § 162, a single unknown force P_1 making a known angle φ (whose tan $=f$) with the normal DE, is put in at D, leaning away from the direction of the impending motion, instead of an N and fN; similarly P_2 acts at C. The present system consisting of but three forces, the most direct method of finding α, without introducing the other two unknowns P_1 and P_2 at all, is to use the principle that if three forces balance, their lines of action must intersect in a point. That is, P_2 must intersect the vertical containing G, the weight, in the same point as P_1, viz., A.

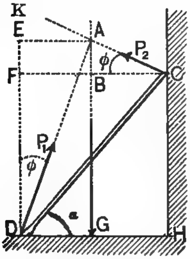

FIG. 172.

Now, since CF is ┐ to FD and the two angles ϕ are equal, CA is ┐ to DA and therefore DAC is a right triangle. We also note that if CA be prolonged to meet DF in some point K, A must be the mid-point of CK, since B is the mid-point of CF; and therefore it follows that in triangle DCK not only does DA bisect the side KC but is ┐ to it. In other words, KDC is an isosceles triangle, with DK and DC as the two equal sides, and therefore the line DA bisects the angle KDC. Hence the angle $KDC=2\phi$. That is to say, the angle α, which was to be determined, is the *complement of double the friction-angle*, or

$$\alpha=90^\circ-2\phi.$$

Problem 3.—Fig. 173. Given the resistance Q, acting parallel to the fixed guide C, the angle α, and the (equal) coefficients of friction at the rubbing surfaces, required the

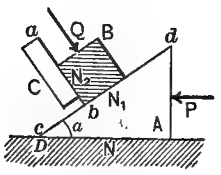

Fig. 173.

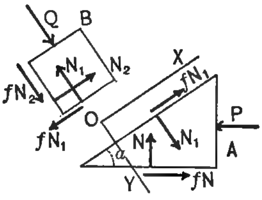

Fig. 174.

amount of the horizontal force P, at the head of the block A (or *wedge*), to overcome Q and the frictions. D is fixed, and ab is perpendicular to cd. Here we have four unknowns, viz., P, and the three pressures N, N_1, and N_2, between the blocks. Consider A and B as free bodies, separately (see Fig. 174), remembering Newton's law of action and reaction. The full values (e.g., fN) of the frictions are put in, since we suppose a slow uniform motion taking place.

For A, $\Sigma X = 0$ and $\Sigma Y = 0$ give

$$N_1 - N\cos\alpha + fN\sin\alpha - P\sin\alpha = 0\,; \quad . \quad . \quad (1)$$

$$fN_1 + N\sin\alpha + fN\cos\alpha - P\cos\alpha = 0. \quad . \quad . \quad (2)$$

For B, ΣX and ΣY give

$$Q - N_1 + fN_2 = 0\,; \ldots . \; (3) \qquad \text{and} \qquad N_2 - fN_1 = 0. \ldots (4)$$

Solve (4) for N_2 and substitute in (3), whence

$$N_1(1 - f^2) = Q. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Solve (2) for N, substitute the result in (1), as also the value of N_1 from (5), and the resulting equation contains but one unknown, P. Solving for P, putting for brevity

$$f\cos\alpha + \sin\alpha = m \quad \text{and} \quad \cos\alpha - f\sin\alpha = n,$$

we have $$P = \frac{(m + fn)Q}{(n\,.\cos\alpha + m\,.\sin\alpha)(1 - f^2)}. \quad . \quad . \quad . \quad (6)$$

or $$P = Q(m + fn) \div (1 - f^2). \quad . \quad . \quad . \quad . \quad (7)$$

i.e., $$P = \left(\frac{2f\cos\alpha}{1 - f^2} + \sin\alpha\right)Q. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Numerical Example of Problem 3.—If $Q = 120$ lbs., $f = 0.20$ (*an abstract number, and ∴ the same in any system of units*), while $\alpha = 14°$, whose sine $= 0.240$ and cosine $= .970$, then

$$P = \left[\frac{2 \times 0.2 \times .970}{1 - 0.04} + .240\right] Q$$

or,

$$P = 0.64Q = 76.8 \text{ lbs.}$$

While the wedge moves 2 inches P does the work (or exerts an energy) of $2 \times 76.80 = 153.6$ in.-lbs. $= 12.8$ ft.-lbs.

For a distance of 2 inches described by the wedge horizontally, the block B (and ∴ the resistance Q) has been moved through a distance $= 2 \times \sin 14° = 0.48$ in. along the guide C, and hence the work of $120 \times 0.48 = 57.6$ in.-lbs. has been done upon Q. Therefore for the supposed portion of the motion $153.6 - 57.6 = 96.0$ in.-lbs. of work has been lost in friction (converted into heat).

For the "mechanical efficiency" of this machine (see § 153) we have $57.6 \div 153.6 = 0.375$. Also note that for $f = 1.00$ $P = \infty$; and for $\alpha = 90°$, $P = Q$.

Problem 4. *Numerical.*—With what minimum pressure P should the pulley A be held against B, which it drives by "frictional gearing," to transmit 2 H.P.; if $\alpha = 45°$, f for impending (relative) motion, i.e., for impending slipping $= 0.40$, and the velocity of the pulley-rim is 9 ft. per second?

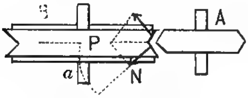

FIG. 175.

The limit-value of the tangential "grip"

$$T = 2fN = 2 \times 0.40 \times P \sin 45°,$$

$$2 \text{ H. P.} = 2 \times 550 = 1100 \text{ ft.-lbs. per second.}$$

Putting $T \times 9$ ft. $= 1100$, we have*

$$2 \times 0.40 \times \sqrt{\tfrac{1}{2}} \times P \times 9 = 1100; \therefore P = 215 \text{ lbs.}$$

Problem 6.—A block of weight G lies on a rough plane, inclined an angle β from the horizontal; find the pull P, making an angle α with the first plane, which will maintain a uniform motion *up* the plane.

---

* In this problem the student should note that, in general, when α is not 45°, we have $N = \frac{1}{2}P \div \cos \alpha$ (since in such a case the parallelogram of forces is not a square).

Problem 7.—Same as 6, except that the pull P is to permit a uniform motion *down* the plane.

Problem 8.—The thrust of a screw-propeller is 15 tons. The ring against which it is exerted has a mean radius of 8 inches, the shaft makes one revolution per second, and $f = 0.06$. Required the H. P. lost in friction from this cause.

Ans. 13.7 H. P.

163. The Bent-Lever with Friction. Worn Bearing.—Fig. 176. Neglect the weight of the lever, and suppose the plumber-block so worn that there is contact along one element only of the shaft. Given the amount and line of action of the resistance R, and the line of action of P, required the amount of the latter for impending slipping in the direction of the dotted arrow. As P gradually increases, the shaft of the lever (or gear-wheel) rolls on its bearing until the line of contact has reached some position A, when rolling ceases and slipping begins. To find A, and the value of P, note that the total action of the bearing upon the lever is some force P_1, applied at A and making a known angle φ ($f = \tan \varphi$) with the normal AC. P_1 must be equal and opposite to the resultant of the known R and the unknown P, and hence graphically (a graphic is much simpler here than an analytical solution) if we describe about C a circle of radius $= r \sin \varphi$, r being the radius of shaft (or gudgeon), and draw a tangent to it from D, we determine DA as the line of action of P_1. If DG is made $= R$, to scale, and GF drawn parallel to $D \ldots P$, P is determined, being $= DE$, while $P_1 = DF$.

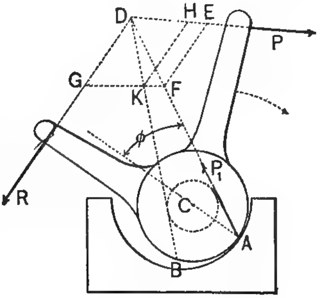

Fig. 176.

If the known force R is capable of acting as a working force, by drawing the other tangent DB from D to the "friction-circle," we have $P = DH$, and $P_1 = DK$, for impending rotation in an opposite direction.

If R and P are the tooth-pressures upon two spur-wheels, keyed upon the same shaft and nearly in the same plane, the same constructions hold good, and for a continuous uniform motion, since the friction $= P_1 \sin \varphi$,

$$\left.\begin{array}{c}\text{the work lost in friction}\\ \text{per revolution,}\end{array}\right\} = [P_1 \sin \varphi] 2\pi r.$$

It is to be remarked, that without friction P_1 would pass through C, and that the moments of R and P would balance about C (for rest or uniform rotation); whereas with friction they balance about the proper tangent-point of the friction-circle.

Another way of stating this is as follows: So long as the resultant of P and R falls within the "dead-angle" BDA, motion is impossible in either direction.

If the weight of the lever is considered, the resultant of it and the force R can be substituted for the latter in the foregoing.

164. Bent-Lever with Friction. Triangular Bearing.—Like the preceding, the gudgeon is much exaggerated in the figure (177). For impending rotation in direction of the force P, the total actions at A_1 and A_2 must lie in known directions, making angles $= \varphi$ with the respective normals, and inclined away from the slipping. Join the intersections D and L. Since the resultant of P and R at D must act along DL to balance that of P_1 and P_2, having given one force, say R, we easily find $P = DE$, while P_1 and $P_2 = LM$ and LN respectively, LO having been made $= DF$, and the parallelogram completed.

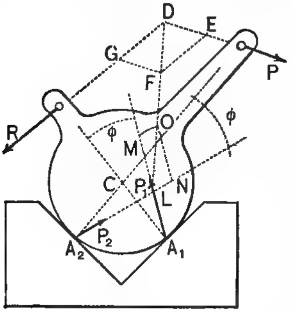

FIG. 177.

(If the direction of impending rotation is reversed, the change in the construction is obvious.) If $P_2 = 0$, the case reduces to that in Fig. 176; if the construction gives P_2 negative, the supposed contact at A_2 is not realized, and the angle A_2CA_1 should be increased, or shifted, until P_2 is positive.

As before, P and R may be the tooth-pressures on two spur-wheels nearly in the same plane and on the same shaft; if so, then, for a uniform rotation,

Work lost in fric. per revol. $= [P_1 \sin \varphi + P_2 \sin \varphi] 2\pi r$.

165. Axle-Friction.—The two foregoing articles are introductory to the subject of axle-friction. When the bearing is new, or nearly so, the elements of the axle which are in contact with the bearing are infinite in number, thus giving an infinite number of unknown forces similar to P_1 and P_2 of the last paragraph, each making an angle φ with its normal. Refined theories as to the law of distribution of these pressures are of little use, considering the uncertainties as to the value of $f\,(= \tan \varphi)$; hence for practical purposes axle-friction may be written

$$F' = f'R,$$

in which f' is a *coefficient of axle-friction* derivable from experiments with axles, and R the resultant pressure on the bearing. In some cases R may be partly due to the tightness of the bolts with which the cap of the bearing is fastened.

As before, the work lost in overcoming axle-friction *per revolution* is $= f'R2\pi r$, in which r is the radius of the axle. f', like f, is an abstract number. As in Fig. 176, a "friction-circle," of radius $= f'r$, may be considered as subtending the "dead-angle."

166. Experiments with Axle-Friction.—Prominent among ? recent experiments have been those of Professor Thurston (1872–73), who invented a special instrument for that purpose, shown (in principle only) in Fig. 178. By means of an internal spring, the amount of whose compression is read on a scale, a weighted bar or pendulum is caused to exert pressure on a projecting axle from which it is suspended. The axle is made to rotate at any desired velocity by some source of power, the axle-friction causing

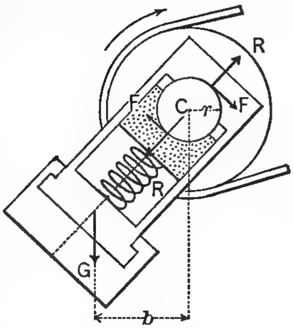

FIG. 178.

the pendulum to remain at rest at some angle of deviation from the vertical. The figure shows the pendulum free, the action of gravity upon it being G, that of the axle consisting of the two pressures,* each $= R$, and of the two frictions (each being $F = f'R$), due to them. Taking moments about C, we have for equilibrium

$$2f'Rr = Gb,$$

in which all the quantities except f' are known or observed. The temperature of the bearing is also noted, with reference to its effect on the lubricant employed. Thus the instrument covers a wide range of relations.

General Morin's experiments as interpreted by Weisbach give the following practical results: (See also p. 192).

$$\left.\begin{array}{c}\text{For iron axles, in iron or}\\ \text{brass bearings}\end{array}\right\} f' = \left\{\begin{array}{l}0.054 \text{ for well-sustained lubrication;}\\ 0.07 \text{ to } .08 \text{ for ordinary lubrication.}\end{array}\right.$$

By "pressure per square inch on the bearing" is commonly meant the quotient of the total *pressure in lbs.* by the area in *square inches* obtained by multiplying the width of the axle by the length of bearing (this length is quite commonly four times the diameter); call it p, and the velocity of rubbing in *feet per minute*, v. Then, according to Rankine, to prevent overheating, we should have

$$p(v + 20) < 44800 \ldots \text{(not homog.)}.$$

Still, in marine-engine bearings pv alone often reaches 60,000, as also in some locomotives (Cotterill). Good practice keeps p within the limit of 800 (lbs. per sq. in.) for other metals than steel (Thurston), for which 1200 is sometimes allowed.

With $v = 200$ (feet per min.) Professor Thurston found that for ordinary lubricants p should not exceed values ranging from 30 to 75 (lbs. per sq. in.).

The product pv is obviously proportional to the power expended in wearing the rubbing surfaces, per unit of area.

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* The weight G being small compared with the compressive force R in the spring, each pressure is practically equal to R.

167. Friction-Wheels—(Or, rather, *anti-friction* wheels). In Fig. 179, M and M (and two more behind) are the "friction-wheels," with axles C_1 and C_1 in fixed bearings.

G is the weight of a heavy wheel, P_1 is a known vertical resistance (tooth-pressure), and P an unknown vertical working force, whose value is to be determined to maintain a uniform rotation. The utility of the friction-wheels is also to be shown. The resultant of P_1, G, and P is a vertical force R, passing nearly through the centre C of the main axle which rolls on the four friction-wheels. R, resolved along CA and CB, produces (nearly) equal pressures, each being $N = R \div 2 \cos \alpha$, at the two axles of the friction-wheels, which rub against their fixed plumber-blocks. $R = P + P_1 + G_1$, and $\therefore$ contains the unknown P, but approximately $= G + 2P_1$, i.e., is nearly the same (in this case) whether friction-wheels are employed or not.

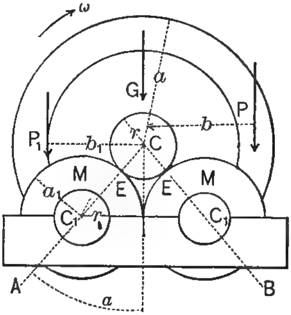


FIG. 179.

When G makes one revolution, the friction $f'N$ at each axle C_1 is overcome through a distance $= (r_1 : a_1)\, 2\pi r$, and

$$\left.\begin{array}{c}\text{Work lost per revol.}\\ \text{with}\\ \text{friction-wheels,}\end{array}\right\} = 2\, f'N \frac{r_1}{a_1} 2\pi r = \frac{r_1}{a_1}\,\frac{1}{\cos \alpha} f'R 2\pi r.$$

Whereas, if C revolved in a fixed bearing,

$$\left.\begin{array}{c}\text{Work lost per revol.}\\ \text{without}\\ \text{friction-wheels,}\end{array}\right\} = f'R 2\pi r.$$

Apparently, then, there is a saving of work in the ratio $r_1 : a_1 \cos \alpha$, but strictly the R is not quite the same in the two cases; for with friction-wheels the force P is less than without, and R depends on P as well as on the known G and P_1. By diminishing the ratio $r_1 : a_1$, and the angle α, the saving is increased. If α were so large that $\cos \alpha < r_1 : a_1$, there would be no saving, but the reverse.

As to the value of P to maintain uniform rotation, we have for equilibrium of moments about C, with friction-wheels (considering the large wheel and axle *free*),

$$Pb = P_1b_1 + 2Tr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which T is the tangential action, or "grip," between one pair of friction-wheels and the axle C which rolls upon them. T would not equal fN unless slipping took place or were impending at E, but is known by considering a pair of friction-wheels free, when $\Sigma\,(Pa)$ about C_1 gives

$$Ta_1 = f'Nr_1 = f'\frac{R}{2}\cdot\frac{r_1}{\cos\alpha},$$

which in (1) gives finally

$$P = \frac{b_1}{b}P_1 + \frac{r_1}{a_1\cos\alpha}f'R\frac{r}{b}. \quad . \quad . \quad . \quad . \quad (2)$$

Without friction-wheels, we would have

$$P = \frac{b_1}{b}P_1 + f'R\frac{r}{b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The last term in (2) is seen to be less than that in (3) (unless α is too large), in the same ratio as already found for the saving of work, supposing the R's equal.

If P_1 were on the same side of C as P, it would be of an opposite direction, and the pressure R would be diminished. Again, if P were horizontal, R would not be vertical, and the friction-wheel axles would not bear equal pressures. Since P depends on P_1, G, and *the frictions*, while the friction depends on R, and R on P_1, G, and P, an exact analysis is quite complex, and is not warranted by its practical utility.

Example.—If an empty vertical water-wheel weighs 25,000 lbs., required the force P to be applied at its circumference to maintain a uniform motion, with $a = 15$ ft., and $r = 5$ inches. Here $P_1 = 0$, and $R = G$ (nearly; neglecting the influence of P on R), i.e., $R = 25{,}000$ lbs.

First, without friction-wheels (adopting the foot-pound-second system of units), with $f' = .07$ (abstract number). From eq. (3) we have

$$P = 0 + 0.07 \times 25{,}000 \times (\tfrac{5}{12} \div 15) = 48.6 \text{ lbs.}$$

The work lost in friction per revolution is

$$f'R2\pi r = 0.07 \times 25{,}000 \times 2 \times 3.14 \times \tfrac{5}{12} = 4580 \text{ ft.-lbs.}$$

Secondly, with friction-wheels, in which $r_1 : a_1 = \frac{1}{5}$ and $\cos \alpha = 0.80$ (i.e., $\alpha = 36°$). From eq. (2)

$$P = 0 + \tfrac{1}{5} \cdot \tfrac{10}{8} \times 48.6 = \text{only } 12.15 \text{ lbs.},$$

while the work lost per revolution

$$= \tfrac{1}{5} \cdot \tfrac{10}{8} \times 4580 = 1145 \text{ ft.-lbs.}$$

Of course with friction-wheels the wheel is not so steady as without.

In this example the force P has been simply enough to overcome friction. In case the wheel is in actual use, P is the weight of water actually in the buckets at any instant, and does the work of overcoming P_1, the resistance of the mill machinery, and also the friction. By placing P_1 pointing upward on the same side of C as P, and making b_1 nearly $= b$, R will $= G$ nearly, just as when the wheel is running empty; and the foregoing numerical results will still hold good for practical purposes.

168. Friction of Pivots.—In the case of a vertical shaft or axle, and sometimes in other cases, the extremity requires support against a thrust along the axis of the axle or pivot. If the end of the pivot is *flat* and also the surface against which it rubs, we may consider the pressure, and therefore the friction, as uniform over the surface. With a flat circular pivot, then, Fig. 180, the frictions on a small sector of the circle form a system of parallel forces whose resultant is equal to their sum, and is applied a distance of $\frac{2}{3}r$ from the centre. Hence the sum of the moments of all the frictions about the centre $= fR\frac{2}{3}r$, in which R is the axial pressure. Therefore a force P necessary to overcome the friction with uniform rotation must have a moment

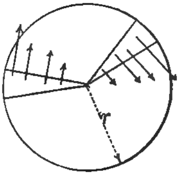


FIG. 180.

$$Pa = fR\tfrac{2}{3}r,$$

and the work lost in friction per revolution is

$$=fR2\pi \cdot \tfrac{2}{3}r = \tfrac{4}{3}\pi fRr. \quad . \quad . \quad . \quad . \quad (1)$$

As the pivot and step become worn, the resultant frictions in the small sectors probably approach the centre; for the greatest wear occurs first near the outer edge, since there the product pv is greatest (see § 166). Hence for $\frac{2}{3}r$ we may more reasonably put $\frac{1}{2}r$.

Example.—A vertical flat-ended pivot presses its step with a force of 12 tons, is 6 inches in diameter, and makes 40 revolutions per minute. Required the H. P. absorbed by the friction. Supposing the pivot and step new, and f for good lubrication $= 0.07$, we have, from eq. (1) (*foot-lb.-second*),

Work lost per revolution

$$= .07 \times 24{,}000 \times 6.28 \times \tfrac{2}{3} \cdot \tfrac{1}{4} = 1758.4 \text{ ft.-lbs.},$$

and ∴ work per second

$$= 1758.4 \times \tfrac{40}{60} \qquad\qquad = 1172.2 \text{ ft.-lbs.},$$

which ÷ 550 gives 2.13 H. P. absorbed in friction. If ordinary axle-friction also occurs its effect must be added.

If the flat-ended pivot is *hollow*, with radii r_1 and r_2, we may put $\frac{1}{2}(r_1 + r_2)$ instead of the $\frac{2}{3}r$ of the preceding.

It is obvious that the smaller the lever-arm given to the resultant friction in each sector of the rubbing surface the smaller the power lost in friction. Hence pivots should be made as small as possible, consistently with strength.

For a *conical pivot* and step, Fig. 181, the resultant friction in each sector of the conical bearing surface has a lever-arm $= \frac{2}{3}r_1$ about the axis A, and a value > than for a flat-ended pivot; for, on account of the wedge-like action of the bodies, the pressure causing friction is greater. The sum of the moments of these resultant frictions about A is the same as if only two elements of the cone received pressure (each $= N = \frac{1}{2}R \div \sin\alpha$). Hence the

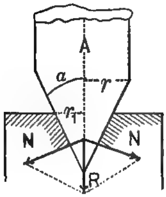


FIG. 181.

moment of friction of the pivot, i.e., the moment of the force necessary to maintain uniform rotation, is

$$Pa = f2N\frac{2}{3}r_1 = f\frac{R}{\sin\alpha}\,\frac{2}{3}r_1,$$

and work lost per revolution $= \frac{4}{3}\pi f\frac{R}{\sin\alpha}r_1.$

By making r_1 small enough, these values may be made less than those for a flat-ended pivot of the same diameter $= 2r$.

In Schiele's "anti-friction" pivots the outline is designed according to the following theory for securing uniform vertical wear. Let p = the pressure per *horizontal* unit of area (i.e., $= R \div$ horizontal projection of the *actual rubbing surface*); this is assumed constant. Let the unit of area be small, for algebraic simplicity. The friction on the rubbing surface, whose horizontal projection = unity, is $= fN = f(p \div \sin\alpha)$ (see Fig. 182; the horizontal component of p is annulled by a corresponding one opposite). The work per revolution in producing wear on this area $= fN2\pi y$. But the *vertical depth* of wear per revolution is to be the same at all parts of the surface; and this implies that the same volume of material is worn away under each horizontal unit of area. Hence $fN2\pi y$, i.e., $f\frac{p}{\sin\alpha}2\pi y$, is to be constant for all values of y; and since fp and 2π are constant, we must have, as the law of the curve,

$$\frac{y}{\sin\alpha}, \text{ i.e., } \textit{the tangent } BC = \textit{the same at all points.}$$

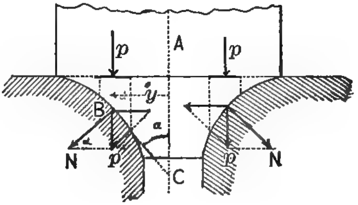


FIG. 182.

This curve is called the "*tractrix*." Schiele's pivots give a very uniform wear at high speeds. The smoothness of wear prevents leakage in the case of cocks and faucets.

169. Normal Pressure of Belting.—When a perfectly flexible cord, or belt, is stretched over a smooth cylinder, both at rest,

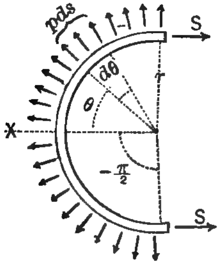


FIG. 183.

the action between them is normal at every point. As to its amount, p, per linear unit of arc, the following will determine. Consider a semicircle of the cord free, neglecting its weight. Fig. 183. The forces holding it in equilibrium are the tensions at the two ends (these are equal, manifestly, the cylinder being smooth; for they are the only two forces having moments about C, and each has the *same lever-arm*), and the normal pressures, which are infinite in number, but have an intensity, p, per linear unit, which must be constant along the curve since S is the same at all points. The normal pressure on a single element, ds, of the cord is $= pds$, and its X component $= pds \cos\theta = prd\theta \cos\theta$. Hence $\Sigma X = 0$ gives

$$rp\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\cos\theta d\theta - 2S = 0, \text{ i.e., } rp\Big[_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\sin\theta = 2S;$$

$$\therefore rp[1-(-1)] = 2S \quad \text{or} \quad p = \frac{S}{r}. \quad . \quad . \quad . \quad . \quad (1)$$

170. Belt on Rough Cylinder. Impending Slipping.—If friction is possible between the two bodies, the tension may vary along the arc of contact, so that p also varies, and consequently

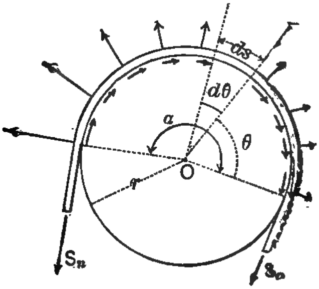


FIG. 184.

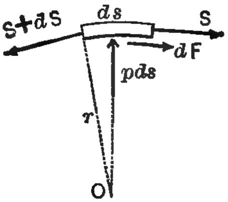


FIG. 185.

the friction on an element ds being $= fpds = f(S \div r)ds$, also varies. If *slipping is impending*, the law of variation of the tension S may be found, as follows: Fig. 184, in which the impending slipping is toward the left, shows the cord free. For any element, ds, of the cord, we have, putting Σ (moms. about O) $= 0$ (Fig. 185),

$$(S + dS)r = Sr + dFr\text{; i.e., } dF = dS,$$

or (see above) $dS = f(S \div r)ds$.

But $ds = rd\theta$; hence, after transforming,

$$fd\theta = \frac{dS}{S}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In (1) the two variables θ and S are separated; (1) is therefore ready for integration.

$$\therefore f\int_0^\alpha d\theta = \int_{S_0}^{S_n} \frac{dS}{S}\text{; i.e.,}$$

$$f\alpha = \log_e S_n - \log_e S_0 = \log_e\left[\frac{S_n}{S_0}\right]. \quad (2)$$

Or, by inversion, $$S_0 e^{f\alpha} = S_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

e, denoting the Naperian base, $= 2.71828 +$; α of course is in π-measure.

Since S_n evidently increases very rapidly as α becomes larger, S_0 remaining the same, we have the explanation of the well-known fact that a comparatively small tension, S_0, exerted by a man, is able to prevent the slipping of a rope around a pile-head, when the further end is under the great tension S_n due to the stopping of a moving steamer. For example, with $f = \frac{1}{3}$, we have (Weisbach)

$$\begin{aligned} \text{for } \alpha &= \tfrac{1}{4} \text{ turn, or } \alpha = \tfrac{1}{2}\pi, & S_n &= 1.69 S_0; \\ &= \tfrac{1}{2} \text{ turn, or } \alpha = \pi, & S_n &= 2.85 S_0; \\ &= 1 \text{ turn, or } \alpha = 2\pi, & S_n &= 8.12 S_0; \\ &= 2 \text{ turns, or } \alpha = 4\pi, & S_n &= 65.94 S_0; \\ &= 4 \text{ turns, or } \alpha = 8\pi, & S_n &= 4348.56 S_0. \end{aligned}$$

If slipping actually occurs, we must use a value of f for friction of motion.

Example.—A leather belt drives an iron pulley, covering one half the circumference. What is the limiting value of the ratio of S_n (tension on driving-side) to S_0 (tension on following side) if the belt is not to slip, taking the low value of $f = 0.25$ for leather on iron?

We have given $f\alpha = 0.25 \times \pi = .7854$, which by eq. (2) is the Naperian log. of $(S_n : S_0)$ when slipping occurs. Hence the common log. of $(S_n : S_0) = 0.7854 \times 0.43429 = 0.34109$; i.e., if

$$(S_n : S_0) = 2.193, \text{ say } 2.2,$$

the belt will (barely) slip (for $f = 0.25$).

(0.43429 is the modulus of the common system of logarithms, and $= 1 : 2.30258$. See example in § 48.)

At very high speeds the relation $p = S \div r$ (in § 169) is not strictly true, since the tensions at the two ends of an element ds are partly employed in furnishing the necessary deviating force to keep the element of the cord in its circular path, the remainder producing normal pressure.

171. Transmission of Power by Belting or Wire Rope.—In the simple design in Fig. 186, it is required to find the *motive weight* G, necessary to overcome the given resistance R at a uniform velocity $= v_1$; also the proper stationary *tension weight* G_0 to prevent slipping of the belt on its pulleys, and the amount of power, L, transmitted.

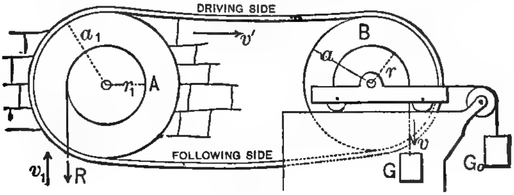


FIG. 186.

In other words,

Given: $\left\{ \begin{array}{l} R, a, r, a_1, r_1; \alpha = \pi \text{ for both pulleys,} \\ v_1; \text{ and } f \text{ for both pulleys;} \end{array} \right\}$

Required: $\left\{ \begin{array}{l} L; G, \text{ to furnish } L; G_0 \text{ for no slip; } v \text{ the velocity} \\ \text{of } G; v' \text{ that of belt; and the tensions in belt.} \end{array} \right.$

Neglecting axle-friction and the rigidity of the belting, the power transmitted is that required to overcome R through a distance $= v_1$ every second, i.e.,

$$L = Rv_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

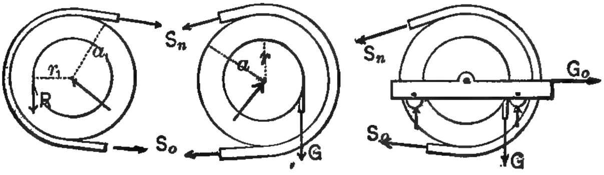
Since (if the belts do not slip)

$$a : r :: v' : v, \quad \text{and} \quad a_1 : r_1 :: v' : v_1,$$

we have

$$v' = \frac{a_1}{r_1} v_1, \quad \text{and} \quad v = \frac{r}{a} \frac{a_1}{r_1} v_1. \quad . \quad . \quad . \quad . \qquad (2)$$

Neglecting the mass of the belt, and assuming that each pulley revolves on a gravity-axis, we obtain the following, by considering the free bodies in Fig. 187:



(A free) (B free) (B and truck free)

FIG. 187.

$$\Sigma\,(\text{moms.}) = 0 \text{ in } A\ free \text{ gives } Rr_1 = (S_n - S_0)a_1; \quad . \qquad (3)$$

$$\Sigma\,(\text{moms.}) = 0 \text{ in } B \text{ free gives } Gr = (S_n - S_0)a; \quad . \qquad (4)$$

whence we readily find

$$G = \frac{a}{r} \cdot \frac{r_1}{a_1} R.$$

Evidently R and G are inversely proportional to their velocities v_1 and v; see (2). This ought to be true, since in Fig. 186 G is the only working-force, R the only resistance, and the motions are uniform; hence (from eq. (XVI.), § 142)

$$Gv - Rv_1 = 0.$$

$\Sigma X = 0$, for B and truck free, gives

$$G_0 = S_n + S_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

while, for impending slip,

$$S_n = S_0 e^{f\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

By elimination between (4), (5), and (6), we obtain

$$G_0 = G\frac{r}{a} \cdot \frac{e^{f\pi}+1}{e^{f\pi}-1} = \frac{L}{v'} \cdot \frac{e^{f\pi}+1}{e^{f\pi}-1}, \quad . \quad . \quad . \quad (7)$$

and

$$S_n = \frac{L}{v'} \cdot \frac{e^{f\pi}}{e^{f\pi}-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence G_0 and S_n vary directly as the power transmitted and inversely as the velocity of the belt. For safety G_0 should be made > the above value in (7); corresponding values of the two tensions may then be found from (5), and from the relation (see § 150)

$$(S_n - S_0)v' = L. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

These *new* values of the tensions will be found to satisfy the condition of no slip, viz.,

$$(S_n : S_0) < e^{f\pi} (\S\ 170).$$

For leather on iron, $e^{f\pi} = 2.2$ (see example in § 170), as a low value. The belt should be made strong enough to withstand S_n safely.

As the belt is more tightly stretched, and hence elongated, on the driving than on the following side, it "*creeps*" backward on the driving and forward on the driven pulley, so that the former moves slightly faster than the latter. The loss of work due to this cause does not exceed 2 per cent with ordinary belting (Cotterill).

In the foregoing it is evident that the sum of the tensions in the two sides $= G_0$, i.e., is the same, whether the power is being transmitted or not; and this is found to be true, both in theory and by experiment, when a tension-weight is not used, viz., when an initial tension S is produced in the whole belt before transmitting the power, then after turning on the latter the sum of the two tensions (driving and following) always $= 2S$, since one side elongates as much as the other contracts; it being understood that the pulley-axles preserve a constant distance apart.

172. Rolling Friction.—The few experiments which have been made to determine the resistance offered by a level road way to the uniform motion of a roller or wheel rolling upon it corroborate approximately the following theory. The word friction is hardly appropriate in this connection (except when the roadway is perfectly elastic, as will be seen), but is sanctioned by usage.

First, let the roadway or track be compressible, but *inelastic*, G the weight of the roller and its load, and P the horizontal force* necessary to preserve a uniform motion (both of translation and rotation). The track (or roller itself) being compressed just in front, and not reacting symmetrically from behind, its resultant pressure against the roller is not at O vertically under the centre, but some small distance, $OD = b$, in front. (The successive crushing of small projecting particles has the same effect.) Since for this case of motion the forces have the same relations as if balanced (see § 124), we may put Σ moms. about $D = 0$,

$$\therefore Pr = Gb; \quad \text{or,} \quad P = \frac{b}{r} G. \quad . \quad . \quad . \quad . \quad (1)$$

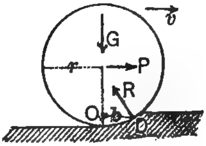


FIG. 188.

According to Professor Goodman we have the following values of b, approximately:

	Inches.		
Iron or steel wheels on iron or steel rails..	$b = 0.007$	to	0.015
“ “ “ “ “ wood............	0.06	“	0.10
“ “ “ “ “ macadam.........	0.05	“	0.20
“ “ “ “ “ soft ground.......	3.0	“	5.0
Pneumatic tires on good road, or asphalt..	0.02	“	0.022
“ “ “ heavy mud	0.04	“	0.06
Solid rubber tires on good road, or asphalt	0.04		
“ “ “ “ heavy mud.........	0.09	“	0.11

According to the foregoing theory, P, the “rolling friction” (see eq. (1)), is directly proportional to G, and inversely to the radius, if b is constant. The experiments of General Morin and others confirm this, while those of Dupuit, Poirée. and Sauvage indicate it to be proportional directly to G, and inversely to the square root of the radius.

---

* Applied at the center of the roller.

Although b is a *distance* to be expressed in linear units, and not an abstract number like the f and f' for sliding and axle-friction, it is sometimes called a "coefficient of rolling friction." In eq. (1), b and r should be expressed in the same unit.

Of course if P is applied at the top of the roller its lever-arm about D is $2r$ instead of r, with a corresponding change in eq. (1).

With ordinary railroad cars the resistance due to axle and rolling frictions combined is about 8 lbs. per ton of weight on a level track. For wagons on macadamized roads $b = \frac{1}{2}$ inch, but on soft ground from 2 to 3 inches.

Secondly, when the roadway is *perfectly elastic*. This is chiefly of theoretic interest, since at first sight no force would be considered necessary to maintain a uniform rolling motion. But, as the material of the roadway is compressed under the roller its surface is first elongated and then recovers its former state; hence some rubbing and consequent sliding friction must occur. Fig. 189 gives an exaggerated view of the circumstances, P being the horizontal force applied at the centre necessary to maintain a uniform motion. The roadway (rubber for instance) is heaped up both in front and behind the roller, O being the point of greatest pressure and elongation of the surface. The forces acting are G, P, the normal pressures, and the frictions due to them, and must form a balanced system. Hence, since G and P, and also the normal pressures, pass through C, the resultant of the frictions must also pass through C; therefore the frictions, or tangential actions, on the roller must be some forward and some backward (and not all in one direction, as seems to be asserted on p. 260 of Cotterill's Applied Mechanics, where Professor Reynolds'

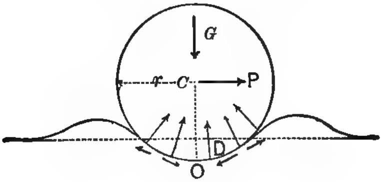


FIG. 189.

explanation is cited). The resultant action of the roadway upon the roller acts, then, through some point D, a distance $OD = b$ ahead of O, and in the direction DC, and we have as before, with D as a centre of moments,

$$Pr = Gb, \quad \text{or} \quad P = \frac{b}{r} G.$$

If rolling friction is encountered *above as well as below* the rollers, Fig. 190, the student may easily prove, by considering three separate free bodies, that for uniform motion

$$P = \frac{b + b_1}{2r} G, \quad \ldots \ldots \quad (2)$$

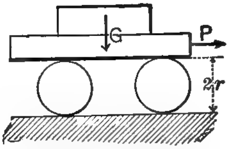


FIG. 190.

where b and b_1 are the respective "coefficients of rolling friction" for the upper and lower contacts. (See Kent's "*Pocket-Book for Mechanical Engineers*" for "friction-rollers," "ball-bearings," and "roller-bearings."

Example 1.—If it is found that a train of cars will move *uniformly* down an incline of 1 in 200, gravity being the only working force, and friction (both rolling and axle) the only resistance, required the coefficient, f', of axle-friction, the diameter of all the wheels being $2r = 30$ inches, that of the journals $2a = 3$ inches, taking $b = 0.02$ inch for the rolling friction. Let us use equation (XVI.) (§ 142), noting that while the train moves a distance s measured on the incline, its weight G does the work $G\frac{1}{200}s$, the rolling friction $\frac{b}{r}G$ (at* the axles) has been overcome through the distance s, and the axle-friction (total) through the (relative) distance $\frac{a}{r}s$ in the journal boxes; whence, the change in kinetic energy being zero,

$$\frac{1}{200} Gs - \frac{b}{r} Gs - \frac{a}{r} f' Gs = 0.$$

Gs cancels out, the ratios $b : r$ and $a : r$ are $= \frac{1}{1500}$ and $\frac{1}{10}$, respectively (being ratios or abstract numbers they have the same numerical values, whether the inch or foot is used), and solving, we have

---

* That is, the ideal resistance, at centre of axles and || to the incline, equivalent to actual rolling resistance.

$$f' = 0.05 - 0.0133 = 0.036.$$

Example 2.—How many pounds of tractive effort per ton of load would the train in Example 1 require for uniform motion on a level track? *Ans.* **10 lbs.**

173. Railroad Brakes.*—During the uniform motion of a railroad car the tangential action between the track and each wheel is small. Thus, in Example 1, just cited, if ten cars of eight wheels each make up the train, each car weighing 20 tons, the backward tangential action of the rails upon each wheel is only 25 lbs. When the brakes are applied to stop the train this action is much increased, and is the only agency by which the rails can retard the train, directly or indirectly: *directly*, when the pressure of the brakes is so great as to prevent the wheels from turning, thereby causing them to "skid" (i.e., slide) on the rails; *indirectly*, when the brake-pressure is of such a value as still to permit perfect rolling of the wheel, in which case the rubbing (and heating) occurs between the brake and wheel, and the tangential action of the rail has a value equal to or less than the friction of rest. In the first case, then (skidding), the retarding influence of the rails is the *friction of motion* between rail and wheel; in the second, a force which may be made as great as the *friction of rest* between rail and wheel. Hence, aside from the fact that skidding produces objectionable flat places on the wheel-tread, the brakes are more effective if so applied that skidding is *impending*, but not actually produced; for the friction of rest is usually greater than that of actual slipping (§ 160). This has been proved experimentally in England. The retarding effect of axle and rolling friction has been neglected in the above theory.

Example 1.—A twenty-ton car with an initial velocity of 80 feet per second (nearly a mile a minute) is to be stopped on a level within 1000 feet; required the necessary friction on each of the eight wheels.

Supposing the wheels not to skid, the friction will occur between the brakes and wheels, and is overcome through the (relative) distance 1000 feet. Eq. (XVI.), § 142, gives (foot-lb.-second system)

---

* See statement on p. 168, as to diminution of the coefficient f with speed.

$$0 - 8F \times 1000 = 0 - \frac{1}{2}\frac{40000}{32.2}(80)^2,$$

from which F (= friction at circumference of each wheel) = 496 lbs.

Note.—This result of 496 lbs. must be looked upon as only an average value. For a given pressure, N, of brake-shoe on wheel-rim on account of the variation of the coefficient f' with changing speed (see p. 168) the friction will be small at first and gradually increase. This same remark applies to Examples 3 and 4, also.

Example 2.—Suppose skidding to be impending in the foregoing, and the coefficient of friction of rest (i.e., impending slipping) between rail and wheel to be $f = 0.20$. In what distance will the car be stopped? *Ans.* 496 ft.

Example 3.—Suppose the car in Example 1 to be on an up-grade of 60 feet to the mile. (In applying eq. (XVI.) here, the weight 20 tons will enter as a resistance.) *Ans.* 439 lbs.

Example 4.—In Example 3, consider all four resistances, viz., gravity, rolling friction, and brake and axle frictions, the distance being 1000 ft., and F the unknown quantity. (Take the wheel dimensions of p. 189.) *Ans.* 414 lbs.

174. Friction of Car-journals in Brass Bearings.—(Prof. J. E. Denton, in Vol. xii Transac. Am. Soc. Mech. Engs., p. 405; also Kent's Pocket-Book.) A new brass dressed with an emery wheel, loaded with 5000 lbs., may have an actual bearing surface on the journal, as shown by the polish of a portion of the surface, of only one sq. inch. With this pressure of 5000 lbs./sq.in. the coefficient of friction may be 0.060 and the brass may be overheated, scarred, and cut; or, on the contrary, it may wear down evenly to a smooth bearing, giving a highly polished area of contact of 3 sq. in., or more, inside of two hours of running, gradually decreasing the pressure per sq. in. of contact, and showing a coefficient of friction of less than 0.005. A reciprocating motion in the direction of the axis is of importance in reducing the friction. With such polished surfaces any oil will lubricate and the coefficient of friction then depends on the viscosity of the oil. With a pressure of 1000 lbs. per sq. in., revolutions from 170 to 320 per min., and temperature of 75° to 113° Fahr., with both sperm and parraffine oils, a coefficient as low as 0.0011 has been obtained, the oil being fed continuously by a pad.

175. Well Lubricated Journals. Laws of Friction.—In the Proc. Inst. Civ. Engs. for 1886 (see also *Engineering News* for Mar. 31, April 7 and 14, 1888) Prof. Goodman presents the conclusions arrived at by him as to the laws of friction of well lubricated journals as based on the experiments made by Thurston, Beauchamp Tower, and Stroudley. They are as follows:

1. The coefficient of friction with surfaces efficiently lubricated is from $\frac{1}{6}$ to $\frac{1}{10}$ that for dry or scantily lubricated surfaces.

2. The coefficient of friction for moderate pressures and speeds varies approximately inversely as the normal pressure; the frictional resistance varies as the area in contact, the normal pressure remaining constant.

3. At very low journal speeds the coefficient of friction is abnormally high, but as the speed of sliding increases from about 10 to 100 ft. per min. the friction diminishes; and again rises when that speed is exceeded, varying approximately as the square root of the speed.

4. The coefficient of friction varies approximately inversely as the temperature, within certain limits, viz., just before abrasion takes place.

In one of Mr. Tower's experiments it was found that when the lubrication was made by a pad under the journal (which received pressure on its upper surface) the coefficient was some seven times as large as when an "oil bath," or copious supply of oil, was provided; (0.0090 as against 0.0014).

176. Rigidity of Ropes.—If a rope or wire cable passes over a pulley or sheave, a force P is required on one side greater than the resistance Q on the other for uniform motion, aside from axle-friction. Since in a given time both P and Q describe the same space s, if P is $>$ Q, then Ps is $>$ Qs, i.e., the work done by P is $>$ than that expended upon Q. This is because some of the work Ps has been expended in bending the stiff rope or cable, and in overcoming friction between the strands, both where the rope passes upon and where it leaves the pulley. With hemp ropes, Fig. 191, the material being nearly inelastic, the energy spent in bending it on at D is nearly all lost, and energy must also be spent in straightening

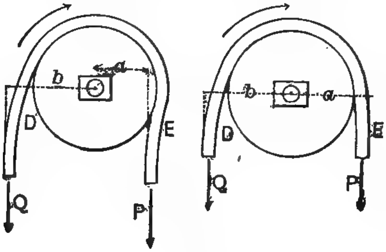


FIG. 191.

it at E; but with a wire rope or cable some of this energy is restored by the elasticity of the material. The energy spent in friction or rubbing of strands, however, is lost in both cases.

The figure shows geometrically why P must be $> Q$ for a uniform motion, for the lever-arm, a, of P is evidently $< b$ that of Q. If axle-friction is also considered, we must have

$$Pa = Qb + f'(P + Q)r,$$

r being the radius of the journal.

Experiments with cordage have been made by Prony, Coulomb, Eytelwein, and Weisbach, with considerable variation in the results and formulæ proposed. (See Coxe's translation of vol. i., Weisbach's Mechanics.)

With pulleys of large diameter the effect of rigidity is very slight. For instance, Weisbach gives an example of a pulley five feet in diameter, with which, Q being $= 1200$ lbs., $P = 1219$. A wire rope $\frac{3}{8}$ in. in diameter was used. Of this difference, 19 lbs., only 5 lbs. was due to rigidity, the remainder, 14 lbs., being caused by axle-friction. When a hemp-rope 1.6 inches in diameter was substituted for the wire one, $P - Q = 27$ lbs., of which 12 lbs. was due to the rigidity. Hence in one case the loss of work was less than $\frac{1}{2}$ of 1%, in the other about 1%, caused by the rigidity. For very small sheaves and thick ropes the loss is probably much greater.

177. Miscellaneous Examples.—*Example* 1. The end of a shaft 12 inches in diameter and making 50 revolutions per minute exerts against its bearing an axial pressure of 10 tons and a lateral pressure of 40 tons. With $f = f' = 0.05$, required the H. P. lost in friction. *Ans.* 22.2 H. P.

Example 2.—A leather belt passes over a vertical pulley, covering half its circumference. One end is held by a spring balance, which reads 10 lbs. while the other end sustains a weight of 20 lbs., the pulley making 100 revolutions per minute. Required the coefficient of friction, and the H. P. spent in overcoming the friction. Also suppose the pulley turned in the other direction, the weight remaining the same. The diameter of the pulley is 18 inches. *Ans.* $\begin{cases} f = 0.22\,; \\ 0.142 \text{ and } .284 \text{ H. P.} \end{cases}$

Example 3.—A grindstone with a radius of gyration = 12 inches has been revolving at 120 revolutions per minute, and at a given instant is left to the influence of gravity and axle friction. The axles are 1½ inches in diameter, and the wheel makes 160 revolutions in coming to rest. Required the coefficient of axle-friction. (Average.) *Ans.* $f = 0.039$.

Example 4.—A board A, weight 2 lbs., rests horizontally on another B; coefficient of friction of rest between them being $f = 0.30$. B is now moved horizontally with a uniformly accelerated motion, the acceleration being = 15 feet per "square second;" will A keep company with it, or not? *Ans.* "**No.**"

PART III.

STRENGTH OF MATERIALS.

[Or Mechanics of Materials].

CHAPTER I.

ELEMENTARY STRESSES AND STRAINS.

178. Deformation of Solid Bodies.—In the preceding portions of this work, what was called technically a "rigid body," was supposed incapable of changing its form, i.e., the positions of its particles relatively to each other, under the action of any forces to be brought upon it. This supposition was made because the change of form which must actually occur does not appreciably alter the distances, angles, etc., measured in any one body, among most of the pieces of a properly designed structure or machine. To show how the individual pieces of such constructions should be designed to avoid undesirable deformation or injury is the object of this division of Mechanics of Engineering, viz., the Strength of Materials.

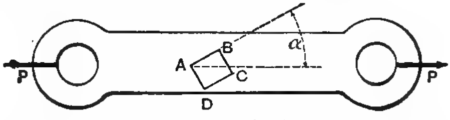


Fig. 192. § 178.

As perhaps the simplest instance of the deformation or distortion of a solid, let us consider the case of a prismatic rod in a state of tension, Fig. 192 (eye-bar of a bridge truss, e.g.). The pull at each end is P, and the body is said to be under a tension of P (lbs., tons, or other unit), not $2P$. Let $ABCD$ be the end view of an elementary parallelopiped, originally of square section and with faces at 45° with the axis of the prism. It is now deformed, the four faces perpendicular to the paper being longer* than before, while the angles BAD and BCD, originally right angles, are now smaller by a certain amount δ, ABC and ADC larger by an equal amount δ. The element is said to be in a state of **strain**, viz.: the elongation of its edges (parallel to paper) is called a **tensile strain**, while the alteration in the angles between its faces is called a *shearing strain*, or angular distortion (sometimes also called a sliding, or tangential, strain, since BC has been made to slide, relatively to AD, and thereby caused the change of angle). [This use of the word strain, to signify change of form and not the force producing it, is of recent adoption among many, though not all, technical writers.]

179. Strains. Two Kinds Only.—Just as a curved line may be considered to be made up of small straight-line elements, so the substance of any solid body may be considered to be made up of small contiguous parallelopipeds, whose angles are each 90° before the body is subjected to the action of forces, but which are not necessarily cubes. A line of such elements forming an elementary prism is sometimes called a *fibre*, but this does not necessarily imply a fibrous nature in the material in question. The system of imaginary cutting surfaces by which the body is thus subdivided need not consist entirely of planes ; in the subject of Torsion, for instance, the parallelopipedical elements considered lie in concentric cylindrical shells, cut both by transverse and radial planes.

Since these elements are taken so small that the only possible changes of form in any one of them, as induced by a system of external forces acting on the body, are elongations or contractions of its edges, and alteration of its angles, there are but two kinds of strain, **elongation** (contraction, if negative) and **shearing**.

---

* When α is nearly 0° (or 90°) BC and AD (or AB and DC) are shorter than before, on account of lateral contraction; see § 193.

180. Distributed Forces or Stresses.—In the matter preceding this chapter it has sufficed for practical purposes to consider a force as applied at a *point* of a body, but in reality it must be distributed over a definite area; for otherwise the material would be subjected to an infinite force per unit of area. (Forces like gravity, magnetic attraction, etc., we have already treated as distributed over the mass of a body, but reference is now had particularly to the pressure of one body against another, or the action of one portion of the body on the remainder.) For instance, sufficient surface must be provided between the end of a loaded beam and the pier on which it rests to avoid injury to either. Again, too small a wire must not be used to sustain a given load, or the tension per unit of area of its cross section becomes sufficient to rupture it.

Stress is distributed force, and its intensity at any point of the area is

$$p=\frac{dP}{dF} \quad . \quad . \quad . \qquad (1)$$

where dF is a small area containing the point and dP the force coming upon that area. If equal dP's (all parallel) act on equal dF's of a plane surface, the stress is said to be of uniform intensity, which is then

$$p=\frac{P}{F} \quad . \quad . \quad . \quad . \qquad (2)$$

where $P=$ total force and F the total area over which it acts. The steam pressure on a piston is an example of stress of uniform intensity.

For example, if a force $P=28800$ lbs, is uniformly distributed over a plane area of $F=72$ sq. inches, or ½ of a sq. foot, the intensity of the stress is

$$p=\frac{28800}{72}=400 \text{ lbs. per sq. inch,}$$

(or $p=28800\div\frac{1}{2}=57600$ lbs. per sq. foot, or $p=14.400\div\frac{1}{2}=28.8$ tons per sq. ft., etc.)

181. Stresses on an Element: of Two Kinds Only.—When a solid body of any material is in equilibrium under a system of forces which do not rupture it. not only is its shape altered (i.e. its elements are *strained*). and stresses produced on those planes on which the forces act, but other stresses also are induced on some or all internal surfaces which separate element from element. (over and above the forces with which the elements may have acted on each other before the application of the external stresses or "applied forces"). So long as the whole solid is the "*free body*" under consideration, these internal stresses, being the forces with which the portion on one side of an imaginary cutting plane acts on the portion on the other side, do not appear in any equation of equilibrium (for if introduced they would cancel out); but if we consider free a portion only, some or all of whose bounding surfaces are cutting planes of the original body. the stresses existing on these planes are brought into the equations of equilibrium.

Similarly, if a single element of the body is treated by itself, the stresses on all six of its faces, together with its weight, form a balanced system of forces, the body being supposed at rest.

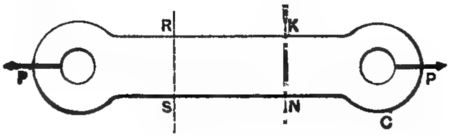


Fig. 191.

As an example of internal stress, consider again the case of a bar in tension ; Fig. 193 shows the whole bar (or eye-bar) free, the forces P being the pressures of the pins in the eyes, and causing external stress (compression here) on the surfaces of contact. Conceive a right section made through RS, far enough from the eye, C, that we may consider the internal stress to be uniform * in this section, and consider the portion RSC as a free body, in Fig. 194. The stresses on RS, now one of the bounding surfaces of the free body, must be parallel to P, i.e., normal to RS; (otherwise they would have components perpendicular to P, which is precluded by the necessity of ΣY being $= 0$, and the supposition of uniformity.) Let $F =$ the sectional area RS, and $p =$ the stress per unit of area ; then

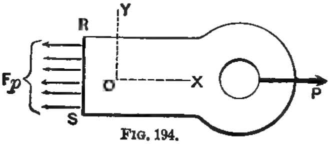


Fig. 194.

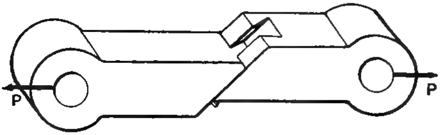


Fig. 195.

$$\Sigma Y = 0 \text{ gives } P = Fp, \text{ i.e., } p = \frac{P}{F} \quad . \quad . (2)$$

The state of internal stress, then, is such that on planes perpendicular to the axis of the bar the stress is *tensile* and *normal* (to those planes). Since if a section were made oblique to the axis of the bar, the stress would still be parallel to the axis for reasons as above, it is evident that on an oblique section, the stress has components both *normal* and *tangential* to the section, the normal component being a tension.

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* As will be shown later (§ 295) the line of the two P's in Fig. 193 must pass through the centre of gravity of the cross-section RS (plane figure) of the bar, for the stress to be uniform over the section.

The presence of the *tangential* or *shearing* stress in oblique sections is rendered evident by considering that if an oblique dove-tail joint were cut in the rod, Fig. 195, the shearing stress on its surfaces may be sufficient to overcome friction and cause sliding along the oblique plane.

If a short prismatic block is under the compressive action of two forces, each $=P$ and applied centrally in one base, we may show that the state of internal stress is the same as that of the rod under tension, except that the normal stresses are of contrary sign, i.e., compressive instead of tensile, and that the shearing stresses (or tendency to slide) on oblique planes are opposite in direction to those in the rod.

Since the resultant stress on a given internal plane of a body is fully represented by its normal and tangential components, we are therefore justified in considering but two kinds of internal stress, *normal* or *direct*, and *tangential* or *shearing*.

182. Stress on Oblique Section of Rod in Tension.—Consider free a small cubic element whose edge $=a$ in length; it has two faces parallel to the paper, being taken near the middle of the rod in Fig. 192. Let the angle which the face AB, Fig. 196, makes with the axis of the rod be $=\alpha$. This angle, for our present purpose, is considered to remain the same while the two forces P are acting, as before their action. The resultant stress on the face AB having an intensity $p=P\div F$, (see eq. 2) per unit of *transverse section* of rod, is $=p\,(a\sin\alpha)\,a$. Hence its component normal to AB is $pa^2\sin^2\alpha$; and the tangential or shearing component along $AB=pa^2\sin\alpha\cos\alpha$. Dividing by the area, a^2, we have the following:

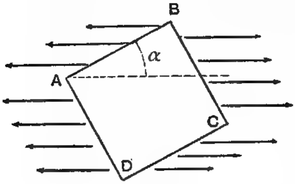

FIG. 196.

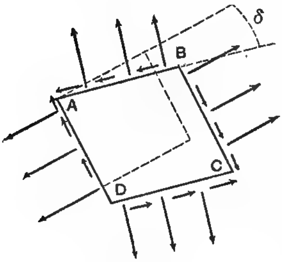

FIG. 197.

For a rod in simple tension we have, on a plane making an angle, α, with the axis:

a Normal Stress $=p \sin^2 \alpha$ per unit of area . . (1)
and a Shearing Stress $=p \sin \alpha \cos \alpha$ per unit of area . (2)

"Unit of area" here refers to the oblique plane in question, while p denotes the normal stress per unit of area of a transverse section, i.e., when $\alpha=90°$, Fig. 194.

The stresses on CD are the same in value as on AB, while for BC and AD we substitute $90°—\alpha$ for α. Fig. 197 shows these normal and shearing stresses, and also, much exaggerated, the strains or change of form of the element (see Fig. 192).

182a. Relation between Stress and Strain.—Experiment shows that so long as the stresses are of such moderate value that the piece recovers its original form completely when the external forces which induce the stresses are removed, the following is true and is known as *Hooke's Law* (stress proportional to strain). As the forces P in Fig. 193 (rod in tension) are gradually increased, the elongation, or additional length, of RK increases in the same ratio as the normal stress, p, on the sections RS and KN, per unit of area [§ 191].

As for the distorting effect of shearing stresses, consider in Fig. 197 that since

$$p \sin \alpha \cos \alpha = p \cos (90°—\alpha) \sin (90°—\alpha)$$

the shearing stress per unit of area is of *equal value on all four of the faces* (perpendicular to paper) in the elementary block, and is evidently accountable for the shearing strain, i.e., for the angular distortion, or difference, δ, between 90° and the present value of each of the four angles. According to Hooke's Law then, as P increases within the limit mentioned above, δ varies proportionally to

$$p \sin \alpha \cos \alpha, \text{ i.e. to the stress.}$$

182b. Example.—Supposing the rod in question were of a kind of wood in which a shearing stress of 200 lbs. per sq. inch along the grain, or a normal stress of 400 lbs. per sq. inch, perpendicular to a fibre-plane will produce rupture, required the value of α the angle which the grain must make with the axis that, as P increases, the danger of rupture from each source may be the same. This requires that $200:400::p \sin \alpha \cos \alpha : p \sin^2\alpha$, i.e. tan. α must $=2.000.\because \alpha = 63\frac{1}{2}°$. If the cross section of the rod is 2 sq. inches, the force P at each end necessary to produce rupture of either kind, when $\alpha = 63\frac{1}{2}°$, is found by putting $p \sin \alpha \cos \alpha = 200.\because p = 500.0$ lbs. per sq. inch. Whence, since $p = P \div F$, $P = 1000$ lbs. (Units, inch and pound.)

183. Elasticity is the name given to the property which most materials have, to a certain extent, of regaining their original form when the external forces are removed. If the state of stress exceeds a certain stage, called the **Elastic Limit**, the recovery of original form on the part of the elements is only partial, the permanent deformation being called the **Set.**

Although theoretically the elastic limit is a perfectly definite stage of stress, experimentally it is somewhat indefinite, and is generally considered to be reached when the permanent set becomes well marked as the stresses are increased and the test piece is given ample time for recovery in the intervals of rest.

The **Safe Limit** of stress, taken well within the elastic limit, determines the *working strength* or safe load of the piece under consideration. E.g., the tables of safe loads of the structural steel beams for floors, made by the Cambria Steel Co., at Johnstown, Pa., are computed on the basis that the greatest normal stress (tension or compression) occurring on any internal plane shall not exceed 16,000 lbs. per sq. inch; and, again, by the building laws of Philadelphia, the greatest shearing stress to be permitted in "web plates" of "mild steel" is 8750 lbs./in.^{2}

The Ultimate Limit is reached when rupture occurs.

184. The Modulus of Elasticity (sometimes called co-efficient of elasticity) is the number obtained by dividing the stress per unit of area by the corresponding relative strain.

Thus, a rod of wrought iron $\frac{1}{2}$ sq. inch sectional area being subjected to a tension of $2\frac{1}{2}$ tons $=5{,}000$ lbs., it is found that a length which was six feet before tension is $=6.002$ ft. during tension. The relative longitudinal strain or elongation is then$=(0.002)\div 6=1:3{,}000$ and the corresponding stress (being the normal stress on a transverse plane) has an intensity of

$$p_t = P \div F = 5{,}000 \div \tfrac{1}{2} = 10{,}000 \text{ lbs., per sq. inch.}$$

Hence by definition the modulus of elasticity is (for tension), if we denote the relative elongation by ε,

$$E_t = p_t \div \varepsilon = 10{,}000 \div \tfrac{1}{3{,}000} = 30{,}000{,}000 \text{ lbs. per sq. inch, (the}$$

sub-script "t" refers to tension).

It will be noticed that since ε is an abstract number, E_t is of the same quality as p_t, i.e., lbs. per sq. inch, or one dimension of force divided by two dimensions of length. (In the subject of strength of materials the inch is the most convenient English linear unit, when the pound is the unit of force ; sometimes the foot and ton are used together.)

The foregoing would be called *the modulus of elasticity of wrought iron in tension* in the direction of the fibre, as given by the experiment quoted. But by Hooke's Law p and ε vary together, for a given direction in a given material, hence *within the elastic limit E is constant* for a given direction in a given material. Experiment confirms this approximately.

Similarly, the modulus of elasticity for *compression* E_c in a given direction in a given material may be determined by experiments on short blocks, or on rods confined laterally to prevent flexure.

As to the modulus of elasticity for shearing, E_s, we divide the shearing stress per unit of area in the given direction by δ (in radians) the corresponding angular strain or distortion; e.g., for an angular distortion of $0.10°$ or $\delta=.00174$, and a shearing stress of 15,660 lbs. per sq. inch, we have $E=\frac{15.660}{.00174}=9{,}000{,}000$ lbs. per sq. inch.

184a. Young's Modulus is a name frequently given to E_t and E_c, it being understood that in the experiments to determine these moduli the elastic limit is not passed, and also that the rod or prism tested *is not subjected to any stress on the sides.* See p. 507.

185. Isotropes.—This name is given to materials which are homogenous as regards their elastic properties. In such a material the moduli of elasticity are individually the same for all directions. E.g., a rod of rubber cut out of a large mass will exhibit the same elastic behavior when subjected to tension, whatever its original position in the mass. Fibrous materials like wood and wrought iron are not isotropic; the direction of grain in the former must always be considered. The "piling" and welding of numerous small pieces of iron prevent the resultant forging from being isotropic.

186. Resilience refers to the potential energy stored in a body held under external forces in a state of stress which does not pass the elastic limit. On its release from constraint, by virtue of its elasticity it can perform a certain amount of work called the resilience, depending in amount upon the circumstances of each case and the nature of the material. See § 148.

187. General Properties of Materials.—In view of some definitions already made we may say that a material is *ductile* when the ultimate limit is far removed from the elastic limit; that it is *brittle* like glass and cast iron, when those limits are near together. A small modulus of elasticity means that a comparatively small force is necessary to produce a given change of form, and vice versâ, but implies little or nothing concerning the stress or strain at the elastic limit; thus Weisbach gives E_c, lbs. per sq. inch for wrought iron $=28,000,000=$ double the E_c for cast iron while the compressive stresses at the elastic limit are the same for both materials (nearly).

188. Element with Normal Stress on Sides as well as on End-Faces. Ellipse of Stress.—In Fig. 193, p. 198, the parallelopiped *RKNS* is subjected to stress on the two end-faces only. Let us now consider a small square-cornered element of material subjected to a normal stress p_1 (tension) on the two vertical end-faces, while on the horizontal side faces there acts a normal (also tensile) stress of p_2 lbs./in.^{2}; (but no stress on the vertical side faces). In Fig. 197*a* is shown, as a free body in equilibrium, a triangular prism *ABC*, which is the upper right-hand half of such an element; obtained by passing the cutting plane *AC* along a diagonal of the side plane (plane of paper) on which there is no stress, and ⊥ to it. The angle θ *may have any value* and it is desired to determine the unit stress q_0 induced on the oblique plane *AC* by the normal stresses p_1 and p_2 acting respectively on the end face *BC* and on the side face *AB*. The unit stress q_0 on the face *AC* is not ⊥ to that face but makes with it some angle μ. Let $\overline{AB}=b$ inches, $\overline{BC}=n$ in., and $\overline{AC}=c$ in.; each of the rectangular areas having a common dimension, $=d$ in., ⊥ to the paper. Then the total (oblique) stress on face *AC* is q_0cd lbs., that on *AB* is p_2bd, and that on *BC* is p_1nd lbs. Since the total stress on *AC* is the anti-resultant of the other two, and these are ⊥ to each other, we have

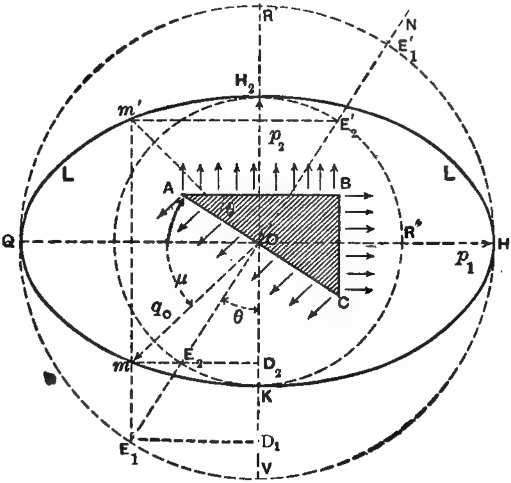

FIG. 197*a*.

$$(q_0cd)^2=(p_1nd)^2+(p_2bd)^2; \quad \text{i.e., } q_0^2=\left(p_1\frac{n}{c}\right)^2+\left(p_2\frac{b}{c}\right)^2.$$

But, since $n\div c=\sin\theta$, and $b\div c=\cos\theta$, this may be written

$$q_0^2=(p_1\sin\theta)^2+(p_2\cos\theta)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Eq. (1) gives the magnitude of q_0 for any value of angle θ; but both position and magnitude are best shown by a geometrical construction. O being any point on *AC*, draw a circle with center at O and radius, OH_1, equal by scale to the unit stress p_1. Similarly, with radius OH_2, equal (on same scale) to the unit stress p_2, draw the circle H_2E_2. Through O draw E_1ON normal to the face *AC* on which the stress q_0 is to be determined, and note the intersections E_1 and E_2 (both on left of O) with the two circles respectively. A vertical line through E_1 and a horizontal through E_2 intersect at some point m. $\overline{Om}$ is the magnitude and position of the stress q_0; since $\overline{mD_2}=\overline{E_1D_1}=p_1\sin\theta$, and $\overline{OD_2}=p_2\cos\theta$; hence from eq. (1) $\overline{Om}=q_0$..

The point m is a point in an *ellipse* whose semi-principal axes are $\overline{OH_1}$ and $\overline{OH_2}$, i.e., p_1 and p_2. This ellipse is called the **Ellipse of Stress;** Om being a semi-diameter, determined in the way indicated. (Similarly, if the elementary right parallelopiped is subjected to the action of three normal stresses, p_1, p_2, and p_3, on all *three* pairs of faces, respectively, the unit stress on any oblique plane is a semi-diameter of an *Ellipsoid of Stress*).

The unit shearing stress on the oblique face AC is $q_s=q_0 \cos \mu$; and the unit normal stress is $q=q_0 \sin \mu$.

In case the normal stress p_2 on the face AB were *compressive*, p_1 being tensile, a horizontal would be drawn through E'_2 on the circle of radius OH_2, instead of through E_2, to meet the vertical through E_1, and would thus determine Om', instead of Om, as the stress on AC. If, in such a case, p_2 were numerically equal to p_1, and θ were 45°, q_0 would $=p_1=p_2$, and would lie *in the surface AC* (*pure shear*; compare with Exam. 5, p. 242). With $p_1=p_2$, and both tensile, or both compressive, q_0 would be equal to p_1, $=p_2$, *for all values of* θ.

189. Classification of Cases.—Although in almost any case whatever of the deformation of a solid body by a balanced system of forces acting on it, normal and shearing stresses are both developed in every element which is affected at all (according to the plane section considered,) still, cases where the body is prismatic, and the external system consists of two equal and opposite forces, one at each end of the piece and directed away from each other, are commonly called cases of **Tension;** (Fig. 192); if the piece is a short prism with the same two terminal forces directed *toward* each other, the case is said to be one of **Compression**; a case similar to the last, but where the prism is quite long ("long column"), is a case of **Flexure** or bending, as are also most cases where the "applied forces" (i.e., the external forces), are not directed along the axis of the piece. Riveted joints and "pin-connections" present cases of **Shearing;** a twisted shaft one of **Torsion.** When the gravity forces due to the weights of the elements are also considered, a combination of two or more of the foregoing general cases may occur.

In each case, as treated, the principal objects aimed at are, so to design the piece or its loading that the greatest stress,* in whatever element it may occur, shall not exceed a safe value; and sometimes, furthermore, to prevent too great deformation on the part of the piece. The first object is to provide sufficient **strength**; the second sufficient **stiffness.**

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* See § 405*b* for mention of the "elongation theory" of safety. This is based on considerations of *strain*, or deformation, instead of stress

TENSION.

191. Hooke's Law by Experiment.—As a typical experiment in the tension of a long rod of ductile metal such as wrought iron and the mild steels, the following table is quoted from Prof. Cotterill's "Applied Mechanics." The experiment is old, made by Hodgkinson for an English Railway Commission, but well adapted to the purpose. From the great length of the rod, which was of wrought iron and 0.517 in. in diameter, the portion whose elongation was observed being 49 ft. 2 in. long, the small increase in length below the elastic limit was readily measured. The successive loads were of such a value that the tensile stress $p = P \div F$, or normal stress per sq. in. in the transverse section, was made to increase by equal increments of 2667.5 lbs. per sq. in., its initial value. After each application of load the elongation was measured, and after the removal of the load, the permanent set, if any.

TABLE OF ELONGATIONS OF A WROUGHT IRON ROD, OF A LENGTH = 49 FT. 2 IN.

p	λ	$\Delta\lambda$	$\varepsilon = \lambda \div l$	λ'
Load (lbs. per square inch.)	Elongation. (inches.)	Increment of Elongation.	ε, the relative elongation, (abstract number.)	Permanent Set. (inches.)
1×2667.5	.0485	.0485	0.000082	
2× "	.1095	.061	.000186	
3× "	.1675	.058	.000283	0.0015
4× "	.224	.0565	.000379	.002
5× "	.2805	.0565	.000475	.0027
6× "	.337	.0565	.000570	.003
7× "	.393	.056		.004
8× "	.452	.059	.000766	.0075
9× "	.5155	.0635		.0195
10× "	.598	.0825		.049
11× "	.760	.162		.1545
12× "	1.310	.550		.667
etc.				

Referring now to Fig. 198, the notation is evident. P is the total load in any experiment, F the cross section of the rod; hence the normal stress on the transverse section is $p=P\div F$. When the loads are increased by equal increments, the corresponding increments of the elongation λ should also be equal if Hooke's law is true. It will be noticed in the table that this is very nearly true up to the 8th loading, i.e., that $\Delta\lambda$, the difference between two consecutive values of λ, is nearly constant. In other words the proposition holds good:

$$P:P_1::\lambda:\lambda_1$$

if P and P_1 are any two loads below the 8th, and λ and λ_1 the corresponding elongations.

The permanent set is just perceptible at the 3d load, and increases rapidly after the 8th, as also the increment of elongation. Hence at the 8th load, which produces a tensile stress on the cross section of $p=8\times2667.5=21340.0$ lbs. per sq. inch, the *elastic limit* is reached.

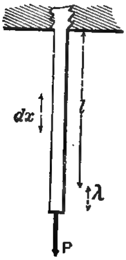

Fig. 198

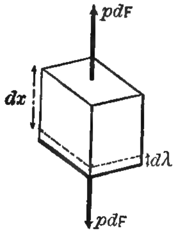

Fig. 199.

As to the state of stress of the individual elements, if we conceive such sub-division of the rod that four edges of each element are parallel to the axis of the rod, we find that it is in equilibrium between two normal stresses on its end faces (Fig. 199) of a value $=pdF=(P\div F)dF$ where dF is the horizontal section of the element. If dx was the original length, and $d\lambda$ the elongation produced by pdF, we shall have, since all the dx's of the length are equally elongated at the same time,

$$\frac{d\lambda}{dx}=\frac{\lambda}{l}$$

where $l=$total (original) length. But $d\lambda\div dx$ is the relative elongation ε, and by definition (§ 184) the *Modulus of Elasticity for Tension*, $E_{t},=p\div\varepsilon$, (*Young's Modulus*, § 184*a*).

$$\therefore E_t = \frac{p}{\frac{d\lambda}{dx}}; \text{ or } E_t = \frac{Pl}{F\lambda} \quad . \quad . \quad . \quad . \qquad (1)$$

Eq. (1) enables us to solve problems involving the elongation of a prism under tension, so long as the elastic limit is not surpassed.

The values of E_t computed from experiments like those just cited should be the same for any load under the elastic limit, if Hooke's law were accurately obeyed, but in reality they differ somewhat, especially if the material lacks homogeneity. In the present instance (see Table) we have from the

2d Exper. $E = p \div \varepsilon = 28{,}680{,}000$ lbs. per sq. in.
5th " $E =$ " $= 28{,}009{,}000$ " "
8th " $E_t =$ " $= 27{,}848{,}000$ " "

192. Stress-Strain Diagrams.—If the relative elongations or "strains" (ε) corresponding to a series of values of the tensile unit-stresses (p) (lbs./in.^{2}) to which a rod of metal has been subjected in a testing machine, are plotted as abscissæ, and the unit-stresses themselves (p) as ordinates, we have in the curve joining these points a useful graphic representation of the results of experiment.

Fig. 200 shows some of these curves, giving average results for the principal "ferrous" metals. On the left, in (I), the scale adopted (horizontal) for the "strain" (ε) or "unit-elongation" is one hundred times as great as that used in the right-hand diagram, (II); while the vertical scale for stress (p) in (I) is only twice as great as that in (II). The change of form within the elastic limit is so small compared with that beyond, that this difference in scale is quite necessary in order that diagram (I) may show what occurs within the elastic limit and a little beyond. Diagram (II) shows the remainder of the curves of wrought iron and soft steel, up to the point of rupture.

We have here the means of comparing the properties of the four typical metals represented, as to elasticity and tenacity. Up to the respective elastic limits, B, B', B'', and B''', stress is fairly proportional to strain, and a straight line is the result;

the "true elastic limit" being regarded as the point where such proportionality ceases. In the case of wrought iron and soft steel there is a point Y, called the "*yield point,*" a little above the true elastic limit, and sometimes called the "apparent elastic limit," or "commercial elastic limit," immediately beyond which further slight increments of stress produce relatively great increments of strain, permanent set becoming very marked; i.e., the part YD of the curve is almost horizontal. Beyond D the curve rises again, more steeply, but just before rupture [see (II)] may descend somewhat; since, on account of the lateral contraction mentioned in the next paragraph, here plotted, the stress being computed by dividing the total pull by the *original* sectional area, is less toward rupture than at stages closely preceding.

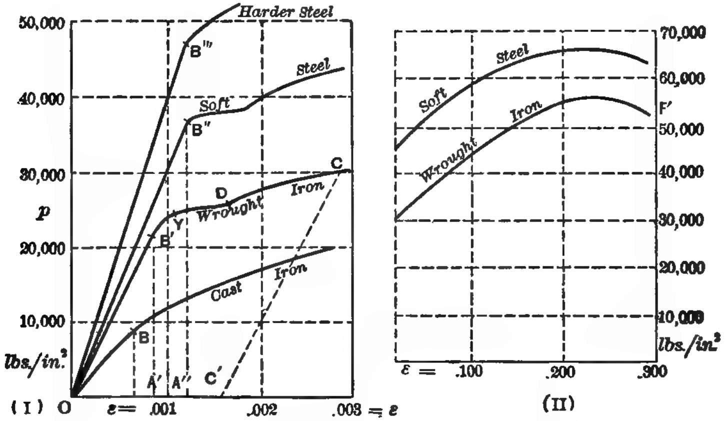

FIG. 200.

If at any point beyond the elastic limit, as at C (see curve for wrought iron) in (I), the stress be gradually removed, the relations of stress and strain during this gradual diminution of stress, are shown by the straight line CC'. The position of the point C' indicates that there is in the rod (now under no stress) a *permanent set,* or relative elongation, of $\varepsilon = 0.0015$, or 15 parts in 10,000, an elastic recovery having occurred from 0.0028 to 0.0015 (see horizontal scale).

Since by definition the modulus of elasticity $E = p \div \varepsilon$, the values of the respective moduli for the metals in diagram (I) are proportional to the tangent of the angle which the corresponding straight portions OB, OB', etc., make with the horizontal axis. From the various ordinates and abscissæ for the points B, B', etc., we find E for cast iron to be 14,000,000 lbs./in.^{2} and for the other three metals 28,000,000, 30,000,000, and 40,000,000, respectively. The curve for the "harder steel" is not shown in (II), being beyond the limits of the diagram, as to stress; and the complete curve for cast iron is contained within the limits of diagram (I), since the elongation at rupture is very small in the case of this metal, only about 3/10 of one per cent, or 3 parts in 1000; whereas that for wrought iron or soft steel is 300 parts in 1000 (or 30 per cent). In the case of cast iron the elastic limit is very ill-defined and the proportion of carbon and the mode of manufacture have much influence on its behavior under test.

"Soft steel" is another name for "structural steel," used in construction on a large scale, as in buildings and bridge trusses; "medium steel" being a somewhat harder grade of the same. Many grades of steel are made which are much stronger and harder than these, such as tool steel, nickel steel, and piano wire (whose rupturing stress may be as high as 300,000 lbs./in.^{2}). Wrought iron in the form of wire is much stronger than in bars.

Note.—Such a line as CC', showing the relation of stress and strain as the stress is gradually removed, will be called an "elasticity line" on p. 241. In § 206 some mention will be made of the phenomena of "overstraining" a test-piece of iron or steel, showing that on re-applying stress after a certain period of rest the plotted results of stress and strain relations show that the line $C'C$ is retraced to C and continues in the same straight line prolonged, to a *new elastic limit higher than* C, before curving off to the right.

193. Lateral Contraction.—In the stretching of prisms of nearly all kinds of material, accompanying the elongation of length is found also a diminution of width whose relative amount in the case of the three metals just treated is about $\frac{1}{3}$ or $\frac{1}{4}$ of the relative elongation (within elastic limit). Thus, in the third experiment in the table of § 191, this relative lateral contraction or decrease of diameter $= \frac{1}{3}$ to $\frac{1}{4}$ of ε, i.e., about 0.00008. In the case of cast iron and hard steels contraction is not noticeable except by very delicate measurements, both within and without the elastic limit; but the more ductile metals, as wrought iron and the soft steels, when stretched beyond the elastic limit show this feature of their deformation in a very marked degree. Fig. 201 shows by dotted lines the original contour of a wrought iron rod, while the continuous lines indicate that at rupture. At the cross section of rupture, whose position is determined by some local weakness, the drawing out is peculiarly pronounced.

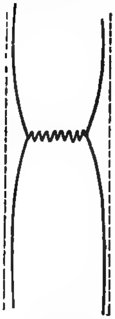

FIG. 201.

The contraction of area thus produced is sometimes as great as 50 or 60% at the fracture.

194. "Flow of Solids."—When the change in relative position of the elements of a solid is extreme, as occurs in the making of lead pipe, drawing of wire, the stretching of a rod of ductile metal as in the preceding article, we have instances of what is called the *Flow of Solids*, interesting experiments on which have been made by Tresca.

195. Moduli of Tenacity.—The tensile stress per square inch (of original sectional area) required to rupture a prism of a given material will be denoted by T and called the *modulus of ultimate tenacity*; similarly, the *modulus of safe tenacity*, or greatest safe tensile stress on an element, by T'; while the tensile stress at elastic limit may be called T''. The ratio of T' to T'' is not fixed in practice but depends upon circumstances (from ⅓ to ⅔).

Hence, if a prism of any material sustains a total pull or load P, and has a sectional area $=F$, we have

$$\left.\begin{array}{ll} P=FT & \text{for the ultimate or breaking load.} \\ P'=FT' & \text{" " safe load.} \\ P''=FT'' & \text{" " load at elastic limit.} \end{array}\right\} \quad . \quad . \quad (2)$$

Of course T' should always be less than T''. (The handbook of the Cambria Steel Co., in quoting from the building laws of various cities of the U. S., gives allowable unit-stresses for ordinary materials, both in tension and compression.)

196. Resilience of a Stretched Prism.—Fig. 202. In the gradual stretching of a prism, *fixed at one extremity*, the value of the tensile force P at the other necessarily depends on the elongation λ at each stage of the lengthening, according to the relation [eq. (1) of § 191.]

$$\lambda = \frac{Pl}{FE_t} \quad . \quad . \quad . \quad . \qquad (3)$$

within the elastic limit. (If we place a weight G on the flanges of the unstretched prism and then leave it to the action of gravity and the elastic action of the prism, the weight begins to sink, meeting an increasing pressure P, proportional to λ, from the flanges). Suppose the stretching to continue until P reaches some value P'' (at elastic limit say), and λ a value λ''. Then the work done so far is (see p. 155)

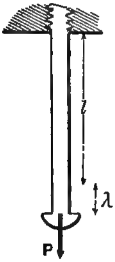

FIG. 202.

$$U = \text{mean force} \times \text{space} = \tfrac{1}{2}\, P''\, \lambda'' \quad . \quad . \qquad (4)$$

But from (2) $P'' = FT''$, and (see §§ 184 and 191)

$$\lambda'' = \varepsilon'' l$$

$$\therefore \text{(4) becomes } U = \tfrac{1}{2}\, T''\, \varepsilon''.\, Fl = \tfrac{1}{2}\, T''\, \varepsilon''\, V \quad . \quad . \qquad (5)$$

where V is the volume of the prism. The quantity $\frac{1}{2} T'' \varepsilon''$, or work done in stretching to the elastic limit a cubic inch (or other unit of volume) of the given material, may be called the *Modulus of Resilience for tension*. From (5) it appears that the amounts of work done in stretching to the elastic limit prisms of the same material but of different dimensions are proportional to *their volumes* simply.

The quantity $\frac{1}{2} T'' \varepsilon''$ is graphically represented by the area of one of the triangles such as $OA'B'$, $OA''B''$ in Fig. 200; for (in the curve for wrought iron for instance) the modulus of tenacity at elastic limit is represented by $A'B'$, and ε'' (i.e., ε for elastic limit) by OA'. The remainder of the area $OB'F'$ included between the curve and the horizontal axis, i.e., from B' to F', represents the work done in stretching a cubic unit from the elastic limit to the point of rupture, for each vertical strip having an altitude $=p$ and a width $=d\varepsilon$, has an area $=pd\varepsilon$, i.e., the work done by the stress p on one face of a cubic unit through the distance $d\varepsilon$, or increment of elongation.

If a weight or load $= G$ be "suddenly" applied to stretch the prism, i.e., placed on the flanges, barely touching them, and then allowed to fall, when it comes to rest again it has fallen through a height λ_1, and experiences at this instant some pressure P_1 from the flanges; $P_1 = ?$. Applying to this body the "Work and Energy" method (p. 138), noting that its initial and final kinetic energy are each zero and that the force G is constant, while the upward force P (from the flanges) is variable, with an average value of $\frac{1}{2}P_1$, we have

$$G\lambda_1 = \tfrac{1}{2}P_1\lambda_1 + 0 - 0\,; \text{ whence } P_1 = 2G.$$

Since $P_1 = 2G$, i.e., is $> G$, the body does not remain in this position but is pulled upward by the elasticity of the prism. In fact, the motion is *harmonic* (see §§ 59 and 138). Theoretically, the elastic limit not being passed, the oscillations should continue indefinitely.

Hence a load G "suddenly applied" occasions *double the tension* it would if compelled to sink gradually by a support underneath, which is not removed until the tension is just $= G$, oscillation being thus prevented.

If the weight G sinks through a height $=h$ before striking the flanges, Fig. 202, we shall have similarly, within elastic limit, if $\lambda_1 =$ greatest elongation, (the mass of rod being small compared with that of G).

$$G(h+\lambda_1) = \tfrac{1}{2}P_1\lambda_1 \quad . \quad . \quad . \quad . \quad (6)$$

If the elastic limit is to be just reached we have from eqs. (5) and (6), neglecting λ_1 compared with h,

$$Gh = \tfrac{1}{2}T''\varepsilon''V \quad . \quad . \quad . \quad (7)$$

an equation of condition that the prism shall not be injured.

Example.—If a steel prism have a sectional area of ¼ sq. inch and a length $l=10$ ft. $=120$ inches, what is the greatest allowable height of fall of a weight of 200 lbs., that the final tensile stress induced may not exceed $T''=$ 30,000 lbs. per sq. inch, if $\epsilon''=.002$? From (7), using the inch and pound, we have

$$h=\frac{T''\varepsilon''V}{2G}=\frac{30{,}000\times .002\times \frac{1}{4}\times 120}{2\times 200}=4.5 \text{ inches.}$$

197. Stretching of a Prism by Its Own Weight.—In the case of a very long prism such as a mining-pump rod, its weight must be taken into account as well as that of the terminal load P, see Fig. 203. At (*a.*) the prism is shown in its unstrained condition; at (*b*) strained by the load P_1 and its own weight. Let the cross section be $=F$, the heaviness of the prism $=\gamma$. Then the relative extension of any element at a distance x from o is *

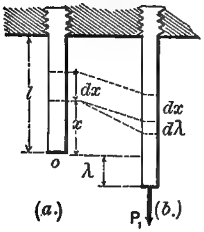

FIG. 203.

$$\varepsilon=\frac{d\lambda}{dx}=\frac{(P_1+\gamma Fx)}{FE_t} \quad . \quad . \quad . \quad (1)$$

(See eq. (1) § 191); since $P_1+F\gamma x$ is the load hanging upon the cross section at that locality. Equal dx's, therefore, are unequally elongated, x varying from 0 to l. The total elongation is

$$\lambda=\int_0^l d\lambda=\frac{1}{FE_t}\int_0^l [P_1dx+\gamma Fxdx] = \frac{P_1l}{FE_t}+\frac{\frac{1}{2}Gl}{FE_t}$$

I.e., $\lambda=$ the amount due to P_1, plus an extension which half the weight of the prism would produce, hung at the lower extremity.

---

* In $\lambda=\frac{Pl}{FE_t}$ put $d\lambda$ for λ, dx for l, and $(P_1+\gamma Fx)$ for P.

The foregoing relates to the deformation of the piece, and is therefore a problem of *stiffness*. As to the *strength* of the prism, the relative elongation $\varepsilon = d\lambda \div dx$ [see eq. (1)], which is variable, must nowhere exceed a safe value $\varepsilon' = T' \div E_t$ (from eq. (1) § 191, putting $P = FT'$, and $\lambda = \lambda'$). Now the greatest value of the ratio $d\lambda : dx$, by inspecting eq. (1), is seen to be at the upper end where $x = l$. The proper cross section F, for a given load P_1, is thus found.

$$\text{Putting } \frac{P_1 + \gamma F l}{F E_t} = \frac{T'}{E_t}, \text{ we have } F = \frac{P_1}{T' - \gamma l} \quad . \quad (2)$$

198. Solid of Uniform Strength in Tension, or *hanging body of minimum material supporting its own weight and a terminal load* P_1. Let it be a solid of revolution. If every cross-section F at a distance $= x$ from the lower extremity, bears its safe load FT', every element of the body is doing full duty, and its form is the most economical of material.

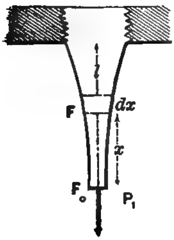

FIG. 204.

The lowest section must have an area $F_0 = P_1 \div T'$, since P_1 is its safe load. Fig. 204. Consider any horizontal lamina; its weight is $\gamma F dx$, ($\gamma =$ heaviness of the material, supposed homogenous), and its lower base F must have $P_1 + G$ for its safe load, i.e.

$$G + P_1 = FT' \quad . \quad . \quad . \quad (1)$$

in which G denotes the weight of the portion of the solid below F. Similarly for the upper base $F + dF$, we have

$$G + P_1 + \gamma F dx = (F + dF) T' \quad . \quad . \quad (2)$$

By subtraction we obtain

$$\gamma F dx = T' dF; \quad \text{i.e. } \frac{\gamma}{T'} dx = \frac{dF}{F}$$

in which the two variables x and F are separated. By integration we now have

$$\frac{\gamma}{T'}\int_0^x dx = \int_{F_0}^{F}\frac{dF}{F}\ ;\ \text{or}\ \frac{\gamma x}{T'} = \log_e \frac{F}{F_0} \quad . \quad . \quad (3)$$

$$\text{i.e., } F = F_0 e^{\frac{\gamma x}{T'}} = \frac{P_1}{T'}\ e^{\frac{\gamma x}{T'}} \quad . \ . \ . \ . \ . \ (4)$$

from which F may be computed for any value of x.

The *weight* of the portion below any F is found from (1) and (4); i.e.

$$G = P_1\left(e^{\frac{\gamma x}{T'}} - 1\right); \quad . \ . \ . \ . \ . \ . \ (5)$$

while the total extension λ will be

$$\lambda = \varepsilon'' \frac{T'}{T''} l \quad . \ . \ . \ . \ . \ . \ . \ . \ (6)$$

the relative elongation $d\lambda \div dx$ being the same for every dx and bearing the same ratio to ε'' (at elastic limit), as T' does to T''.

199. Tensile Stresses Induced by Temperature.—If the two ends of a prism are immovably fixed, when under no strain and at a temperature t, and the temperature is then lowered to a value t', the body suffers a tension proportional to the fall in temperature (within elastic limit). If for a rise or fall of 1° Fahr. (or Cent.) a unit of length of the material would change in length by an amount η (called the co-efficient of expansion) a length $=l$ would be contracted an amount $\lambda = \eta l(t - t')$ during the given fall of temperature if one end were free. Hence, if this contraction is prevented by fixing both ends, the rod must be under a tension P, equal in value to the force which would be necessary to produce the elongation λ, just stated, under ordinary circumstances at the lower temperature.

From eq. (1) §191, therefore, we have for this tension due to fall of temperature

$$P = \frac{E_t F}{l} \eta l (t - t') = E_t F (t - t') \eta$$

For 1° Cent. we may write

For Cast iron $\eta = .0000111$;
" Wrought iron $= .0000120$;
" Steel $= .0000108$ to $.0000114$;
" Copper $\eta = .0000172$;
" Zinc $\eta = .0000300$.

COMPRESSION OF SHORT BLOCKS.

200. Short and Long Columns.—In a prism in tension, its own weight being neglected, all the elements between the localities of application of the pair of external forces producing the stretching are in the same state of stress, if the external forces act axially (excepting the few elements in the immediate neighborhood of the forces; these suffering local stresses dependent on the manner of application of the external forces), and the prism may be of any length without vitiating this statement. But if the two external forces are directed *toward* each other the intervening elements will not all be in the same state of compressive stress unless the prism is comparatively short (or unless numerous points of lateral support are provided). A long prism will buckle out sideways, thus even inducing tensile stress, in some cases, in the elements on the convex side.

Hence the distinction between *short blocks* and *long columns.* Under compression the former yield by crushing or splitting, while the latter give way by flexure (i.e. bending). *Long columns,* then will be treated separately in a subsequent chapter. In the present section the blocks treated being about three or four times as long as wide, all the elements will be considered as being under equal compressive stresses at the same time.

201. Notation for Compression.—By using a subscript *c*, we may write

E_c = Modulus of Elasticity;* i.e. the quotient of the compressive stress per unit of area divided by the relative shortening. (*Young's Modulus;* no stress on sides);

C = Modulus of crushing; i.e. the force per unit of sectional area necessary to rupture the block by crushing;

C' = Modulus of safe compression, a safe compressive stress per unit of area; and

C'' = Modulus of compression at elastic limit.

For the absolute and relative shortening in length we may still use λ and ε, respectively, and within the elastic limit may write equations similar to those for tension, F being the sectional area of the block and P one of the terminal forces, while p = compressive stress per unit of area of F, viz.:

$$E_c = \frac{p}{\varepsilon} = \frac{P \div F}{d\lambda \div dx} = \frac{P \div F}{\lambda \div l} = \frac{Pl}{F\lambda} \quad . \quad . \quad . \quad . \quad (1)$$

within the elastic limit.

Also for a short block

$$\left.\begin{array}{l} \text{Crushing force } = FC \\ \text{Compressive force at elastic limit } = FC'' \\ \text{Safe compressive force } = FC' \end{array}\right\} \quad . \quad (2)$$

202. Remarks on Crushing.—As in § 182 for a tensile stress, so for a compressive stress we may prove that a shearing stress $=p.\sin\alpha\cos\alpha$ is produced on planes at an angle α with the axis of the short block, p being the compressive stress per unit of area of transverse section. Experiment shows, however, that, although the above value for the shearing stress is a maximum for $\alpha=45°$, in the crushing of short blocks of rather brittle materials like cast iron and stone, the surface along which separation takes place makes an angle smaller than 45° with the axis (35° for cast iron, according to Hodgkinson's experiments); but the block must be two or three times as long as wide to enable this phenomenon to take place. This seems to show that the presence of the compressive stress on the 45° plane is sufficient to strengthen the material against rupture by shearing on that plane, causing the separation to occur along a plane on which the compressive stress is considerably less. Crushing by splitting into pieces parallel to the axis sometimes occurs.

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* [NOTE.—It must be remembered that the modulus of elasticity, whether for normal or shearing stresses, is a number indicative of *stiffness*, not of strength, and has no relation to the elastic limit (except that experiments to determine it must not pass that limit).]

Blocks of *ductile* material, however, yield by swelling out, or bulging, laterally, resembling plastic bodies somewhat in this respect.

The elastic limit is more difficult to locate than in tension, but seems to have a position corresponding to that in tension, in the case of wrought iron and steel. With cast iron, however, the relative compression at elastic limit is about double the relative extension (at elastic limit in tension), but the force producing it is also double. For all three metals it is found that $E_c=E_t$ quite nearly, so that the single symbol E may be used for both.

EXAMPLES IN TENSION AND COMPRESSION.

203. Tables for Tension and Compression.—The round numbers in the following tables are to be taken as rude averages only; the scope and design of the present work admitting of nothing more. For abundant details of the more important experiments and researches of recent years, the reader is referred to Professor J. B. Johnson's "Materials of Construction" and the works of Professors Thurston, Burr, and Lanza; also to "Testing of Materials" by Unwin, and Martens' work of similar title. Another column might have been added giving the Modulus of Resilience, viz.: $\frac{1}{2}\varepsilon''T''$, $(=\frac{1}{2}T''^2\div E$; see § 196). ε is an abstract number, and $=\lambda \div l$, while E_t, T'', and T are given in pounds per square inch:

TABLE OF THE MODULI, ETC., OF MATERIALS IN TENSION.

Material.	ε'' (Elastic limit.)	ε At Rupture.	E_t Mod. of Elast.	T'' Elastic limit.	T Rupture.
	abst. number.	abst. number.	lbs. per sq. in.	lbs. per sq. in.	lbs. per sq. in.
Soft Steel,	.00120	.3000	30,000,000	35,000	60,000
Hard Steel,	.00200	.0500	40,000,000	60,000	120,000
Cast Iron,	.00066	.0020	14,000,000	9,000	18,000
Wro't Iron,	.00080	.3000	28,000,000	22,000	45 000 to 60,000
Brass,	.00100		10,000,000	7,000 to 19,000	16,000 to 50,000
Glass,			9,000,000		3,500
Wood, with the fibres,	.00200 to .01100	.0070 to .0150	200,000 to 2,000,000	3,000 to 19,000	6,000 to 28,000
Hemp rope,					7,000

[N.B.—Expressed in *kilograms per square centim.*, E_t, T and T'' would be numerically about $^1/_{14}$ as large as above, while ϵ and ϵ'' would be unchanged.]

TABLE OF MODULI, ETC.; COMPRESSION OF SHORT BLOCKS.

Material.	ε'' Elastic limit.	ε At Rupture.	E_c Mod. of Elast.	C'' Elastic limit.	C Rupture.
	abst. number	abst. number.	lbs. per sq. in.	lbs. per sq. in.	lbs. per sq. in.
Soft Steel,	0.00100		30,000,000	30,000	
Hard Steel,	0.00120	0.3000	40,000,000	50,000	200,000
Cast Iron,	0.00150		14,000,000	20,000	90,000
Wro't Iron,	0.00080	0.3000	28,000,000	24,000	40,000
Glass,					20,000
Granite,			See § 213a		10,000
Sandstone,					5,000
Brick,					3,000
Wood, with the fibres,		0.0100 to 0.0400	350,000 to 2,000,000		2,000 to 10,000
Portland Cement,			(§ 213a)		4,000

204. Examples. No. 1.—A bar of tool steel, of sectional area $=0.097$ sq. inches, is ruptured by a tensile force of 14,000 lbs. A portion of its length, originally ½ a foot, is now found to have a length of 0.532 ft. Required T, and ε at rupture. Using the inch and pound as units (as in the foregoing tables) we have $T=\frac{14.000}{.097}=144326$ lbs. per sq. in.; (eq. (2) § 195); while

$$\varepsilon=(0.532-0.5)\times12\div(0.50\times12)=0.064.$$

EXAMPLE 2.—Tensile test of a bar of "Hay Steel" for the Glasgow Bridge, Missouri. The portion measured was originally 3.21 ft. long and 2.09 in. × 1.10 in. in section. At the elastic limit P was 124,200 lbs., and the elongation was 0.064 ins. Required E_t, T'', and ε'' (for elastic limit).

$$\varepsilon''=\frac{\lambda}{l}=\frac{0.064}{3.21\times12}=.00166 \text{ at elastic limit.}$$

$$T''=124,200\div(2.09\times1.10)=54,000 \text{ lbs. per sq. in.}$$

$$E_t=\frac{p}{\varepsilon}=\frac{P}{F\varepsilon}=\frac{124,200}{2.30\times.00166}=32,520,000 \text{ lbs. per sq. in.}$$

Nearly the same result for E_t would probably have been obtained for values of p and ε below the elastic limit.

The *Modulus of Resilience* of the above steel (see § 196) would be $\frac{1}{2}\,\varepsilon''\,T''=44.82$ inch-pounds of work per cubic inch of metal, so that the whole work expended in stretching to the elastic limit the portion above cited is

$$U=\tfrac{1}{2}\,\varepsilon''\,T''\,V=3968. \text{ inch-lbs.}$$

An equal amount of work will be done by the rod in recovering its original length.

EXAMPLE 3.—A hard steel rod of ½ sq. in. section and 20 ft. long is under no stress at a temperature of 130°

Cent., and is provided with flanges so that the slightest contraction of length will tend to bring two walls nearer together. If the resistance to this motion is 10 tons how low must the temperature fall to cause any motion? η being $=.0000110$ (Cent. scale). From § 199 we have, expressing P in lbs. and F in sq. inches, since $E_t = 40{,}000{,}000$ *lbs. per sq. inch,*

$10 \times 2{,}000 = 40{,}000{,}000 \times \frac{1}{2} \times (130 - t') \times 0.000011$; whence $t' = 39.0°$ Centigrade.

EXAMPLE 4.—If the ends of an iron beam bearing 5 tons at its middle rest upon stone piers, required the necessary bearing surface at each pier, putting C' for stone $=200$ lbs. per sq. inch. 25 sq. in., Ans.

EXAMPLE 5.—How long must a wrought iron wire* be, supported vertically at its upper end, to break with its own weight? 216,000 inches, Ans.

EXAMPLE 6.—One voussoir (or block) of an arch-ring presses its neighbor with a force of 50 tons, the joint having a surface of 5 sq. feet; required the compression per sq. inch. 138.8 lbs. per sq. in.; Ans.

205. Factor of Safety.—When, as in the case of stone, the value of the stress at the elastic limit is of very uncertain determination by experiment, it is customary to refer the value of the safe stress to that of the ultimate by making it the n'th portion of the latter. n is called a *factor of safety*, and should be taken large enough to make the safe stress come within the elastic limit. For stone, n should not be less than 10, i.e. $C' = C \div n$; (see Ex. 6, just given).

206. Practical Notes.—It was discovered independently by Commander Beardslee and Prof. Thurston, in 1873, that if wrought iron rods were strained considerably beyond the elastic limit and allowed to remain free from stress for at least one day thereafter, a second test would show *higher limits* both elastic and ultimate.

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* Take $T = 60{,}000$ lbs. per square inch.

In 1899 Mr. James Muir discovered that this recovery of elasticity and raising of both the yield-point and ultimate strength, in the case of iron and steel after "overstraining," may be brought about by simply heating the metal *for a few minutes* in a bath of boiling water. In one experiment a bar of a kind of mild steel which under ordinary tests broke at 39 tons/in.^{2} with 20% elongation on 8 in., was stretched just to its yield-point, then relieved and heated for a few minutes to 100° Cent., then stretched just to its new yield-point, then relieved and heated as before; and so on, for three times more. The first yield-point was at 27, the others at 33, 38, 43¼, and 47 tons/in.^{2} The bar was then broken at 49 tons/in.^{2} with total extension of 12%. The diminished ultimate extension shows the hardening effect of the treatment. (See Prof. Ewing's "*Strength of Materials*," pp. 38 and 40.)

By *fatigue of metals* we understand the fact, recently discovered by Wöhler in experiments made for the Prussian Government, that rupture may be produced by causing the stress on the elements to vary repeatedly between two limiting values, the highest of which may be considerably below T (or C), the number of repetitions necessary to produce rupture being dependent both on the range of variation and the higher value.

For example, in the case of Phœnix iron in tension, rupture was produced by causing the stress to vary from 0 to 52,800 lbs. per sq. inch, 800 times; also, from 0 to 44,000 lbs. per sq. inch 240,853 times; while 4,000,000 variations between 26,400 and 48,400 per sq. inch did not cause rupture. Many other experiments were made and the following conclusions drawn (among others):

Unlimited repetitions of variations of stress (lbs. per sq. in.) between the limits given below will not injure the metal (Prof. Burr's Materials of Engineering).

Wrought iron.	From 17,600	Comp.	to 17,600	Tension.
	" 0		to 33,000	"
Axle Cast Steel.	From 30,800	Comp.	to 30,800	Tension.
	" 0		to 52,800	"
	" 38500	Tens.	to 88,000	"

(See p. 232 for an *addendum* to this paragraph.)

SHEARING.

207. Rivets.—The angular distortion called shearing strain in the elements of a body, is specially to be provided for in the case of *rivets* joining two or more plates. This distortion is shown, in Figs. 205 and 206, in the elements near the plane of contact of the plates, much exaggerated.

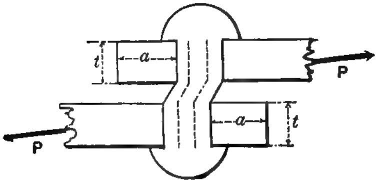

FIG. 205.

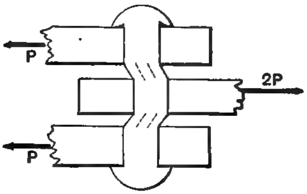

FIG. 206.

In Fig. 205 (a lap-joint) the rivet is said to be in *single shear*; in Fig. 206 in *double shear.* If P is just great enough to shear off the rivet, the *modulus of ultimate shearing*, which may be called S, (being the shearing force per unit of section when rupture occurs) is

$$S=\frac{P}{F}=\frac{P}{\frac{1}{4}\pi d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which F= the cross section of the rivet, its diameter being $=d$. For safety a value $S'=\frac{1}{4}$ to $\frac{1}{6}$ of S should be taken for metal, in order to be within the elastic limit.

As the width of the plate is diminished by the rivet hole the remaining sectional area of the plate should be ample to sustain the tension P, or $2P$, (according to the plate considered, see Fig. 206), P being the safe shearing force for the rivet. Also the thickness t of the plate should be such that the side of the hole shall be secure against crushing; P must not be $> C'td$, Fig. 205.

Again, the distance a, Fig. 205, should be such as to prevent the tearing or shearing out of the part of the plate between the rivet and edge of the plate.

For economy of material the seam or joint should be no more liable to rupture by one than by another, of the four modes just mentioned. The relations which must then subsist will be illustrated in the case of the "butt-joint" with two cover-plates, Fig. 207. Let the dimensions be denoted as in the figure and the total tensile force on the joint be $=Q$. Each rivet (see also Fig. 206) is exposed in each of two of its sections to a shear of $\frac{1}{12}$ Q, hence for safety against shearing of rivets we put

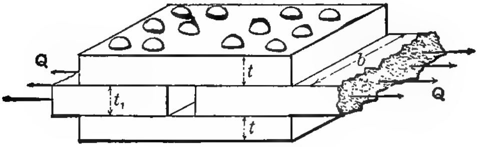

FIG. 207.

$$\tfrac{1}{12}Q = \tfrac{1}{4}\pi d^2 S' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Along one row of rivets in the main plate the sectional area for resisting tension is reduced to $(b-3d)t_1$, hence for safety against rupture of that plate by the tension Q, we put

$$Q = (b-3d)t_1 T' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equations (1) and (2) suffice to determine d for the rivets and t_1 for the main plates, Q and b being given; but the values thus obtained should also be examined with reference to the compression in the side of the rivet hole, i.e., $\frac{1}{6}Q$ must not be $> C' t_1 d$. [The distance a, Fig. 205, to the edge of the plate is recommended by different authorities to be from d to $3d$.]

Similarly, for the cover-plate we must have

$$\tfrac{1}{2}Q \overset{=}{\underset{<}{\text{or}}} (b-3d)tT' \quad . \quad . \quad . \quad . \quad (3)$$

and $\frac{1}{12}Q$ not $> C'td$.

If the rivets do not fit their holes closely, a large margin should be allowed in practice. Again, in boiler work, the *pitch*, or distance between centers of two consecutive rivets may need to be smaller, to make the joint steam-tight, than would be required for strength alone.

208. Shearing Distortion.—The change of form in an element due to shearing is an angular deformation and will be measured in π-measure. This angular change or difference between the value of the corner angle during strain and $\frac{1}{2}\pi$, its value before strain, will be called δ, and is proportional (within elastic limit) to the shearing stress per unit of area, p_s, existing on all the four faces whose angles with each other have been changed.

Fig. 208. (See § 181). By § 184 the **Modulus of Shearing Elasticity** is the quotient obtained by dividing p_s by δ; i.e. (*elastic limit not passed*),

$$E_s = \frac{p_s}{\delta} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or inversely,

$$\delta = p_s \div E_s. \quad . \quad . \quad . \quad . \quad . \quad (1)'$$

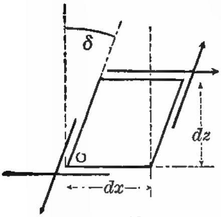

FIG. 208.

The value of E_s for different substances is most easily determined by experiments on torsion in which shearing is the most prominent stress.* (This prominence depends on the position of the bounding planes of the element considered; e.g., in Fig. 208, if another element were considered within the one there shown and with its planes at 45° with those of the first, we should find tension alone on one pair of opposite faces, compression alone on the other pair.) It will be noticed that shearing stress cannot be present on two opposite faces only, but exists also on another pair of faces (those perpendicular to the stress on the first), forming a couple of equal and opposite moment to the first, this being necessary for the equilibrium of the element, even when tensile or compressive stresses are also present on the faces considered.

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* For instance, see numerical example on p. 237, giving a value of E_s as resulting from a torsion test made by students in the Civil Engineering Laboratory at Cornell University, April, 1904.

209. Shearing Stress is Always of the Same Intensity on the Four Faces of an Element.—(By *intensity* is meant *per unit of area;* and the four faces referred to are those perpendicular to the paper in Fig. 208, the shearing stress being parallel to the paper.)

Let dx and dz be the width and height of the element in Fig. 208, while dy is its thickness perpendicular to the paper. Let the *intensity* of the shear on the right hand face be $=q_s$, that on the top face $=p_s$. Then for the element as a free body, taking moments about the axis O perpendicular to paper, we have

$$q_s\,dz\,dy \times dx - p_s\,dx\,dy \times dz = 0 \;\therefore\; q_s = p_s$$

(dx and dz being the respective lever arms of the forces $q_s\,dz\,dy$ and $p_s\,dx\,dy$.)

Even if there were also tensions (or compressions) on one or both pairs of faces their moments about O would balance (or fail to do so by a differential of a higher order) independently of the shears, and the above result would still hold.

210. Table of Moduli for Shearing.

Material.	δ'' i.e. δ at elastic limit.	E_s Mod. of Elasticity for Shearing.	S'' (Elastic limit.)	S (Rupture.)
	arc in radians.	lbs. per sq. in.	lbs. per sq. in	lbs. per sq. in.
Soft Steel,		9,000,000	30,000	70,000
Hard Steel,	0.0032	14,000,000	45,000	90,000
Cast Iron,	0.0021	7,000,000	15,000	30,000
Wrought Iron,	0.0022	9,000,000	20,000	50,000
Brass,		5,000,000		
Glass,				
Wood, across fibre,				1,500 to 8,000
Wood, along fibre,				500 to 1,200

As in the tables for tension and compression, the above values are averages. The true values may differ from these as much as 30 per cent. in particular cases, according to the quality of the specimen.

211. Punching rivet holes in plates of metal requires the overcoming of the shearing resistance along the convex surface of the cylinder punched out. Hence if d = diameter of hole, and t = the thickness of the plate, the necessary force for the punching, the surface sheared being $F=t\pi d$, is

$$P=St\pi d \quad . \quad . \quad . \quad . \quad (2)$$

Another example of shearing action is the "stripping" of the threads of a screw, when the nut is forced off longitudinally without turning, and resembles punching in its nature.

212. E and E$_s$; Theoretical Relation.—In case a rod is in tension within the elastic limit, the relative (linear) lateral contraction (let this $=m$) is so connected with E_t and E_s that if two of the three are known the third can be deduced theoretically. This relation is proved as follows, by Prof. Burr. Taking an elemental cube with four of its faces at 45° with the axis of the piece, Fig. 209, the axial half-diagonal AD becomes of a length $AD'=AD+\varepsilon.\overline{AD}$ under stress, while the transverse half diagonal contracts to a length $B'D'=AD-m.AD$. The angular distortion δ

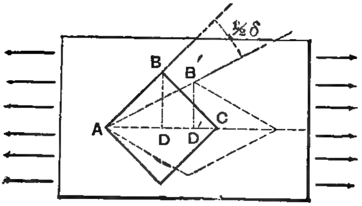

FIG. 209. § 212.

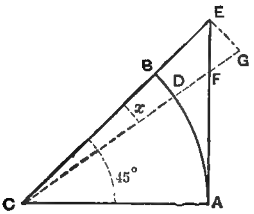

FIG. 210.

is supposed very small compared with 90° and is due to the shear p_s per unit of area on the face BC (or BA). From the figure we have

$$\tan(45^\circ - \frac{\delta}{2}) = \frac{B'D'}{AD'} = \frac{1-m}{1+\varepsilon} = 1-m-\varepsilon, \text{ approx.}$$

[But, Fig. 210, $\tan(45^\circ - x) = 1-2x$ nearly, where x is a small angle, for, taking $\overline{CA}$= unity $= AE$, $\tan AD = AF = AE - EF$. Now approximately $\overline{EF} = \overline{EG}.\sqrt{2}$ and $\overline{EG} = \overline{BD}\sqrt{2} = x\sqrt{2}$. $\therefore AF = 1-2x$ nearly.] Hence

$$1-\delta = 1-m-\varepsilon; \text{ or } \delta = m+\varepsilon \quad . \quad . \quad (2)$$

Eq. (2) holds good whatever the stresses producing the deformation, but in the present case of a rod in tension, if it is an isotrope, and if p = tension per unit of area on its transverse section, (see § 182, putting $\alpha = 45°$), we have $E_t = p \div \varepsilon$ and $E_s = (p_s \text{ on } BC) \div \delta = \frac{1}{2}p \div \delta$. Putting also $(m:\varepsilon) = k$, whence $m = k\varepsilon$, eq. (2) may finally be written*

$$\frac{1}{2E_s} = (k+1)\frac{1}{E_t}; \text{ i.e., } E_s = \frac{E_t}{2(1+k)} \quad . \quad . \quad (3)$$

Prof. Bauschinger, experimenting with cast iron rods, found that in tension the ratio $m:\varepsilon$ was $= \frac{23}{100}$, as an average, which in eq. (3) gives

$$E_s = \frac{100}{246}E_t = \frac{2}{5}E_t \text{ nearly.} \quad . \quad . \quad . \quad (4)$$

His experiments on the torsion of cast iron rods gave E_s = 6,000,000 to 7,000,000 lbs. per sq. inch. By (4), then, E_t should be 15,000,000 to 17,500,000 which is approximately true (§ 203).

Corresponding results may be obtained for short blocks in compression, the lateral change being a dilatation instead of a contraction.

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* This ratio, $m \div \varepsilon$, denoted by k, is called *Poisson's Ratio*. For metals its value lies approximately between 0.20 and 0.35. See also p. 507.

213. Examples in Shearing.—EXAMPLE 1.—Required the proper length, a, Fig. 211, to guard against the shearing off, along the grain, of the portion ab, of a wooden tie-rod, the force P being $= 2$ tons, and the width of the tie $=$ 4 inches. Using a value of $S' = 100$ lbs. per sq. in., we put $baS' = 4{,}000 \cos 45°$; i.e. $a = (4{,}000 \times 0.707) \div (4 \times 100) = 7.07$ inches.

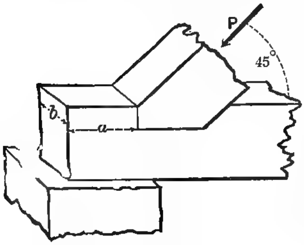

FIG. 211.

EXAMPLE 2.—A $\frac{7}{8}$ in. rivet of wrought iron, in *single shear* (see Fig. 205) has an ultimate shearing strength $P = FS = \frac{1}{4}\pi d^2 S = \frac{1}{4}\pi(\frac{7}{8})^2 \times 50{,}000 = 30{,}050$ lbs. For safety, putting $S' = 8{,}000$ instead of S, $P' = 4{,}800$ lbs. is its safe shearing strength in single shear.

The wrought iron plate, to be secure against the side-crushing in the hole, should have a thickness t, computed thus:

$$P' = tdC'; \text{ or } 4{,}800 = t.\tfrac{7}{8} \times 12{,}000 \ \therefore\ t = 0.46 \text{ in.}$$

If the plate were only 0.23 in. thick the safe value of P would be only $\frac{1}{2}$ of 4,800.

EXAMPLE 3.—Conversely, given a lap-joint, Fig. 205, in which the plates are $\frac{1}{4}$ in. thick and the tensile force on the joint $=$ 600 lbs. per linear inch of seam, how closely must $\frac{3}{4}$ inch rivets be spaced in one row, putting $S' = 8{,}000$ and $C' = 12{,}000$ lbs. per sq. in.? Let the distance between centres of rivets be $= x$ (in inches), then the force upon each rivet $= 600x$, while its section $F = 0.44$ sq. in. Having regard to the shearing strength of the rivet we put $600x = 0.44 \times 8{,}000$ and obtain $x = 5.86$ in.; but considering that the safe crushing resistance of the hole is $= \frac{1}{4} \cdot \frac{3}{4} \cdot 12{,}000 = 2{,}250$ lbs., $600x = 2{,}250$ gives $x = 3.75$ inches, which is the pitch to be adopted. What is the tensile strength of the reduced sectional area of the plate, with this pitch?

EXAMPLE 4.—Double butt-joint; (see Fig. 207); $^3/_8$ inch plate; $\frac{3}{4}$ in. rivets; $T' = C' = 12,000$; $S' = 8,333$; width of plates=14 inches. Will one row of rivets be sufficient at each side of joint, if $Q = 30,000$ lbs.? The number of rivets = ? Here each rivet is in double shear and has therefore a double strength as regards shear. In double shear the safe strength of each rivet $= 2FS' = 7,333$ lbs. Now $30,000 \div 7,333 = 4.0$ (say). With the four rivets in one row the reduced sectional area of the main plate is $= [14 - 4 \times \frac{3}{4}] \times {}^3/_8 = 4.12$ sq. in., whose safe tensile strength is $= FT' = 4.12 \times 12,000 = 49,440$ lbs.; which is $> 30,000$ lbs. $\therefore$ main plate is safe in this respect. But as to side-crushing in holes in main plate we find that $C't_1d$ (i.e. $12,000 \times {}^3/_8 \times \frac{3}{4} = 3,375$ lbs.) is $< \frac{1}{4} Q$ i.e. $< 7,500$ lbs., the actual force on side of hole. Hence four rivets in one row are too few unless thickness of main plate be doubled. Will eight in one row be safe?

213a. (Addendum to § 206.) **Elasticity of Stone and Cements.**—Experiments by Gen. Gillmore with the large Watertown testing-machine in 1883 resulted as follows (see p. 221 for notation):

With cubes of Haverstraw Freestone (a homogeneous brownstone) from 1 in. to 12 in. on the edge, E_c was found to be from 900,000 to 1,000,000 lbs. per sq. in. approximately; and C about 4,000 or 5,000 lbs. per sq. in. Cubes of the same range of sizes of Dyckerman's Portland cement gave E_c from 1,350,000 to 1,630,000, and C from 4,000 to 7,000, lbs. per sq. in. Cubes of concrete of the above sizes, made with the Newark Co.'s Rosendale cement, gave E_c about 538,000, while cubes of cement-mortar, and some of concrete, both made with National Portland cement, showed E_c from 800,000 to 2,000,000 lbs. per sq. in.

The compressibility of *brick piers* 12 in. square in section and 16 in. high was also tested. They were made of common North River brick with mortar joints $\frac{3}{8}$ in. thick, and showed a value for E_c of about 300,000 or 400,000, while at elastic limit C'' was on the average 1,000, lbs. per sq. in.

CHAPTER II.

TORSION.

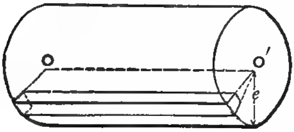

FIG. 212.

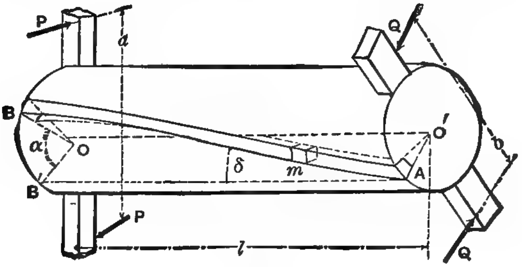

FIG. 213.

214. Angle of Torsion and of Helix. When a cylindrical beam or shaft is subjected to a twisting or torsional action, *i.e.* when it is the means of holding in equilibrium two couples in parallel planes and of equal and opposite moments, the longitudinal axis of symmetry remains straight and the elements along it experience no stress (whence it may be called the "line of no twist"), while the lines originally parallel to it assume the form of helices, each element of which is distorted in its angles (originally right angles), the amount of distortion being assumed proportional to the radius of the helix. The directions of the faces of any element were originally as follows: two radial, two in consecutive transverse sections, and the other two tangent to two consecutive circular cylinders whose common axis is that of the shaft. E.g. in Fig. 212 we have an unstrained shaft, while in Fig. 213 it holds the two couples (of equal moment $Pa = Qb$) in equilibrium. These couples act in parallel planes perpendicular to the axis of the prism and a distance, l, apart. Assuming that the transverse sections *remain plane and parallel* during torsion, any surface element, m, which in Fig. 212 was entirely right-angled, is now distorted. Two of its angles have been increased, two diminished, by an amount δ, the angle between the helix and a line parallel to the axis. Supposing m to be the most distant of any element from the axis, this distance being e, any other element at a distance z from the axis experiences an angular distortion $= \frac{z}{e}\delta$.

If now we draw OB' parallel to $O'A$ the angle BOB', $=\alpha$, is called the **Angle of Torsion**, while δ may be called the *helix angle*; the former lies in a transverse plane, the latter in a plane tangent to the cylinder. Now

$\tan \delta = (\text{linear arc } BB') \div l$; but lin. arc $BB' = e\alpha$; hence, putting δ for $\tan \delta$, (δ being small)

$$\delta = \frac{e\alpha}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(δ and α both in radians).

215. Shearing Stress on the Elements. The angular distortion, or shearing strain, δ, of any element (bounded as already described) is due to the shearing stresses exerted on it by its neighbors on the four faces perpendicular to the tangent plane of the cylindrical shell in which the element is situated. Consider these neighboring elements of an *outside element* removed, and the stresses put in; the latter are accountable for the distortion of the element and

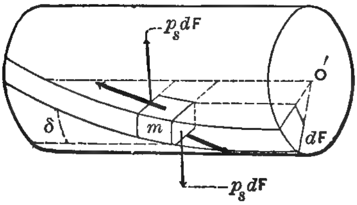

FIG. 214.

hold it in equilibrium. Fig. 214 shows this element "free." Within the elastic limit δ is known to be proportional to p_s, the shearing stress per unit of area on the faces whose relative angular positions have been changed. That is, from eq. (1), § 208, $\delta = p_s \div E_s$; whence, see (1) of § 214,

$$v_s = \frac{e\alpha E_s}{l} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

In (2) p_s and e both refer to a surface element, e being the radius of the cylinder, and p_s the greatest intensity of shearing stress existing in the shaft. Elements lying nearer the axis suffer shearing stresses of less intensity in proportion to their radial distances, i.e., to their helix-angles. That is, the shearing stress on that face of the element which forms a part of a transverse section and whose distance from the axis is z, is $p = \frac{z}{e}\, p_s$, per unit of area, and the total shear on the face is pdF, dF being the area of the face.

216. Torsional Strength.—We are now ready to expose the full transverse section of a shaft under torsion, to deduce formulæ of practical utility. Making a right section of the shaft of Fig. 213 anywhere between the two couples and considering the left hand portion as a free body, the forces holding it in equilibrium are the two forces P of the left-hand couple and an infinite number of shearing forces, each tangent to its circle of radius z, on the cross section exposed by the removal of the right-hand portion. The cross section is assumed to remain plane during torsion, and is composed of an infinite number of dF's, each being the area of an exposed face of an element; see Fig. 215.

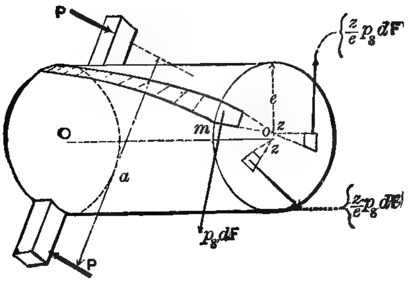

FIG. 215.

Each elementary shearing force $=\frac{z}{e}p_s dF$, and z is its lever arm about the axis Oo. For equilibrium, Σ (mom.) about the axis Oo must $=0$; i.e. in detail

$$-P\tfrac{1}{2}a-P\tfrac{1}{2}a+\int\left(\frac{z}{e}p_s dF\right)z=0$$

or, reducing,

$$\frac{p_s}{e}\int z^2dF=Pa\,;\ \text{or,}\ \frac{p_sI_p}{e}=Pa \quad . \quad . \quad (3)$$

Eq. (3) relates to *torsional strength*, since it contains p_s, the greatest shearing stress induced by the torsional couple, whose moment Pa is called the **Moment of Torsion**, the stresses in the cross section forming a couple of equal and opposite moment. Pa is also called the "*torque*."

I_p is recognized as the **Polar Moment of Inertia** of the cross section, discussed in § 94; e is the radial distance of the outermost element, and = the radius for a circular shaft.

217. Torsional Stiffness.—In problems involving the angle of torsion, or deformation of the shaft, we need an equation connecting Pa and α, which is obtained by substituting in eq. (3) the value of p_s in eq. (2), whence

$$\frac{\alpha I_p E_s}{l}=Pa. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From this is appears that the angle of torsion, α, is proportional to the moment of torsion, or "torque," Pa inch-lbs., within the elastic limit; α must be expressed in *radians*.

Example.—A portion 3.4 ft. long, of a solid cylindrical shaft of soft steel, of diam. $=1.5$ in., is found by the use of "Torsion Clinometers" (see frontispiece) to be held at an angle of torsion of $\alpha=5.41°$, $=0.0944$ radians, just before the elastic limit is reached, by a "*torque*," $=Pa$, of 10,200 in.-lbs. Compute the Modulus of Elasticity for Shearing.

Substituting in eq. (4), with $I_p=\pi r^4/2$, (§ 94), $=\pi(0.75)^4\div 2$, $=0.497$ in.^{4}, and $l=3.4\times 12=40.8$ in., we have

$$E_s=\frac{10{,}200\times 40.8}{0.0944\times 0.497}=8{,}870{,}000 \text{ lbs. per sq. in.}$$

218. Torsional Resilience is the work done in twisting a shaft from an unstrained state until the elastic limit is reached in the outermost elements. If in Fig. 213 we imagine the right-hand extremity to be fixed, while the other end is gradually twisted through an angle α each force P of the couple must be made to increase gradually from a zero value up to the value P_1, corresponding to α_1. In this motion each end of the arm a describes a space $=\frac{1}{2}a\alpha_1$, and the mean value of the force $=\frac{1}{2}P_1$ (compare § 196). Hence the work done in twisting is

$$U_1=\tfrac{1}{2}P_1\times\tfrac{1}{2}a\alpha_1\times 2=\tfrac{1}{2}P_1a\alpha_1 \quad . \quad . \quad (5)$$

By the aid of preceding equations, (5) can be written

$$U_1=\frac{\alpha_1^2E_sI_p}{2l}\text{, or }=\frac{P_1^2a^2l}{2I_pE_s}\text{, or }=\frac{p_s^2I_pl}{2E_se^2} \quad . \quad . \quad (6)$$

If for p_s we write S' (Modulus of safe shearing) we have for the *safe resilience* of the shaft

$$U'=\frac{S'^2I_pl}{2E_se^2} \quad . \quad . \quad . \quad . \quad (7)$$

If the torsional elasticity of an originally unstrained shaft is to be the means of arresting the motion of a moving mass whose weight is G, (large compared with the parts intervening) and velocity $=v$, we write (§ 133)

$$U'=\frac{G}{g}\cdot\frac{v^2}{2};$$

as the condition that the shaft shall not be injured.

219. Polar Moment of Inertia.—For a shaft of circular cross section (see § 94) $I_p=\frac{1}{2}\pi r^4$; for a hollow cylinder $I_p=\frac{1}{2}\pi(r_1^4-r_2^4)$; while for a square shaft $I_p=\frac{1}{6}b^4$, b being the side of the square; for a rectangular cross-section sides b and h, $I_p=\frac{1}{12}bh(b^2+h^2)$. For a cylinder $e=r$; if hollow, $e=r$, the greater radius. For a square, $e=\frac{1}{2}b\sqrt{2}$.

220. Non-Circular Shafts.—If the cross-section is not circular it becomes warped, in torsion, instead of remaining plane. Hence the foregoing theory does not strictly apply. The celebrated investigations of St. Venant, however, cover many of these cases. (See § 708 of Thompson and Tait's Natural Philosophy; also, Prof. Burr's Elasticity and Strength of the Materials of Engineering). His results give for a square shaft (instead of the

$$\frac{ab^4E_s}{6l}=Pa \text{ of eq. (4) of § 217),}$$

$$Pa=0.841\,\frac{ab^4E_s}{6l} \quad . \quad . \quad . \quad . \quad (1)$$

and $Pa=\frac{1}{5}b^3p_s$, instead of eq. (3) of § 216, p_s being the greatest shearing stress.

The elements under greatest shearing strain are found at the middles of the sides, instead of at the corners, when the prism is of square or rectangular cross-section. The warping of the cross-section in such a case is easily verified by the student by twisting a bar of india-rubber in his fingers.

221. Transmission of Power.—Fig. 216. Suppose the cog-wheel B to cause A, on the same shaft, to revolve uniformly and overcome a resistance Q, the pressure of the teeth of another cog-wheel, B being driven by still another wheel. The shaft AB is under torsion, the moment of torsion being $=Pa=Qb$. (P_1 and Q_1 the bearing reactions have no moment about the axis of the shaft). If the shaft makes u revolutions per unit-time, the work transmitted (*transmitted*; not *expend-*

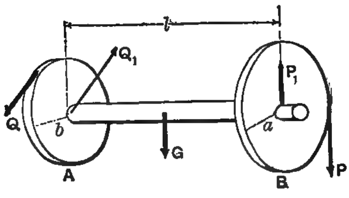

FIG. 216.

ed in twisting the shaft whose angle of torsion remains constant, corresponding to Pa) per unit-time, i.e. the **Power**, is

$$L = P.2\pi a.u = 2\pi uPa \quad . \quad . \quad . \quad (8)$$

To reduce L to Horse Power (§ 132), we divide by N, the number of units of work per unit-time constituting one H. P. in the system of units employed, i.e.,

$$\text{Horse Power} = \text{H. P.} = \frac{2\pi uPa}{N}$$

For example N=33,000 ft.-lbs. per minute, or =396,000 inch-lbs. per minute; or =550 ft.-lbs. per second. Usually the rate of rotation of a shaft is given in revolutions per minute.

But eq. (8) happens to contain Pa the moment of torsion acting to maintain the constant value of the angle of torsion, and since for safety (see eq. (3) § 216) $Pa = S'I_p \div e$, with $I_p = \frac{1}{2}\pi r^4$ and $e = r$ for a solid circular shaft, we have for such a shaft

$$\text{(Safe), H. P.} = \frac{\pi^2 S'ur^3}{N} \quad . \quad . \quad . \quad (9)$$

which is the safe H. P., which the given shaft can transmit at the given speed. S' may be made 7,000 lbs. per sq. inch for wrought iron; 10,000 for steel, and 5,000 for cast-iron. If the value of Pa fluctuates periodically, as when a shaft is driven by a connecting rod and crank, for (H. P.) we put $m\times$(H. P.), m being the ratio of the maximum to the mean torsional moment; m = about $1\frac{1}{2}$ under ordinary circumstances (Cotterill).

With a **hollow cylindrical shaft**, of outer radius $=r_1$, and inner $=r_2$ the r^3 of eq. (9) must be replaced by $(r_1^4 - r_2^4) \div r_1$. If, furthermore, the *thickness of metal is small*, we may proceed thus, taking numerical data: Let the radius to the middle of the thickness be $r_0 = 10$ in., the thickness $t = \frac{1}{4}$ in., and the (steel) shaft make $u = 120$ revs./min.; with $S' = 5000$ lbs./in.^{2}; then the total safe shearing stress in the cross-section is $R' = 2\pi r_0 tS' = 2\pi 10 \times \frac{1}{4} \times 5000 = 78{,}540$ lbs., while the velocity of the mid-thickness is $v' = 2\pi r_0 u = 2\pi\ 10 \times 2 = 125.6$ in./sec. $= 10.47$ ft./sec. Hence the (safe) power that may be transmitted at given speed is $L = R'v'$ $= 78{,}540 \times 10.47 = 822{,}100$ ft.-lbs. per sec.; or, $(\div 550)$, $= 1495$ H.P.

222. Autographic Testing Machine.—The principle of Prof Thurston's invention bearing this name is shown in Fig 217. The test-piece is of a standard shape and size, its central cylinder being subjected to torsion. A jaw, carrying a handle (or gear-wheel turned by a worm) and a drum on which paper is wrapped, takes a firm hold of one end of the test-piece, whose further end lies in another jaw rigidly connected with a heavy pendulum carrying a pencil free to move axially. By a continuous slow motion of the handle the pendulum is gradually deviated more and more from the vertical, through the intervention of the test-piece, which is thus subjected to an increasing torsional moment. The axis of the test-piece lies in the axis of motion. This motion of the pendulum by means of a properly curved guide, *WR*, causes an axial (i.e., parallel to axis of test-piece) motion of the pencil *A*, as well as an angular deviation β equal to that of the pendulum, and this axial distance $CF, = sT$, of the pencil from its initial position measures the moment of torsion $= Pa = Pc \sin \beta$. As the piece twists, the drum and paper move relatively to the pencil through an angle sUo equal to the angle of torsion α so far attained. The abscissa so and ordinate sT of the curve thus marked on the paper, measure, when the paper is unrolled, the values of α and Pa through

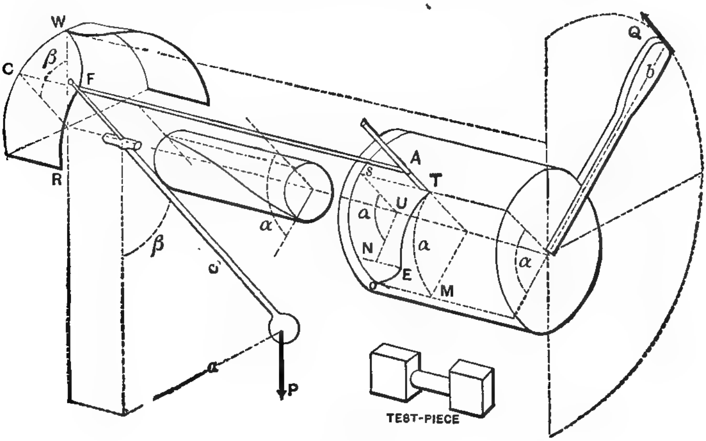

FIG. 217.

all the stages of the torsion. Fig. 218 shows typical curves thus obtained. Many valuable indications are given by these strain diagrams as to homogeneousness of composition, ductility, etc., etc. On relaxing the strain at any stage within the elastic limit, the pencil retraces its path; but if beyond that limit, a new path is taken called an "elasticity-line," in general parallel to the first part of the line, and showing the amount of angular recovery, BC, and the permanent angular set, OB.

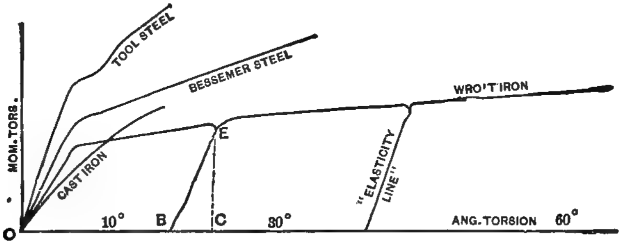

Fig. 218.

222a. Torsion Clinometers.—When the test-piece used in the Thurston testing machine is short, the indicated angles of torsion below the elastic limit are far in excess of the actual values, on account of the initial yielding of the wedges in the jaws. By the use of "*torsion clinometers*," however (see frontispiece) the angle of torsion can be measured accurately within one minute of arc.

223. Examples in Torsion.—The modulus of safe shearing strength, S', as given in § 221, is expressed in pounds per square inch; hence these two units should be adopted throughout in any numerical examples where one of the above values for S' is used. The same statement applies to the modulus of shearing elasticity, E_s, in the table of § 210.

Example 1.—Fig. 216. With $P = 1$ ton, $a = 3$ ft., $l = 10$ ft., and the radius of the cylindrical shaft $r=2.5$ inches, required the max. shearing stress per sq. inch, p_s, the shaft being of wrought iron. From eq. (3) § 216

$$p_s = \frac{Pae}{I_p} = \frac{2,000 \times 36 \times 2.5}{\frac{1}{2}\pi \times (2.5)^4} = 2,930 \text{ lbs. per sq. inch},$$

which is a safe value for any ferrous metal.

EXAMPLE 2.—What H. P. is the shaft in Ex. 1 transmitting, if it makes 50 revolutions per minute? Let $u =$ number of revolutions per unit of time, and $N =$ the number of units of work per unit of time constituting one horse-power. Then H. P.$=Pu2\pi a \div N$, which for the foot-pound-minute system of units gives

$$\text{H. P.}=2{,}000\times 50\times 2\pi\times 3\div 33{,}000=57\tfrac{1}{4}\ \text{H. P.}$$

EXAMPLE 3.—What different radius should be given to the shaft in Ex. 1, if two radii at its extremities, originally parallel, are to make an angle of 2° when the given moment of torsion is acting, the strains in the shaft remaining constant. From eq. (4) § 217, and the table 210, with $\alpha=\frac{2^\circ}{180^\circ}\pi=$ 0.035 *radians* (i.e. π-measure), and $I_p={}^1/_2\pi r^4$, we have

$$r^4=\frac{2{,}000\times 36\times 120}{\tfrac{1}{2}\pi 0.035\times 9{,}000{,}000}=17.45\ \therefore\ r=2.04\ \text{inches.}$$

(This would bring about a different p_s, but still safe.) The foregoing is an example in *stiffness.*

EXAMPLE 4.—A working shaft of steel (solid) is to transmit 4,000 H. P. and make 60 rev. per minute, the maximum twisting moment being $1\tfrac{1}{2}$ times the average; required its diameter. $d=14.74$ inches. Ans.

EXAMPLE 5.—In example 1, $p_s = 2{,}930$ lbs. per square inch; what tensile stress does this imply on a plane at 45° with the pair of planes on which p_s acts? Fig. 219 shows

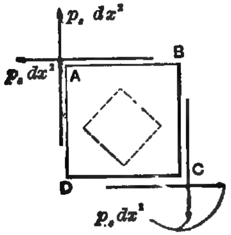

FIG. 219

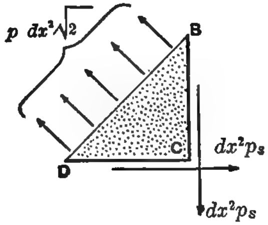

FIG. 220.

a small cube, of edge $=dx$, (taken from the outer helix of Fig. 215,) free and in equilibrium, the plane of the paper being tangent to the cylinder; while 220 shows the portion BDC, also free, with the unknown total tensile stress $pdx^2\sqrt{2}$ acting on the newly exposed rectangle of area $=dx\times dx\sqrt{2}$, p being the unknown stress per unit of area. From symmetry the stress on this diagonal plane has no shearing component. Putting Σ [components normal to BD]$=0$, we have

$$pdx^2\sqrt{2}=2dx^2p_s\cos 45°=dx^2p_s\sqrt{2} \therefore p=p_s \quad . \quad (1)$$

That is, a normal tensile stress exists in the diagonal plane BD of the cubical element equal in intensity to the shearing stress on one of the faces, i.e., $=2,930$ lbs. per sq. in. in this case.

Similarly in the plane AC will be found a compressive stress of 2,930 lbs. per sq. in. If a plane surface had been exposed making any other angle than 45° with the face of the cube in Fig. 219, we should have found shearing and normal stresses each less than p_s per sq. inch. Hence the interior dotted cube in 219, if shown "free" is in tension in one direction, in compression in the other, and with no shear, these normal stresses having equal intensities. Since S' is usually less than T' or C', if p_s is made $= S'$ the tensile and compressive actions are not injurious. It follows therefore that when a cylinder is in torsion any helix at an angle of 45° with the axis is a line of tensile, or of compressive stress, according as it is a right or left handed helix, or vice versâ.

Example 6.—A solid and a hollow cylindrical shaft, of equal length, contain the same amount of the same kind of metal, the solid one fitting the hollow of the other.

Compare their torsional strengths, used separately. The solid shaft has only $\frac{471}{1,000}$ the strength of the hollow one. **Ans.**

Example 7.—Compare the shafts of Example 6 as to torsional *stiffness* (i.e., the angles of torsion due to equal moments). The solid shaft is only one-third as stiff as the other; an equal moment produces three times the angle. Ans.

CHAPTER III.

FLEXURE OF HOMOGENEOUS PRISMS UNDER PERPENDICULAR FORCES IN ONE PLANE.

224. Assumptions of the Common Theory of Flexure.—When a prism is bent, under the action of external forces perpendicular to it and in the same plane with each other, it may be assumed that the longitudinal fibres are in tension on the convex side, in compression on the concave side, and that the relative stretching or contraction of the elements is proportional to their distances from a plane intermediate between, with the understanding that the flexure is slight and that the elastic limit is not passed in any element.

This " common theory " is sufficiently exact for ordinary engineering purposes if the constants employed are properly determined by a wide range of experiments, and involves certain assumptions of as simple a nature as possible, consistently with practical facts. These assumptions are as follows, (for prisms, and for solids with variable cross sections, when the cross sections are similarly situated as regards a central straight axis) and are approximately borne out by experiment:

(1.) The external or " applied " forces are all perpendicular to the axis of the piece and lie in one plane, which may be called the force-plane; the force-plane contains the axis of the piece and cuts each cross-section symmetrically;

(2.) The cross-sections remain plane surfaces during flexure;

(3.) There is a surface (or, rather, sheet of elements) which is parallel to the axis and perpendicular to the force-plane, and along which the elements of the solid experience no tension nor compression in an axial direction, this being called the **Neutral Surface**;

(4.) The projection of the neutral surface upon the force plane (or a ∥ plane) being called the **Neutral Line** or **Elastic Curve**, the bending or flexure of the piece is so slight that an elementary division, ds, of the neutral line may be put $=dx$, its projection on a line parallel to the direction of the axis before flexure;

(5.) The elements of the body contained between any two consecutive cross-sections, whose intersections with the neutral surface are the respective **Neutral Axes** of the sections, experience elongations (or contractions, according as they are situated on one side or the other of the neutral surface), in an axial direction, whose amounts are proportional to their distances from the neutral axis, and indicate corresponding tensile or compressive stresses;

(6.) $E_t = E_c$;

(7.) The dimensions of the cross-section are small compared with the length of the piece;

(8.) There is no shear perpendicular to the force plane on internal surfaces perpendicular to that plane.

In the locality where any one of the external forces is applied, local stresses are of course induced which demand separate treatment. These are not considered at present.

225. Illustration.—Consider the case of flexure shown in **Fig.** 221. The external forces are three (neglecting the

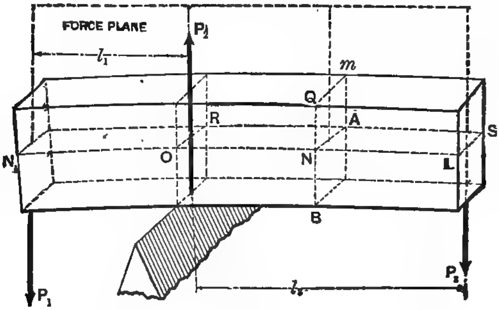

FIG. 221.

weight of the beam), viz.: P_1, P_2, and P_3. P_1 and P_3 are loads, P_2 the reaction of the support.

The force plane is vertical. N_1L is the neutral line or elastic curve. NA is the neutral axis of the cross-section at *m*; this cross-section, originally perpendicular to the sides of the prism, is during flexure ⏋ to their tangent planes drawn at the intersection lines; in other words, the side view QNB, of any cross-section is perpendicular to the neutral line. In considering the whole prism free we have the system P_1, P_2, and P_3 in equilibrium, whence from $\Sigma Y=0$ we have $P_2=P_1+P_3$, and from Σ (mom. about O) $=0$, $P_3l_3=P_1l_1$. Hence given P_1 we may determine the other two external forces. A reaction such as P_2 is sometimes called a supporting force. The elements above the neutral surface N_1OLS are in tension; those below in compression (in an axial direction).

226. The Elastic Forces.—Conceive the beam in Fig. 221 separated into two parts by any transverse section such as QA, and the portion N_1ON, considered as a free body in Fig. 222. Of this free body the surface QAB is one of

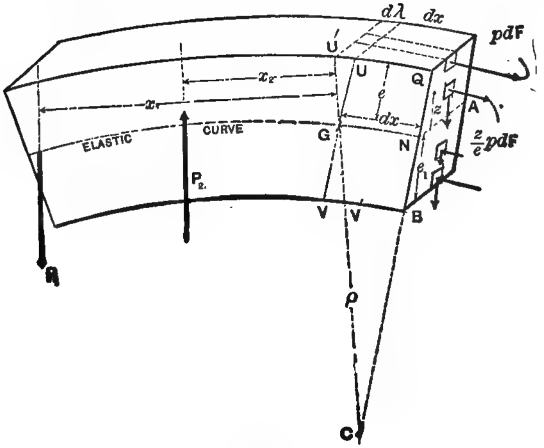

FIG. 222.

the bounding surfaces, but was originally an internal surface of the beam in Fig. 221. Hence in Fig. 222 we must put in the stresses acting on all the dF's or elements of area of QAB. These stresses represent the actions of the body taken away upon the body which is left, and according to assumptions (5), (6) and (8) consist of normal stresses (tension or compression) proportional per unit of area, to the distance, z, of the dF's from the neutral axis, and of shearing stresses parallel to the force-plane (which in most cases will be vertical).

The intensity of this shearing stress on any dF varies with the position of the dF with respect to the neutral axis, but the law of its variation will be investigated later (§§ 253 and 254). These stresses, called the **Elastic Forces** of the cross-section exposed, and the external forces P_1 and P_2, form a system in equilibrium. We may therefore apply any of the conditions of equilibrium proved in § 38.

227. The Neutral Axis Contains the Centre of Gravity of the Cross-Section.—Fig. 222. Let $e=$ the distance of the *outermost* element of the cross-section from the neutral axis, and the normal stress per unit of area upon it be $=p$, whether tension or compression. Then by assumptions (5) and (6), § 224, the intensity of normal stress on any dF is $= \frac{z}{e} p$ and the actual

$$\text{normal stress on any } dF \text{ is} = \frac{z}{e}\, p dF \quad . \quad (1)$$

This equation is true for dF's having negative z's, i.e. on the other side of the neutral axis, the negative value of the force indicating normal stress of the opposite character; for if the relative elongation (or contraction) of two axial fibres is the same for equal z's, one above, the other below, the neutral surface, the stresses producing the changes in length are also the same, provided $E_t=E_c$; see §§ 184 and 201.

For this free body in equilibrium put $\Sigma X=0$ (X is a horizontal axis). Put the normal stresses equal to their X components, the flexure being so slight, and the X component of the shears = 0 for the same reason. This gives (see eq. (1))

$$\int \frac{z}{e} p dF = 0 \text{ ; i.e. } \frac{p}{e} \int dFz = 0 \text{ ; or, } \frac{p}{e} F\bar{z} = 0 \qquad (2)$$

In which $\bar{z}$= distance of the centre of gravity of the cross-section from the neutral axis, from which, though unknown in position, the z's have been measured (see eq. (4) § 23).

In eq. (2) neither $p \div e$ nor F can be zero $\therefore$ $\bar{z}$ must $= 0$; i.e. the neutral axis contains the centre of gravity. Q. E. D. [If the external forces were not all perpendicular to the beam this result would not be obtained, necessarily.]

228. The Shear.—The "total shear," or simply the "shear," in the cross-section is the sum of the vertical shearing stresses on the respective dF's. Call this sum J, and we shall have from the free body in Fig. 222, by putting $\Sigma Y=0$ (Y being vertical)

$$P_2 - P_1 - J = 0 \therefore J = P_2 - P_1 \quad . \quad . \quad (3)$$

That is, the shear equals the algebraic sum of the external forces acting on one side (only) of the section considered. This result implies nothing concerning its mode of distribution over the section.

229. The Moment.—By the "Moment of Flexure" or simply the *Moment*, at any cross-section is meant the sum of the moments of the elastic forces of the section, taking the neutral axis as an axis of moments. In this summation the normal stresses appear alone, the shear taking no part, having no lever arm about the axis NA. Hence, Fig. 222, the *moment of flexure* (or "moment of resistance")

$$= \int \left(\frac{z}{e} p dF \right) z = \frac{p}{e} \int dFz^2 = \frac{pI}{e}. \quad . \quad . \quad . \quad . \quad (4)$$

This function, $\int dFz^2$, of the cross-section or plane figure is the quantity called **Moment of Inertia** of a plane figure, § 85. For the free body in Fig. 222, by putting Σ(mom.s about the neutral axis NA)=0, we have then

$$\frac{pI}{e}-P_1x_1+P_2x_2=0, \text{ or } \textit{in general, } \frac{pI}{e}=M \quad . \quad (5)$$

in which M signifies the sum of moments,* *about the neutral axis of the section*, of all the forces acting on the free body considered, exclusive of the elastic forces of the exposed section itself. M is also called the "**Bending Moment.**"

Example.—In Fig. 222 let $P_1=3$ and $P_2=4$ tons, $x_1=1$ ft. 8 in. and $x_2=5$ in.; the section of the beam being a rectangle, with $NA=b=3$ in. and $QB=h=6$ in. Then I about axis NA is, (p. 94), $bh^3\div12=54$ in.^{4}; and $e=3$ in. Hence the "*bending moment,*" M, $=3\times20-4\times5=40$ in.-tons. Equating M to the "*moment of resistance*" [or moment of the "stress couple" (see § 230)] we obtain, from eq. (5), $p=Me\div I=40\times3\div54=2.22$ tons/in.^{2}, for the unit normal stress in the outer fibre at Q, or B. We find also, for the shear at section QB, $J=4-3=1$ ton.

230. Strength in Flexure.—Eq. (5) is available for solving problems involving the **Strength** of beams and girders, since it contains p, the greatest normal stress per unit of area to be found in the section.

In the cases of the present chapter, where all the external forces are perpendicular to the prism or beam, and have therefore no components parallel to the beam, i.e. to the axis X, it is evident that the normal stresses in any section, as QB Fig. 222, are equivalent to a couple; for the condition $\Sigma X=0$ falls entirely upon them and cannot be true unless the resultant of the tensions is equal, parallel, and opposite to that of the compressions. These two equal and parallel resultants, not being in the same line, form a couple (§ 28), which we may call the *stress-couple*. The moment of this couple is the "moment of flexure" $\frac{pI}{e}$, and it is further evident that the remaining forces in Fig. 222, viz.: the shear J and the external forces P_1 and P_2, are equivalent to a couple of equal and opposite moment to the one formed by the normal stresses.

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* It is evident, therefore, that M (ft.-lbs., or in.-lbs.) is numerically equal to the "moment of flexure," or moment of the "stress couple"; so that occasionally it may be convenient to use "M" to denote the value of the latter moment also.

231. Flexural Stiffness.—The neutral line, or **elastic curve,** containing the centres of gravity of all the sections, was originally straight; its radius of curvature at any point, as N, Fig. 222, during flexure may be introduced as follows. QB and $U'V'$ are two consecutive cross-sections, originally parallel, but now inclined so that the intersection C, found by prolonging them sufficiently, is the centre of curvature of the ds (put $=dx$) which separates them at N, and $CG=\rho=$ the radius of curvature of the elastic curve at N. From the similar triangles $U'UG$ and GNC we have $d\lambda : dx :: e : \rho$, in which $d\lambda$ is the elongation, $U'U$, of a portion, originally $=dx$, of the outer fibre. But the relative elongation $\varepsilon = \frac{d\lambda}{dx}$ of the latter is, by §184, within the elastic limit, $\doteq \frac{p}{E}$. $\therefore \frac{p}{E} = \frac{e}{\rho}$ and eq. (5) becomes

$$\frac{EI}{\rho} = M \quad . \quad . \quad . \quad . \qquad (6)$$

From (6) the radius of curvature can be computed. $E=$ the value of $E_t = E_c$, as ascertained from experiments in bending.

To obtain a differential equation of the elastic curve, (6) may be transformed thus, Fig. 223. The curve being very flat, consider two consecutive ds's with equal dx's; they may be put $=$ their dx's. Produce the first to intersect the dy of the second, thus cutting off the d^2y, i.e. the difference between two consecutive dy's. Drawing a perpendicular to each ds at its left extremity, the centre of curvature C is determined by their intersection, and thus the radius of curvature ρ. The two shaded triangles have their small angles equal, and d^2y is nearly perpendicular to the prolonged ds; hence, considering them similar, we have

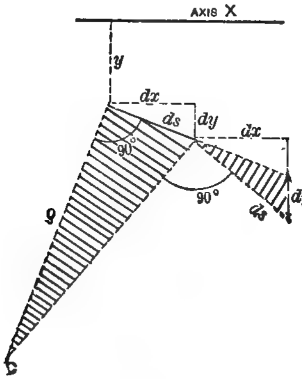


FIG. 223.

$$\rho : dx :: dx : d^2y \quad \therefore \frac{1}{\rho} = \frac{d^2y}{dx^2},$$

and hence from eq. (6) we may write $\Big\}$ (*approx.*) $\pm EI\frac{d^2y}{dx^2}=M$. (7)

as a differential equation of the elastic curve. From this the equation of the elastic curve may be found, the deflections at different points computed, and an idea thus formed of the stiffness. All beams in the present chapter being *prismatic* and *homogeneous* both E and I are the same (i.e. constant) at all points of the elastic curve. In using (7) the axis X must be taken parallel to the length of the beam before flexure, which must be slight ; the minus sign in (7) provides for the case when $d^2y \div dx^2$ is essentially negative.

232. Resilience of Flexure.—If the external forces are made to increase gradually from zero up to certain maximum values, some of them may do work, by reason of their points of application moving through certain distances due to the yielding, or flexure, of the body. If at the beginning and also at the end of this operation the body is at rest, this work has been expended on the elastic resistance of the body, and an equal amount, called the work of resilience (or springing-back), will be restored by the elasticity of the body, if released from the external forces, provided the elastic limit has not been passed. The energy thus temporarily stored is of the potential kind; see §§ 148, 180, 196 and 218.

232a. Distinction Between Simple, and Continuous, Beams (or "Girders").—The external forces acting on a beam consist generally of the loads and the "reactions" of the supports. If the beam is horizontal and rests on two supports only, the reactions of those supports are easily found by elementary statics [§ 36] alone, without calling into account the theory of flexure, and the beam is said to be a **Simple Beam,** or girder; whereas if it is in contact with more than two supports, being "continuous," therefore, over some of them, it is a **Continuous Girder** (§ 271). The remainder of this chapter will deal only with simple beams.

ELASTIC CURVES.

233. Case I. Horizontal Prismatic Beam, Supported at Both Ends, With a Central Load, Weight of Beam Neglected.—Fig. 224. First considering the whole beam free, we find each reaction to be $=\frac{1}{2}P$. AOB is the neutral line; required the equation of the portion OB referred to O as an origin, and to the tangent line through O as the axis of X. To do this consider as free the portion mB between any section m on the right of O and the near support, in Fig. 225 The forces holding this free body in equilibrium are the one external force $\frac{1}{2}P$, and the elastic forces acting on the exposed surface. The latter consist of J, the shear, and the tensions and compressions represented in the figure by their equivalent "stress-couple." Selecting N, the neutral axis of m, as an axis of moments (that J may not appear in the moment equation) and putting Σ (mom) $=0$ we have

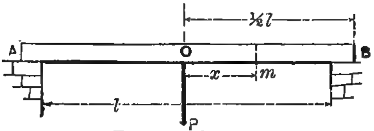


FIG. 224. § 233.

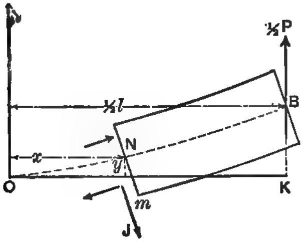


FIG. 225.

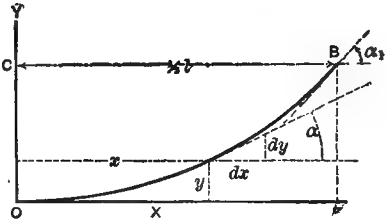


FIG. 226.

$$\frac{P}{2}\left(\frac{l}{2}-x\right)-EI\frac{d^2y}{dx^2}=0\;\therefore\;EI\frac{d^2y}{dx^2}=\frac{P}{2}\left(\frac{l}{2}-x\right)\quad . \quad (1)$$

Fig. 226 shows the elastic curve OB in its purely geometrical aspect, much exaggerated. For axes and origin as in figure $d^2y\div dx^2$ is positive.

Eq. (1) gives the second x-derivative of y equal to a function of x. Hence the first x-derivative of y will be equal to the x-anti-derivative of that function, plus a constant, C. (By anti-derivative is meant the converse of derivative, sometimes called integral though not in the sense of summation). Hence from (1) we have (EI being a constant factor remaining undisturbed)

$$EI\frac{dy}{dx}=\frac{P}{2}\left(\frac{l}{2}x-\frac{x^2}{2}\right)+C \quad . \quad . \quad (2)'$$

(2)′ is an equation between two variables $dy\div dx$ and x, and holds good for any point between O and B; $dy\div dx$ denoting the tang. of α, the *slope*, or angle between the tangent line and X. At O the slope is zero, and x also zero; hence at O (2)′ becomes

$$EI\times 0=0-0+C$$

which enables us to determine the constant C, whose value must be the same at O as for all points of the curve. Hence $C=0$ and (2)′ becomes

$$EI\frac{dy}{dx}=\frac{P}{2}\left(\frac{l}{2}x-\frac{x^2}{2}\right) \quad . \quad . \quad . \quad (2)$$

from which the slope, tan. α, (or simply α, in π-measure; since the angle is small) may be found at any point. Thus at B we have $x=\frac{1}{2}l$ and $dy\div dx=\alpha_1$, and

$$\therefore \alpha_1=\frac{1}{16}\cdot\frac{Pl^2}{EI}$$

Again, taking the x-anti-derivative of both members of eq. (2) we have

$$EIy=\frac{P}{2}\left(\frac{lx^2}{4}-\frac{x^3}{6}\right)+C' \quad . \quad . \quad . \quad (3)'$$

and since at O both x and y are zero, C' is zero. Hence the equation of the elastic curve OB is

$$EIy=\frac{P}{2}\left(\frac{lx^2}{4}-\frac{x^3}{6}\right) \quad . \quad . \quad . \quad (3)$$

To compute the deflection of O from the right line joining A and B in Fig. 224, i.e. $BK, =d$, we put $x=\frac{1}{2}l$ in (3), y being then $=d$, and obtain

$$BK=d=\frac{1}{48}\cdot\frac{Pl^3}{EI} \quad . \quad . \quad . \quad (4)$$

Eq. (3) does not admit of negative values for x; for if the free body of Fig. 225 extended to the left of O, the external forces acting would be P, downward, at O; and $\frac{1}{2}P$, upward, at B, instead of the latter alone; thus altering the form of eq. (1). From symmetry, however, we know that the curve AO, Fig. 224, is symmetrical with OB about the vertical through O.

Numerical Illustration.—Let the beam shown in Fig. 224, resting on two unyielding supports at the same level, be of white oak timber and bear a load of $P=200$ lbs. at the middle, its length being $l=12$ ft. and cross-section rectangular with a width (horizontal) of $b=2$ in. and height $h=6$ in. The modulus of elasticity E will be taken as 1,600,000 lbs./in.^{2} Required the radius of curvature, ρ, of the elastic curve at a point 4 ft. from the right-hand pier (or left).

From the free body in Fig. 225 we have, using the form $EI\div\rho$ for the moment of the stress-couple in the section, and putting $\Sigma(\text{moms.})_N=0$, with $x=2$ ft., $EI\div\rho=100\times 48$, the inch and pound being selected as units. Now $I=bh^3\div 12$ (p. 94) which $=36$ in.^{4}; whence, solving, $\rho=1{,}600{,}000\times 36\div 4800=12{,}000$ in. The curve is evidently very flat. The smallest radius of curvature is found at the middle of the beam and is 8000 in.; at either extremity, A or B, it is *infinite*, since at each of these points the moment of the stress-couple is zero.

At the same point (4 ft. from B) the "slope" of the elastic curve, viz., $dy\div dx$, is found by putting $x=2$ ft. $=24$ in., in eq. (2) from which is derived $\tan\alpha=dy/dx=0.0025$, corresponding to an angle of 0° 8′ 36″. At the extremity B we find, from $\alpha_1=Pl^2\div 16EI$, the slope of the tangent line to be $\alpha_1=0.0045$; which is the tangent of 0° 15′ 29″.

The deflection of the middle point is known from eq. (4), viz., $d=Pl^3\div 48EI$; i.e., $d=(200\times 144\times 144\times 144)\div(48\times 1{,}600{,}000\times 36)=0.216$ in.

It now remains to ascertain if the elastic limit is passed in any fibre of the beam. If we put the form $pI\div e$ (for moment of stress-couple) in place of the present left-hand member of eq. (1), and solve for the unit (normal) stress in outer fibre, we find $p=\frac{1}{2}Pe(\frac{1}{2}l-x)\div I$, which shows that p is greatest in the outer fibre of the section for which $\frac{1}{2}l-x$ is greatest, within the limits of the half-length; and this occurs at the middle of the beam, where $x=0$. With this substitution we obtain $p(\max.)=p_m=Ple\div(4I)$; or $p_m=(200\times 12\times 12\times 3)\div(4\times 36)=600$ lbs./in.^{2}, which is well within the elastic limit, for tension or compression, in white oak.

233a. Load Suddenly Applied.—Eq. (4) gives the deflection d corresponding to the force or pressure P applied at the middle of the beam, and is seen to be proportional to it. If a load G hangs at rest from the middle of the beam, $P=G$; but if the load G, being initially placed at rest upon the unbent beam, is suddenly released from the external constraint necessary to hold it there, it sinks and deflects the beam, the pressure P actually felt by the beam varying with the deflection as the load sinks. What is the maximum deflection d_m? and what the pressure P_m between the load and the beam at the instant of maximum deflection? In this motion of the body, or "load," it is acted on by two forces, the constant downward force G (its weight) and the variable upward force P, whose average value is $\frac{1}{2}P_m$; while its initial and final kinetic energy are each zero. G does the work Gd_m, while the work done upon P is $\frac{1}{2}P_m d_m$; hence, by the theorem of "*Work and Energy*" (p. 138), we have

$$Gd_m=\tfrac{1}{2}P_m d_m+0-0. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

That is, $P_m=2G$. Since at this instant the load is subjected to an upward force of $2G$ and to a downward force of only G (gravity) it immediately begins an upward motion, reaching the point whence the motion began, and thus the oscillation continues. We here suppose the elasticity of the beam unimpaired. This is called the "sudden" application of a load, and produces, as shown above, double the pressure on the beam which it does when gradually applied, and a double deflection. The work done by the beam in raising the weight again is called its resilience.

Similarly, if the weight G is allowed to fall on the middle of the beam from a height h, we shall have

$$G\times(h+d_m), \text{ or approx., } Gh,=\tfrac{1}{2}P_m d_m;$$

and hence, since (4) gives d_m in terms of P_m,

$$Gh=\frac{1}{96}\cdot\frac{P_m^2 l^3}{EI}; \text{ or } Gh=\frac{24EId_m^2}{l^3} \quad . \quad (6)$$

This theory supposes the mass of the beam small compared with the falling weight.

234. Case II. Horizontal Prismatic Beam, Supported at Both Ends, Bearing a Single Eccentric Load. Weight of Beam Neglected.—Fig. 227. The reactions of the points of support, P_0 and P_1, are easily found by considering the whole beam free, and putting first $\Sigma(\text{mom.})_o=0$, whence $P_1=Pl\div l_1$, and then $\Sigma(\text{mom.})_B=0$, whence $P_0=P(l_1-l)\div l_1$. P_0 and P_1 will now be treated as known quantities.

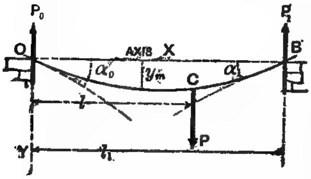


FIG 227.

The elastic curves OC and CB, though having a common tangent line at C (and hence the same slope α_c), and a common ordinate at C, have separate equations and are both referred to the same origin and axes, as shown in the figure. The slope at O, α_0, and that at B, α_1, are unknown constants, to be determined in the progress of the work.

Equation of OC.—Considering as free a portion of the beam extending from B to a section made anywhere on OC, x and y being the co-ordinates of the neutral axis of that section, we conceive the elastic forces put in on the exposed surface, as in the preceding problem, and put Σ(mom. about neutral axis of the section) $=0$ which gives (remembering that here $d^2y\div dx^2$ is negative.)

$$EI\frac{d^2y}{dx^2}=P(l-x)-P_1(l_1-x)\,; \quad . \quad . \quad (1)$$

whence, by taking the x anti-derivatives of both members

$$EI\frac{dy}{dx}=P\left(lx-\frac{x^2}{2}\right)-P_1\left(l_1x-\frac{x^2}{2}\right)+C$$

To find C, write out this equation for the point O, where $dy\div dx=\alpha_0$ and $x=0$, and we have $C=EI\alpha_0$; hence the equation for slope is

$$EI\frac{dy}{dx} = P\left(lx - \frac{x^2}{2}\right) - P_1\left(l_1x - \frac{x^2}{2}\right) + EIa_0 \quad . \quad (2)$$

Again taking the x anti-derivatives, we have from (2)

$$EIy = P\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) - P_1\left(\frac{l_1x^2}{2} - \frac{x^3}{6}\right) + EIa_0x + (C' = 0) \quad (3)$$

(at O both x and y are $=0$ $\therefore$ $C'=0$). In equations (1), (2), and (3) no value of x is to be used <0 or $>l$, since for points in CB different relations apply, thus

Equation of CB.—Fig. 227. Let the free body extend from B to a section made anywhere on CB. Σ(moms.), as before, $=0$, gives (see foot-note on p. 322)

$$EI\frac{d^2y}{dx^2} = -P_1(l_1 - x) \quad . \quad . \quad . \quad (4)$$

(*N.B.* In (4), as in (1), $EId^2y \div dx^2$ is written equal to a negative quantity because itself essentially negative; for the curve is concave to the axis X in the first quadrant of the co-ordinate axes.)

From (4) we have in the ordinary way (x-anti-deriv.)

$$EI\frac{dy}{dx} = -P_1\left(l_1x - \frac{x^2}{2}\right) + C'' \quad . \quad . \quad (5)'$$

To determine C'', consider that the curves CB and OC have the same slope ($dy \div dx$) at C where $x=l$; hence put $x=l$ in the right-hand members of (2) and of (5)' and equate the results. This gives $C'' = \frac{1}{2}Pl^2 + EIa_0$ and $\therefore$

$$EI\frac{dy}{dx} = \frac{Pl^2}{2} + EIa_0 - P_1\left[l_1x - \frac{x^2}{2}\right] \quad . \quad (5)$$

$$\text{and } \therefore EIy = \frac{Pl^2}{2}x + EIa_0x - P_1\left[l_1\frac{x^2}{2} - \frac{x^3}{6}\right] + C''' \quad . \quad (6)'$$

At C, where $x=l$, both curves have the same ordinate; hence, by putting $x=l$ in the right members of (3) and (6)′ and equating results, we obtain $C''' = -\frac{1}{6}Pl^3$. $\therefore$ (6)′ becomes

$$EIy = \tfrac{1}{2}Pl^2x + EI\alpha_0 x - P_1\left[\frac{l_1x^2}{2} - \frac{x^3}{6}\right] - \frac{Pl^3}{6} \quad . \quad (6)$$

as the **Equation of CB**, Fig. 227. But α_0 is still an unknown constant, to find which write out (6) for the point B where $x = l_1$, and $y = 0$, whence we obtain

$$\alpha_0 = \frac{1}{6EIl_1}[Pl^3 - 3Pl^2l_1 + 2P_1l_1^3] \quad . \quad . \quad (7)$$

$\alpha_1 =$ a similar form, putting P_0 for P_1, and $(l_1 - l)$ for l.

235. Maximum Deflection in Case II.—Fig. 227. The ordinate y_m of the lowest point is thus found. Assuming $l > \frac{1}{2}l_1$, it will occur in the curve OC. Hence put the $dy \div dx$ of that curve, as expressed in equation (**2**), $=0$. Also for α_0 write its value from (7), having put $P_1 = Pl \div l_1$, and we have

$$P\left(lx - \frac{x^2}{2}\right) - P\frac{l}{l_1}\left(l_1x - \frac{x^2}{2}\right) + \frac{1}{6}\frac{Pl}{l_1}(l^2 - 3ll_1 + 2l_1^2) = 0$$

$$\text{whence } [x \text{ for max. } y] = \sqrt{\tfrac{1}{3}l(2l_1 - l)}$$

Now substitute this value of x in (3), also α_0 from (7), and put $P_1 = Pl \div l_1$, whence

$$\textbf{Max. Deflec.} = y\text{max} = \tfrac{1}{9} \cdot \frac{P}{EIl_1}[l^3 - 3l^2l_1 + 2ll_1^2]\sqrt{\tfrac{1}{3}l(2l_1 - l)}.$$

236. Case III. Horizontal Prismatic Beam Supported at Both Ends and Bearing a Uniformly Distributed Load along its Whole Length.—(The weight of the beam itself, if considered, constitutes a load of this nature.) Let $l=$ the length of the beam and $w=$ the weight, per unit of length, of the loading; then the load coming upon any length x will be $=wx$, and the whole load $=wl$. By hypothesis w is constant. Fig. 228. From symmetry we know that the

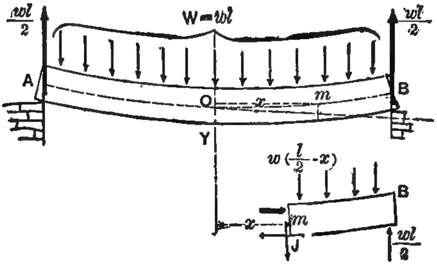


Fig. 228.

reactions at A and B are each $=\frac{1}{2}wl$, that the middle O of the neutral line is its lowest point, and the tangent line at O is horizontal. Conceiving a section made at any point m of the neutral line at a distance x from O, consider as free the portion of beam on the right of m. The forces holding this portion in equilibrium are $\frac{1}{2}wl$, the reaction at B; the elastic forces of the exposed surface at m, viz.: the tensions and compressions, forming a couple, and J the total shear; and a portion of the load, $w(\frac{1}{2}l-x)$. The sum of the moments of these latter forces about the neutral axis of m, is the same as that of their resultant; (i.e., their sum, since they are parallel), and this resultant acts in the middle of the length $\frac{1}{2}l-x$. Hence the sum of these moments $=w(\frac{1}{2}l-x)\frac{1}{2}(\frac{1}{2}l-x)$. Now putting Σ (mom. about neutral axis of m)$=0$ for this free body, we have

$$EI\frac{d^2y}{dx^2}=\frac{1}{2}wl(\frac{1}{2}l-x)-\frac{1}{2}w(\frac{1}{2}l-x)^2\,;$$

$$\text{i.e., } EI\frac{d^2y}{dx^2}=\frac{1}{2}w(\frac{1}{4}l^2-x^2) \quad . \quad . \qquad (1)$$

Taking the x-anti-derivative of both sides of (1),

$$EI\frac{dy}{dx}=\tfrac{1}{2}w(\tfrac{1}{4}l^2x-\tfrac{1}{3}x^3)+(C=0) \qquad (2)$$

as the equation of slope. (The constant is $=0$ since at O both $dy\div dx$ and x are $=0$.) From (2),

$$EIy=\frac{w}{2}(\tfrac{1}{8}l^2x^2-\tfrac{1}{12}x^4)+[C'=0] \quad . \quad . \quad (3)$$

which is the equation of the elastic curve; throughout, i.e., it admits any value of x from $x=+\tfrac{1}{2}l$ to $x=-\tfrac{1}{2}l$. This is an equation of the fourth degree, one degree higher than those for the Curves of Cases I and II, where there were no distributed loads. If w were not constant, but proportional to the ordinates of an inclined right line, eq. (3) would be of the fifth degree; if w were proportional to the vertical ordinates of a parabola with axis vertical, (3) would be of the sixth degree; and so on.

By putting $x=\tfrac{1}{2}l$ in (3) we have the deflection of O below the horizontal through A and B, viz.: (with $W=$ total load $=wl$)

$$d=\frac{5}{384}\cdot\frac{wl^4}{EI}=\frac{5}{384}\cdot\frac{Wl^3}{EI} \quad . \quad . \quad (4)$$

237. Case IV. Cantilevers.—A horizontal beam whose only support consists in one end being built in a wall, as in Fig. 229(a), or supported as in Fig. 229(b) is sometimes called a cantilever. Let the student prove that in Fig. 229(a) with a single end load P, the deflection of B below the tangent at O is $d=\tfrac{1}{3}Pl^3\div EI$; the same statement applies to Fig. 229(b), but the tangent at O is not horizontal if the beam was originally so. It can also be proved that the slope at B, Fig. 229(a) (from the tangent at O) is

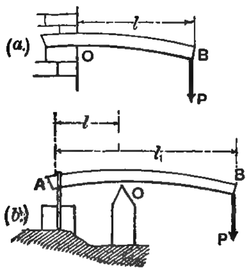


FIG. 229.

$$a_1 = \frac{Pl^2}{2EI}.$$

The greatest deflection of the elastic curve from the right line joining AB, in Fig. 229(b), is evidently given by the equation for y max. in § 235, by writing, instead of P of that equation, the reaction at O in Fig. 229(b). This assumes that the max. deflection occurs between A and O. If it occurs between O and B put $(l_1 - l)$ for l.

If in Fig. 229(a) the loading is uniformly distributed along the beam at the rate of w pounds per linear unit, the student may also prove that the deflection of B below the tangent at O is

$$d = \tfrac{1}{8} wl^4 \div EI = \tfrac{1}{8} \frac{Wl^3}{EI}$$

238. Case V. Horizontal Prismatic Beam Bearing Equal Terminal Loads and Supported Symmetrically at Two Points.—Fig. 231. Weight of beam neglected. In the preceding cases we have made use of the approximate form $EId^2y \div dx^2$ in determining the forms of elastic curves. In the present case the elastic curve from O to C is more directly dealt with by employing the more exact expression $EI \div \rho$ (see § 231) for the moment of the stress-couple in any section. The reactions at O and C are each $= P$, from symmetry. Considering free a portion of the beam extending from A to any section m between O and C (Fig. 232) we have, by putting Σ (mom. about neutral axis of m) $= 0$,

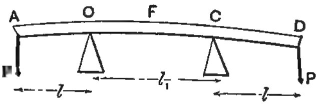


FIG. 231.

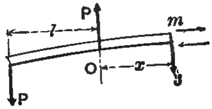


FIG. 232.

$$P(l+x) - \frac{EI}{\rho} - Px = 0 \therefore \rho = \frac{EI}{Pl}$$

That is, the radius of curvature is the same at all points of OC; in other words OC is the *arc of a circle* with the above radius. The upward deflection of F from the right line joining O and C can easily be computed from a knowledge of this fact. This is left to the student as also the value of the slope of the tangent line at O (and C). The deflection of D from the tangent at $C = {}^1/_3 Pl^3 \div EI$, as in Fig. 229(a).

SAFE LOADS IN FLEXURE.

239. Maximum Moment.—As we examine the different sections of a given beam under a given loading we find different values of p, the normal stress per unit of area in the outer element, as obtained from eq. (5) § 229, viz.:

$$\frac{pI}{e} = M. \quad . \quad . \quad . \quad . \quad (1)$$

in which I is the "Moment of Inertia" (§ 85) of the plane figure formed by the section, about its neutral axis, e the distance of the most distant (or outer) fibre from the neutral axis, and M the sum of the moments, about this neutral axis, of all the forces acting on the free body of which the section in question is one end, exclusive of the stresses on the exposed surface of that section. In other words M is the sum of the moments of the forces which balance the stresses of the section, these moments being taken about the neutral axis of the section under examination.

For the prismatic beams of this chapter e and I are the same at all sections, hence p varies with M and becomes a maximum when M is a maximum. In any given case the location of the "*dangerous section*," or section of maximum M, and the amount of that maximum value may be determined by inspection and trial, this being the only method (except by graphics) if the external forces are detached.

If, however, the loading is continuous according to a definite algebraic law the calculus may often be applied, taking care to treat separately each portion of the beam between two consecutive reactions of supports, or detached loads.

As a graphical representation of the values of M along the beam in any given case, these values may be conceived laid off as vertical ordinates (according to some definite scale, e.g. so many inch-lbs. of moment to the linear inch of paper) from a horizontal axis just below the beam. If the upper fibres are in compression in any portion of the beam, so that that portion is convex downwards, these ordinates will be laid off below the axis, and vice versâ; for it is evident that at a section where $M=0$, p also $=0$, i.e., the character of the normal stress in the outermost fibre changes (from tension to compression, or vice versâ) when M changes sign. It is also evident from eq. (6) § 231 that the radius of curvature changes sign, and consequently the curvature is reversed, when M changes sign. These moment ordinates form a **Moment Diagram**, and the extremities a **Moment Curve.**

The maximum moment, M_m, being found, in terms of the loads and reactions, we must make the p of the "dangerous section," where $M=M_m$, equal to a safe value R', and thus may write

$$\frac{R'I}{e}=M_m \quad . \quad . \quad . \quad . \quad (2)$$

Eq. (2) is available for finding any one unknown quantity, whether it be a load, span, or some one dimension of the beam, and is concerned only with the **Strength**, and not with the stiffness of the beam. If it is satisfied in any given case, the normal stress on all elements in all sections is known to be $=$ or $<R'$, and the design is therefore safe in that one respect.

As to danger arising from the shearing stresses in any section, the consideration of the latter will be taken up in a subsequent chapter and will be found to be necessary only in beams composed of a thin web uniting two flanges. The *total shear*, however, denoted by J, bears to the moment M, an important relation of great service in determining M_m. This relation, therefore, is presented in the next article.

240. The Shear is the First x-Derivative of the Moment.—Fig. 233. (x is the distance of any section, *measured parallel to the beam* from an arbitrary origin). Consider as free a vertical slice of the beam included between any two consecutive vertical sections whose distance apart is dx. The forces acting are the elastic forces of the two internal surfaces now laid bare, and, possibly, a portion, wdx, of the loading, which at this part of the beam has some intensity $=w$ lbs. per running linear unit. Putting Σ(mom. about axis N')$=0$ we have (noting that since the tensions and compressions of section N form a couple, the sum of their moments about N' is just the same as about N,)

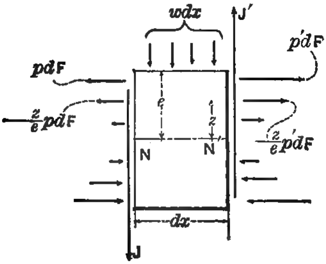


FIG. 233.

$$\frac{pI}{e} - \frac{p'I}{e} + Jdx + wdx.\frac{dx}{2} = 0$$

But $\frac{pI}{e} = M$, the Moment of the left hand section, $\frac{p'I}{e} = M'$, that of the right; whence we may write, after dividing through by dx and transposing,

$$\frac{M'-M}{dx} = J + w\frac{dx}{2} \quad \text{i.e.,} \frac{dM}{dx} = J; \tag{3}$$

for $w\frac{dx}{2}$ vanishes when added to the finite J, and $M'-M = dM =$ increment of the moment corresponding to the increment, dx, of x. This proves the theorem.

Now the value of x which renders M a maximum or minimum would be obtained by putting the derivative $dM \div dx =$ zero; hence we may state as a

Corollary.—*At sections where the moment is a maximum or minimum the shear passes through the value zero.*

The shear J at any section is easily determined by considering free the portion of beam from the section to either end of the beam and putting Σ(vertical components)=0.

In this article the words maximum and minimum are used in the same sense as in calculus; i.e., graphically, they are the ordinates of the moment curve at points where the *tangent line is horizontal.* If the moment curve be reduced to a straight line, or a series of straight lines, it has no maximum or minimum in the strict sense just stated; nevertheless the relation is still practically borne out by the fact that at the sections of greatest and least ordinates in the moment diagram the shear changes sign suddenly. This is best shown by drawing a *shear diagram,* whose ordinates are laid off vertically from a horizontal axis and under the respective sections of the beam. They will be laid off upward or downward according as J is found to be upward or downward, when the free body considered extends from the section toward the right.

In these diagrams the moment ordinates are set off on an arbitrary scale of so many inch-pounds, or foot-pounds, to the linear inch of paper; the shears being simply pounds, or some other unit of *force,* on a scale of so many pounds to the inch of paper. The scale on which the beam is drawn is so many feet, or inches, to the inch of paper.

241. Safe Load at the Middle of a Prismatic Beam Supported at the Ends.—Fig. 234. The reaction at each support is $\frac{1}{2}P$. Make a section n at any distance $x < \frac{l}{2}$ from B. Consider the portion nB free, putting in the proper elastic and external forces. The weight of beam is neglected. From Σ(mom. about n)=0 we have

$$\frac{pI}{e}=\frac{P}{2}x\text{; i.e., } M=\tfrac{1}{2}Px$$

Evidently M is proportional to x, and the ordinates representing it will therefore be limited by the straight line

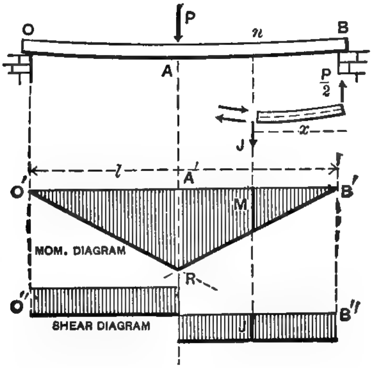


FIG. 234.

$B'R$, forming a triangle $B'RA'$. From symmetry, another triangle $O'RA'$ forms the other half of the moment diagram. From inspection, the maximum M is seen to be in the middle where $x=\tfrac{1}{2}l$, and hence

$$(M\text{ max.})=M_m=\tfrac{1}{4}Pl \quad . \quad . \quad . \quad (1)$$

Again by putting Σ(vert. compons.)$=0$, for the free body nB we have

$$J=\frac{P}{2}$$

and must point downward since $\frac{P}{2}$ points upward. Hence the shear is constant and $=\tfrac{1}{2}P$ at any section in the right hand half. If n be taken in the left half we would have, nB being free, from Σ(vert. com.)$=0$,

$$J=P-\tfrac{1}{2}P=\tfrac{1}{2}P$$

the same numerical value as before; but J must point upward, since $\frac{P}{2}$ at B and J at n must balance the downward P at A. At A, then, the shear changes sign suddenly, that is, passes through *the value zero;* also at A, *M is a maximum,* thus illustrating the statement in § 240. Notice the shear diagram in Fig. 234.

To find the safe load in this case we write the maximum value of the normal stress, $p_{\prime} = R'$, a safe value, (see table in a subsequent article) and solve the equation for P. But the maximum value of p is in the outer fibre at A, since M for that section is a maximum. Hence

$$\frac{R'I}{e} = \tfrac{1}{4} Pl \tag{2}$$

is the equation for safe loading in this case, so far as the normal stresses in any section are concerned.

EXAMPLE.—If the beam is of wood and has a rectangular section with width $b = 2$ in., height $h = 4$ in., while its length $l = 10$ ft., required the safe load, if the greatest normal stress is limited to 1,000 lbs. per sq. in. Use the pound and inch. From § 90 $I = \frac{1}{12} bh^3 = \frac{1}{12} \times 2 \times 64 = 10.66$ biquad. inches, while $e = \frac{h}{2} = 2$ in.

$$\therefore P = \frac{4R'I}{le} = \frac{4 \times 1{,}000 \times 10.66}{120 \times 2} = 177.7 \text{ lbs.}$$

242. Safe Load Uniformly Distributed along a Prismatic Beam Supported at the Ends.—Let the load per lineal unit of the length of beam be $= w$ (this can be made to include the weight of the beam itself). Fig. 235. From symmetry, each reaction $= \tfrac{1}{2} wl$. For the free body nO we have, putting Σ(mom. about n)$=0$,

$$\frac{pI}{e} = \frac{wl}{2} x - (wx)\frac{x}{2} \;\therefore\; M = \frac{w}{2}(lx - x^2)$$

which gives M for any section by making x vary from 0 to l. Notice that in this case the law of loading is continuous along the whole length, and that hence the moment curve is continuous for the whole length.

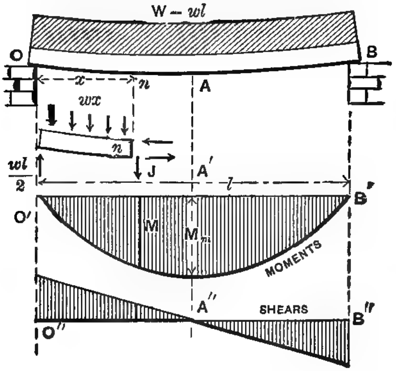


FIG. 235.

To find the shear J, at n, we may either put Σ(vert. compons.) = 0 for the free body, whence $J = \frac{1}{2}wl - wx$, and mus therefore be downward for a small value of x; or, employing § 240, we may write out $dM \div dx$, which gives

$$J = \frac{dM}{dx} = \frac{w}{2}(l - 2x) \qquad (1)$$

the same as before. To find the max. M, or M_m, put $J = 0$, which gives $x = \frac{1}{2}l$. This indicates a maximum, for when substituted in $d^2M \div dx^2$, i.e., in $-w$, a negative result is obtained. Hence M_m occurs at the middle of the beam and its value is

$$M_m = \tfrac{1}{8}wl^2; \ \therefore \ \frac{R'I}{e} = \tfrac{1}{8}wl^2 = \tfrac{1}{8}Wl \qquad (2)$$

the equation of safe loading. W = total load $= wl$.

It can easily be shown that the moment curve is a portion of a parabola, whose vertex is at A'' under the middle of the beam, and axis vertical. The shear diagram consists of ordinates to a single straight line inclined to its axis and crossing it, i.e., giving a zero shear, under the middle of the beam, where we find the max. M.

If a frictionless dove-tail joint with vertical faces were introduced at any locality in the beam and thus divided the beam into two parts, the presence of J would be made manifest by the downward slipping of the left hand part on the right hand part if the joint were on the right of the middle, and vice versâ if it were on the left of the middle. This shows why the ordinates in the two halves of the shear diagram have opposite signs. The greatest shear is close to either support and is $J_m = \frac{1}{2}wl$.

243. Prismatic Beam Supported at its Extremities and Loaded in any Manner. Equation for Safe Loading.—Fig. 236. Given the loads P_1, P_2, and P_3, whose distances from the right support are l_1, l_2, and l_3; ,required the equation for safe loading; i.e., find M_m and write it $= R'I \div e$.

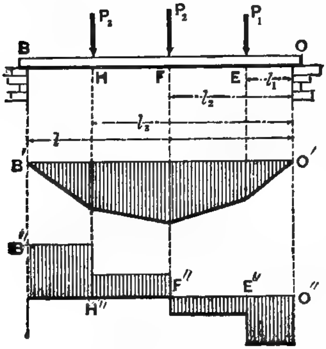


FIG. 236.

If the moment curve were continuous, i.e., if M were a continuous function of x from end to end of the beam, we could easily find M_m by making $dM \div dx = 0$, i.e., $J = 0$, and substitute the resulting value of x in the expression for M. But in the present case of detached loads, J is not zero, necessarily, at any section of the beam. Still there is some one section where it changes sign, i.e., passes suddenly through the value zero, and this will be the section of greatest moment (though not a maximum in the strict sense used in calculus). By considering any portion nO as free, J is found equal to **the Reaction at O Diminished by the Loads Occurring Between** n and O. The reaction at B is

$$P_B=(P_1l_1+P_2l_2+P_3l_3)\div l$$

obtained by treating the whole beam as free (in which case no elastic forces come into play) and putting Σ(mom. about O)$=0$; while that at $O, =P_0=P_1+P_2+P_3-P_B$

If n is taken anywhere between O and E, $J=P_0$

" " " " E " F, $J=P_0-P_1$

" " " " F " H, $J=P_0-P_1-P_2$

" " " " H " B, $J=P_0-P_1-P_2-P_3$

This last value of J also = the reaction at the other support, B. Accordingly, the shear diagram is seen to consist of a number of horizontal steps. The relation $J=dM\div dx$ is such that the *slope* of the moment curve is proportional to the *ordinate* of the shear diagram, and that for a sudden change in the *slope* of the moment curve there is a sudden change in the shear *ordinate*. Hence in the present instance, J being constant between any two consecutive loads, the moment curve reduces to a straight line between the same loads, this line having a different inclination under each of the portions into which the beam is divided by the loads. Under each load the *slope* of the moment curve and the *ordinate* of the shear diagram change suddenly. In Fig. 236 the shear passes through the value zero, i.e., changes sign, at F; or algebraically we are supposed to find that P_0-P_1 is + while $P_0-P_1-P_2$ is −, in the present case. Considering FO, then, as free, we find M_m to be

$M_m=P_0l_2-P_1(l_2-l_1)$ and the equation for safe loading is

$$\frac{R'I}{e}=P_0l_2-P_1(l_2-l_1) \qquad (1)$$

(i.e., if the max. M is at F). It is also evident that the greatest shear is equal to the reaction at one or the other support, whichever is the greater, and that the moment at either support is zero.

The student should not confuse the moment curve, which is entirely imaginary, with the neutral line (or elastic curve) of the beam itself. The greatest moment is not necessarily at the section of maximum deflection of the neutral line (or elastic curve).

For the case in Fig. 236 we may therefore state that the max. moment, and consequently the greatest tension or compression in the outer fibre, will be found in the section under that load for which the sum of the loads (including this load itself) between it and either support first equals or exceeds the reaction of that support. The amount of this moment is then obtained by treating as free either of the two portions of the beam into which this section divides the beam.

244. Numerical Example of the Preceding Article.—Fig. 237. Given P_1, P_2, P_3, equal to ½ ton, 1 ton, and 4 tons, respectively; l_1=5 feet, l_2=7 feet, and l_3=10 feet; while the total length is 15 feet. The beam is of timber, of rectangular cross-section, the horizontal width being b=10 inches, and the value of R' (greatest safe normal stress), = ½ ton per sq. inch, or 1,000 lbs. per sq inch.

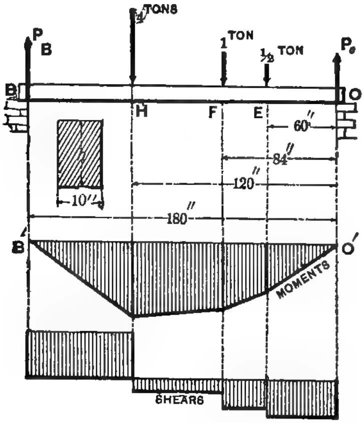


Fig. 237.

Required the proper depth h for the beam, for safe loading.

Solution.—Adopting a definite system of units, viz., the *inch-ton-second* system, we must reduce all distances such as l, etc., to inches, express all forces in tons, write $R'=\frac{1}{2}$ (tons per sq. inch), and *interpret all results by the same system.* Moments will be in inch-tons, and shears in tons. [N. B. In problems involving the strength of materials the inch is more convenient as a linear unit than the foot, since any stress expressed in lbs., or tons, per sq. inch, is numerically 144 times as small as if referred to the square foot.]

Making the whole beam free, we have from moms. about O, $P_B=\frac{1}{180}[\frac{1}{2}\times 60+1\times 84+4\times 120]=3.3$ tons $\therefore$ $P_0=5.5-3.3=2.2$ tons.

The shear anywhere between O and E is $J=P_0=2.2$ tons.

" " " " E and F is $J=2.2-\frac{1}{2}=1.7$ tons.

The shear anywhere between F and H is $J=2.2-\frac{1}{2}-1=0.7$ tons.

The shear anywhere between H and B is $J=2.2-\frac{1}{2}-1-4=-3.3$ tons.

Since the shear changes sign on passing H, $\therefore$ the max. moment is at H; whence making HO free, we have

M at $H=M_m=2.2\times 120-\frac{1}{2}\times 60-1\times 36=198$ inch-tons.

For safety M_m must $=\frac{R'I}{e}$, in which $R'=\frac{1}{2}$ ton per sq. inch, $e=\frac{1}{2}h=\frac{1}{2}$ of unknown depth of beam, and I, §90, $=\frac{1}{12}bh^3$, with $b=10$ inches

$\therefore \frac{1}{12}\cdot\frac{1}{2}\cdot\frac{2}{h}\times 10h^3=198$; or $h^2=237.6$ $\therefore$ $h=15.4$ inches.

245. **Comparative Strength of Rectangular Beams.**—For such a beam, under a given loading, the equation for safe loading is

$$\frac{R'I}{e}=M_m \text{ i. e. } \tfrac{1}{6}R'bh^2=M_m \quad \ldots\ldots (1)$$

whence the following is evident, (since for the same length, mode of support, and distribution of load, M_m is proportional to the safe loading.)

For rectangular prismatic beams of the same length, same material, same mode of support and same arrangement of load:

(1) The safe load is proportional to the width of beams having the same depth (h).

(2) The safe load is proportional to the square of the depth of beams having the same width (b).

(3) The safe load is proportional to the depth of beams having the *same volume* (i. e. the same bh)

(It is understood that the sides of the section are horizontal and vertical respectively and that the material is homogeneous.)

246. Comparative Stiffness of Rectangular Beams.—Taking the deflection under the same loading as an inverse measure of the stiffness, and noting that in §§ 233, 235, and 236, this deflection is inversely proportional to $I = \frac{1}{12}bh^3 =$ the "moment of inertia" of the section about its neutral axis, we may state that:

For rectangular prismatic beams of the same length, same material, same mode of support, and *same loading:*

(1) The stiffness is proportional to the width for beams of the same depth.

(2) The stiffness is proportional to the cube of the height for beams of the same width (b).

(3) The stiffness is proportional to the square of the depth for beams of equal volume (bhl).

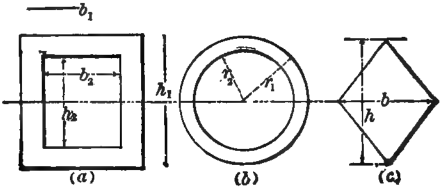
(4) It the length alone vary, the stiffness is inversely proportional to the cube of the length.

247. Table of Moments of Inertia.—These are here recapitulated for the simpler cases, and also the values of e, the distance of the outermost fibre from the axis.

Since the stiffness varies as I (other things being equal).

while the strength varies* as $I \div e$, it is evident that a square beam has the same stiffness in any position (§89), while its strength is greatest with one side horizontal, for then e is smallest, being $= \frac{1}{2}b$.

Since for any cross-section $I = \int dF\, z^2$, in which z = the distance of any element, dF, of area from the neutral axis, a beam is made both stiffer and stronger by throwing most of its material into two flanges united by a vertical web, thus forming a so-called "I-beam" of an I shape. But not without limit, for the web must be thick enough to cause the flanges to act together as a solid of continuous substance, and, if too high, is liable to buckle sideways, thus requiring lateral stiffening. These points will be treated later.



(a) (b) (c)

FIG. 238.

SECTION.	I	e
Rectangle, width = b, depth = h (vertical)	$\frac{1}{12}bh^3$	$\frac{1}{2}h$
Hollow Rectangle, symmet. about neutral axis. See Fig. 238 (*a*)	$\frac{1}{12}[b_1 h_1^3 - b_2 h^3_2]$	$\frac{1}{2}h_1$
Triangle, width = b, height = h, neutral axis parallel to base (horizontal).	$\frac{1}{36}bh^3$	$\frac{2}{3}h$
Circle of radius r	$\frac{1}{4}\pi r^4$	r
Ring of concentric circles. Fig. 238 (*b*)	$\frac{1}{4}\pi(r^4_1 - r^4_2)$	r_1
Rhombus; Fig. 238 (*c*) h = diagonal which is vertical.	$\frac{1}{48}bh^3$	$\frac{1}{2}h$
Square with side b vertical.	$\frac{1}{12}b^4$	$\frac{1}{2}b$
" " " b at 45° with horiz.	$\frac{1}{12}b^4$	$\frac{1}{2}b\sqrt{2}$

248. Moment of Inertia of I-beams, Box-girders, Etc.—In common with other large companies, the Cambria Steel

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* This function, $I \div e$, of the plane figure formed by the cross-section of a beam is evidently of three dimensions of length (cubic inches, for example), and is tabulated in the handbooks of the steel companies for different shapes of section; it is called the "*section-modulus.*" See next page.

Co. of Johnstown, Pa., manufactures prismatic rolled beams and other "shapes," of structural steel, which are variously called I-beams, deck-beams (or "bulb-beams"), rails, angles, T-bars, channels, Z-bars, etc., according to the form of their sections. See Fig. 239 for some of these forms. The company publishes a pocket-book giving tables of quantities relating to the strength and stiffness of beams, such as the safe loads for various spans, moments of inertia of their sections in various positions, etc., etc. The moments of inertia of *I*-beams and deck-beams are computed according to §§ 92 and 93, with the inch as linear unit. The *I*-beams range from 4 in. to 24 inches deep, the deck-beams being about 7 and 8 in. deep. (See foot-note, p. 274.)

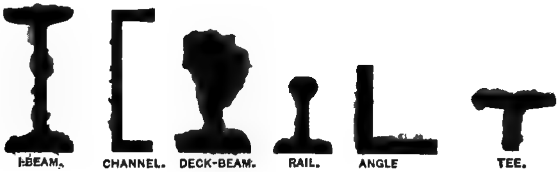

FIG. 239.

For beams of still greater stiffness and strength combinations of plates, channels, angles, etc., are riveted together, forming "built-beams," or "plate girders." The proper design for the riveting of such beams will be examined later. For the present the parts are assumed to act together as a continuous mass. For example, Fig. 240 shows a "box-girder," formed of two "channels" and two plates riveted together. If the axis of symmetry, *N*, is to be horizontal it becomes the neutral axis. Let $C=$ the moment of inertia of one channel (as given in the pocket-book mentioned) about the axis *N* perpendicular to the web of the channel. Then the total *moment of inertia of the combination* is (nearly)

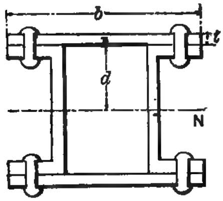

FIG. 240.

$$I_N = 2C + 2btd^2 - 4d't'(d - \tfrac{1}{2}t)^2 \quad . \quad . \quad (1)$$

In (1), b, t, and d are the distances given in Fig. 240 (d extends to the middle of plate) while d' and t' are the length and width of a rivet, the former from head to head (i.e., d' and t' are the dimensions of a rivet-hole).

For example, a box-girder of structural steel is formed of two 15-in. channels (35 lbs. per foot) and two plates 10 in. wide and 1 in. thick; the rivet-holes $\frac{3}{4}$ in. wid[illegible] $1\frac{3}{4}$ in. long. That is, $b=10$; $t=1$; $d=$ [illegible] $t'=\frac{3}{4}$; and $d'=1\frac{3}{4}$ in. Also from the hand-book we fi[illegible]hat for the channel in question [illegible]320 in.^{4} (i[illegible]., bi[illegible]uad. i[illegible]). Hence, e[illegible] (1),

$$I_N=6[illegible]0+2\times[illegible]\times[illegible]4\times[illegible](8-\tfrac{1}{2})^2=1625 \text{ in.}^4$$

In this instance $e=8\frac{1}{2}$ in.; and if 15,000 lbs./in.^{2} ($=7.5$ tons/in.^{2}) be taken as the value of R' (greatest safe normal stress in the extreme fibre of any section) as used by the Cambria Steel Co. for box-girders in buildings, we have

$$\frac{R'I}{e}=\frac{15000\times1625}{8.5}=2,867,500 \text{ inch-lbs.}$$

That is, the max. safe "*moment of resistance*" of the box-girder is $M_m=2,867,000$ inch-lbs. $=1433.7$ inch-tons; this quantity having to do with normal stresses in the section. The greatest "*bending-moment*" due to the amount, and mode, of loading on the beam, must not exceed this. Proper provision for the shearing stresses in the section, and in the rivets, will be considered later.)

249. Strength of Cantilevers.—In Fig. 241 with a single concentrated load P at the projecting extremity, we easily find the moment at n to be $M=Px$, and the max. moment to occur at the section next the wall, its value being $M_m=Pl$.

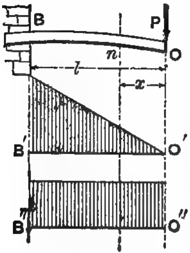

FIG. 241.

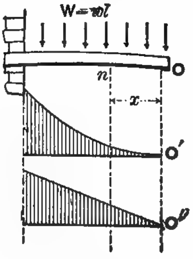

FIG. 242.

The shear, J, is constant, and $=P$ at all sections The moment and shear diagrams are drawn in accordance with these results.

If the load $W = wl$ is uniformly distributed on the cantilever, as in fig. 242, by making nO free we have, putting Σ(mom. about n) $= 0$,

$$\frac{pI}{e} = wx \cdot \frac{x}{2} \therefore M = \tfrac{1}{2}wx^2 \therefore M_m = \tfrac{1}{2}wl^2 = \tfrac{1}{2}Wl.$$

Hence the moment curve is a parabola, whose vertex is at O' and axis vertical. Putting Σ (vert. compons.) $= 0$ we obtain $J = wx$. Hence the shear diagram is a triangle, and the max. $J = wl = W$.

250. Résumé of the Four Simple Cases.—The following table shows the values of the deflections under an arbitrary load P, or W, (within elastic limit), and of the safe load; also the relative strength, the relative stiffness (under the same load), and the relative stiffness under the safe load, *for the same beam.*

	Cantilevers.		Beams with two end supports.	
	With one end load P Fig. 241	With unif. load $W = wl$ Fig. 242	Load P in middle Fig. 234	Unif. load $W = wl$ Fig. 235
Deflection	$\tfrac{1}{3} \cdot \frac{Pl^3}{EI}$	$\tfrac{1}{8} \cdot \frac{Wl^3}{EI}$	$\frac{1}{48} \cdot \frac{Pl^3}{EI}$	$\frac{5}{384} \cdot \frac{Wl^3}{EI}$
Safe load (from $\frac{R'I}{e} = M_m$)	$\frac{R'I}{le}$	$2\frac{R'I}{le}$	$4\frac{R'I}{le}$	$8\frac{R'I}{le}$
Relative strength	1	2	4	8
Relative stiffness under same load	1	$\frac{8}{3}$	16	$\frac{128}{5}$
Relative stiffness under safe load	1	$\frac{4}{3}$	4	$\frac{16}{5}$
Max. shear $= J_m$, (and location,	P, (at wall)	W, (at wall)	$\tfrac{1}{2}P$, (at supp).	$\tfrac{1}{2}$ W, (at supp

The max. shear will be used to determine the proper web-thickness for I-beams and "built-girders." The student should carefully study the foregoing table, noting especially the relative strength, stiffness, and stiffness under safe load, of the same beam.

Thus, a beam with two end supports will bear a double load, if uniformly distributed instead of concentrated in the middle, but will deflect ¼ more; whereas with a given load uniformly distributed the deflection would be only ⅝ of that caused by the same load in the middle, provided the elastic limit is not surpassed in either case.

251. R′, etc. For Various Materials.—The formula $\frac{p_m I}{e} = M_m$, from which in any given case of flexure we can compute the value of p_m, the greatest normal stress in any outer element, provided all the other quantities are known, holds good theoretically within the elastic limit only. Still, some experimenters have used this formula for the rupture of beams by flexure, calling the value of p_m thus obtained the *Modulus of Rupture, R*. R may be found to differ considerably from both the T or C of § 203 with some materials and forms, being frequently much larger. This might be expected, since even supposing the relative extension or compression (i.e., strain) of the fibres to be proportional to their distances from the neutral axis as the load increases toward rupture, the corresponding stresses, not being proportional to these strains beyond the elastic limit, no longer vary directly as the distances from the neutral axis; and the neutral axis does not pass through the centre of gravity of the section, necessarily.

The following table gives average values for R, R', R'', and E for the ordinary materials of construction.* E, the modulus of elasticity for use in the formulæ for deflection, is given as computed from experiments in flexure, and is nearly the same as E_t and E_c.

In any example involving R', e is usually written equal to the distance of the outer fibre from the neutral axis, whether that fibre is to be in tension or compression; since in most materials not only is the tensile equal to the compressive stress for a given strain (relative extension or contraction) but the elastic limit is reached at about the *same strain* both in tension and compression.

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* Wet, or unseasoned, timber is very considerably weaker than that (such as ordinary "dry" timber) containing only 12 per cent. of moisture. Large pieces of timber take a much longer time to season than small ones. (Johnson.)

TABLE FOR USE IN EXAMPLES IN FLEXURE.

	Timber.	Cast Iron.	Wro't Iron.	Structural Steel.
Max. safe stress in outer fibre =R'(lbs. per sq. inch).	600 to 1,200	6,000 in tens. 12,000 in comp.	12,000	15,000
Stress in outer fibre at Elas. limit =R''(lbs. per sq. in.)			17,000* to 35,000	30,000 and upward.
"Modul. of Rupture" =R=lbs. per sq. inch.	4,000 to 10,000	40,000	50,000	60,000
E=Mod. of Elasticity, =lbs. per sq. inch.	1,000,000 to 2,000,000	17,000,000	25,000,000	29,000,000

In the case of cast iron, however, (see § 203) the elastic limit is reached in tension with a stress $=9,000$ lbs. per sq. inch and a relative extension of $\frac{66}{1000}$ of one per cent., while in compression the stress must be about double to reach the elastic limit, the relative change of form (strain) being also double. Hence with cast iron beams, once extensively used but now largely replaced by rolled beams of structural steel, an economy of material was effected by making the outer fibre on the compressed side twice as far from the neutral axis as that on the stretched side. Thus, Fig. 243, cross-sections with unequal flanges were used, so proportioned that the centre of gravity was twice as near to the outer fibre in tension as to that in compression, i.e., $e_2=2e_1$; in other words more material is placed in tension than in compression. The fibre A being in tension (within elastic limit), that at B, since it is twice as far from the neutral axis and on the other side, is contracted twice as much as A is extended; i.e., is under a compressive strain double the tensile strain at A, but in accordance with the above figures its state of stress is proportionally as much within the elastic limit as that of A.

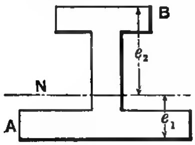

FIG. 243.

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* In the tests by U. S. Gov. in 1879 with I-beams, R'' ranged from 25,000 to 38,000, and the elastic limit was reached with less stress in the large than in the smaller beams. Also, for the same beam, R'' decreased with larger spans.

The great range of values of R for timber is due not only to the fact that the various kinds of wood differ widely in strength, while the behavior of specimens of any one kind depends somewhat on age, seasoning, etc., but also to the circumstance that the size of the beam under experiment has much to do with the result. The experiments of Prof. Lanza at the Mass. Institute of Technology in 1881 were made on full size lumber (spruce), of dimensions such as are usually taken for floor beams in buildings, and gave much smaller values of R (from 3,200 to 8,700 lbs. per sq. inch) than had previously been obtained. The loading employed was in most cases a concentrated load midway between the two supports.

These low values are probably due to the fact that in large specimens of ordinary lumber the continuity of its substance is more or less broken by cracks, knots, etc., the higher values of most other experimenters having been obtained with small, straight-grained, selected pieces, from one foot to six feet in length. See footnote p. 278.

Valuable information and tables relating to timber beams may be found in the hand-book of the Cambria Steel Co.

The value $R'=16,000$ lbs. per sq. inch is employed by the Cambria Steel Co. in computing the safe loads for their rolled I-beams of structural steel; but with the stipulation that the beams (which are high and of narrow width) must be secure against yielding sideways. If such is not the case the ratio of the actual safe load to that computed with $R'=16,000$ is taken less and less as the span increases. The lateral security referred to may be furnished by the brick arch-filling of a fire-proof floor, or by light lateral bracing with the other beams.

252. Numerical Examples.—EXAMPLE 1.—A square bar of wrought iron, $1\frac{1}{2}$ in. in thickness is bent into a circular arc whose radius is 200 ft., the plane of bending being parallel to the side of the square. Required the greatest normal stress p_m in any outer fibre.

Solution. From §§ 230 and 231 we may write

$$\frac{EI}{\rho}=\frac{pI}{e} \quad \therefore\ p=eE\div\rho, \text{ i.e., is constant.}$$

For the units *inch* and *pound* (viz. those of the table in § 251) we have $e=\frac{3}{4}$ in., $\rho=2{,}400$ in., and $E=25{,}000{,}000$ lbs. per sq. inch, and $\therefore$

$$p=p_m=\tfrac{3}{4}\times 25{,}000{,}000\div 2{,}400=7{,}812 \text{ lbs. per sq. in.},$$

which is quite safe. At a distance of $\frac{1}{2}$ inch from the neutral axis, the normal stress is $=[\frac{1}{2}\div\frac{3}{4}]p_m=\frac{2}{3}p_m=$ 5,208 lbs. per sq. in. (If the force-plane (i.e., plane of bending) were parallel to the *diagonal* of the square, e would $=\frac{1}{2}\times\ 1.5\sqrt{2}$ inches, giving $p_m=[7{,}812\times\sqrt{2}\ ]$ lbs. per sq. in.) § 238 shows an instance where a portion, OC, Fig. 231, is bent in a circular arc.

EXAMPLE 2.—A hollow cylindrical cast-iron pipe of radii $3\,{}^1\!/_2$ and 4 inches* is supported at its ends and loaded in middle (see Fig. 234). Required the safe load, neglecting the weight of the pipe. From the table in § 250 we have for safety

$$P=4\ \frac{R'I}{le}$$

From § 251 we put $R'=6{,}000$ lbs. per sq. in.; and from § 247 $I=\frac{\pi}{4}(r_1^4-r_2^4)$; and with these values, r_2 being $=\frac{7}{2}$, $r_1=4$, $e=r_1=4$, $\pi=\frac{22}{7}$ and $l=144$ inches (the inch must be the unit of length since $R'=6{,}000$ lbs. per sq. inch) we have

$$P=4\times 6{,}000\times\tfrac{1}{4}\cdot\tfrac{22}{7}(256-150)\div[144\times 4]\ \therefore\ P=3{,}470 \text{ lbs.}$$

The weight of the beam itself is $G=V\gamma$, (§ 7), i.e.,

$$G=\pi(r_1^2-r_2^2)l\gamma=\tfrac{22}{7}(16-12\tfrac{1}{4})144\times\tfrac{450}{1{,}728}=443 \text{ lbs.}$$

(Notice that γ, here, must be lbs., per *cubic inch*). This weight being a uniformly distributed load is equivalent to half as much, 221 lbs., applied in the middle, as far as the *strength* of the beam is concerned (see § 250), $\therefore$ P must be taken $=3{,}249$ lbs. when the weight of the beam is considered.

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* And length of 12 feet, should be added.

EXAMPLE 3.—A Cambria I-beam, of structural steel, is to be placed horizontally on two supports at its extremities and is to be loaded uniformly (Fig. 235), the span being $l=20$ ft. Its cross-section, Fig. 244, has a depth parallel to the web, of 15 in. In the handbook of the Cambria Steel Co. it is designated as B 53, 15 in. in depth, and weighing 42 lbs. per foot of length; its section having a moment of inertia $I_1=442$ in.^{4} about a gravity axis perpendicular to the web (for use when the web is vertical; the strongest position) and $I_2=14.6$ in.^{4} about a gravity axis parallel to the web (i.e., when the web is placed horizontally).

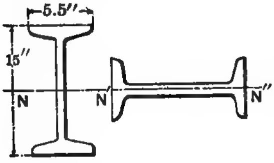

FIG. 244.

First, placing the web vertically, we have from § 250,

$$W_1=\text{safe load, distributed,}\ =8\frac{R'I_1}{le_1}.\quad \text{With } R'=16{,}000,$$

$I_1=442$, $l=240$ *inches*, and $e_1=7\frac{1}{2}$ inches, this gives *

$$W_1=(8\times16{,}000\times442)\div(240\times7.5)=31{,}430 \text{ lbs.}$$

But this includes the weight of the beam, $=20\times42=840$ lbs.; hence a distributed load of 30,600 lbs., or 15.3 tons, may be placed on the beam (secured against lateral yielding). The handbook of the Cambria Steel Co. referred to gives 15.7 tons as the safe load.)

With the web placed *horizontally*, we find as safe load

$$W_2=8\frac{R'I_2}{le_2}=(8\times16{,}000\times14.6)\div\left(240\times\frac{5.5}{2}\right)=2830 \text{ lbs.};$$

or less than 1/10 of W_1. Hence in this position the beam could carry safely only 1990 lbs. above its own weight.

EXAMPLE 4.—Required the deflection at the middle in the first case of Ex. 3. From § 250 this deflection is

$$d_1=\frac{5}{384}\cdot\frac{W_1l^3}{EI_1}=\frac{5}{384}\cdot\frac{8R'I_1}{le_1}\cdot\frac{l^3}{EI_1}=\frac{5}{48}\cdot\frac{R'}{E}\cdot\frac{l^2}{e_1}.$$

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* The handbook of the Cambria Steel Co. also gives in a separate column the quantity $I_1\div e_1$, called the "*section-modulus*," S, (cub. in. or in.^{3}); so that the formula for the safe load would be $W_1=8R'S\div l$, S having the value 58.9 in.^{3} in the present instance.

i.e., $d_1=\frac{5}{48}\cdot\frac{16,000}{29,000,000}\cdot\frac{(240)^2}{\frac{15}{2}}$; (inch and pound)

$\therefore\ d_1=0.441$ in.

EXAMPLE 5.—A rectangular beam of yellow pine, of width $b=4$ inches, is 20 ft. long, rests on two end supports, and is to carry a load of 1,200 lbs. at the middle; required the proper depth h. From § 250

$$P=4\frac{R'I}{le}=4\frac{R'}{l}\,\frac{bh^3}{12}\cdot\frac{1}{\frac{1}{2}h}$$

$\therefore\ h^2=6Pl\div 4R'b$. For variety, use the *inch* and *ton*. For this system of units $P=0.60$ tons, $R'=0.50$ tons per sq. in., $l=240$ inches and $b=$ 4 inches.

$$\therefore\ h^2=(6\times0.6\times240)\div(4\times0.5\times4)=108 \text{ sq. in.}\ \therefore\ h=10.4 \text{ in.}$$

EXAMPLE 6.—Suppose the depth in Ex. 5 to be determined by the condition that the deflection shall be $= {}^1/_{500}$ of the span or length. We should then have from § 250

$$d=\frac{1}{500}\ l=\frac{1}{48}\ \frac{Pl^3}{EI}$$

Using the inch and ton, with $E=1,200,000$ lbs. per sq. in., which $=$ 600 tons per sq. inch, and $I={}^1/_{12}bh^3$, we have

$$h^3=\frac{500\times0.60\times240\times240\times12}{48\times600\times4}=1,800\ \therefore\ h=12.2 \text{ in.}$$

As this is $>$ 10.4 the load would be safe, as well.

EXAMPLE 7.—Required the length of a wro't iron pipe supported at its extremities, its internal radius being 2¼ in., the external 2.50 in., that the deflection *under its own weight* may equal ${}^1/_{100}$ of the length. 579.6 in. Ans.

EXAMPLE 8.—Fig. 245. The wall is 6 feet high and one foot thick, of common brick work (see § 7) and is to be borne by an I-beam in whose outer fibres no greater normal stress than 8,000 lbs. per sq. inch is allowable. If a number of I-beams is available,

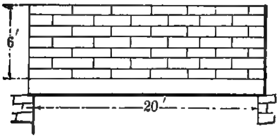

FIG. 245.

ranging in height from 6 in. to 15 in. (by whole inches), which one shall be chosen in the present instance, if their cross-sections are **Similar Figures**, the moment of inertia of the 15-inch beam being 800 biquad. inches?

The 12-inch beam. **Ans.**

SHEARING STRESSES IN FLEXURE.

253. Shearing Stresses in Surfaces Parallel to the Neutral Surface.—If a pile of boards (see Fig. 246) is used to support a load, the boards being free to slip on each other, it is noticeable that the ends overlap, although the boards are of equal length (now see Fig. 247); i.e., slipping has occurred along the surfaces of contact, the combination being no stronger than the same boards side by side. If, however, they are glued together, piled as in the former figure, the slipping is prevented and the deflection is much less under the same load P. That is, the compound beam is both stronger and stiffer than the pile of loose boards, but the *tendency* to slip still exists and is known as the "shearing stress in surfaces parallel to the neutral surface." Its intensity per unit of area will now be determined by the usual "free-body" method. In Fig. 248 let AN' be a portion, considered free, on the left of any

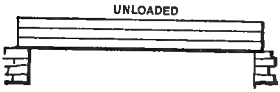

FIG. 246.

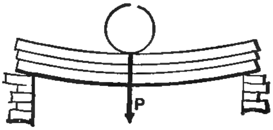

FIG. 247.

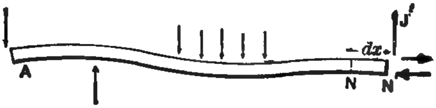

FIG. 248.

section N', of a prismatic beam slightly bent under forces in one plane and perpendicular to the beam. The moment equation, about the neutral axis at N', gives

$$\frac{p'I}{e}=M'; \text{ whence } p'=\frac{M'e}{I} \quad . \quad . \quad (1)$$

Similarly, with AN as a free body, NN' being $=dx$,

$$\frac{pI}{e}=M; \text{ whence } p=\frac{Me}{I} \quad . \quad . \quad . \quad (2)$$

p and p' are the respective normal stresses in the outer fibre in the transverse sections N and N' respectively.

Now separate the block NN', lying between these two consecutive sections, as a free body (in Fig. 249). And

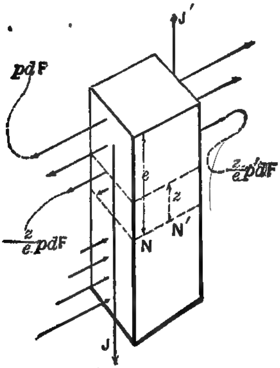

FIG. 249.

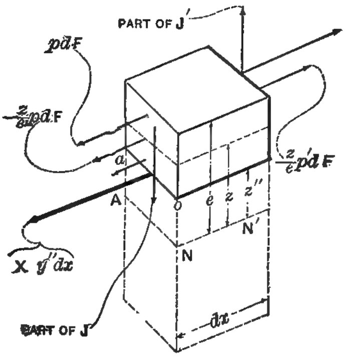

FIG. 250.

furthermore remove a portion of the top of the latter block, the portion lying above a plane passed parallel to the neutral surface and at *any* distance z'' from that surface. This latter free body is shown in Fig. 250, with the system of forces representing the actions upon it of the portions taken away. The under surface, just laid bare, is a portion of a surface (parallel to the neutral surface) in which the above mentioned slipping, or shearing, tendency exists. The lower portion (of the block NN') which is now removed exerted this rubbing, or sliding, force on the remainder along the under surface of the latter. Let the unknown intensity of this shearing force be X (per unit of area); then the shearing force on this under surface is $= Xy''dx$, (y'', $= oa$ in figure, being the horizontal width of the beam at this distance z'' from the neutral axis of N') and takes its place with the other forces of the system, which are the normal stresses between $\left[\begin{matrix} z=e \\ z=z'' \end{matrix}\right.$, and portions of J and J', the respective total vertical shears. (The manner of distribution of J over the vertical section is as yet unknown; see next article.)

Putting Σ (horiz. compons.) $= 0$ in Fig. 250, we have

$$\int_{z''}^{e} \frac{z}{e} p'dF - \int_{z''}^{e} \frac{z}{e} pdF - Xy''dx = 0$$

$$\therefore Xy''dx = \frac{p'-p}{e} \int_{z''}^{e} zdF$$

But from eqs. (1) and (2), $p'-p = (M'-M)\frac{e}{I} = \frac{e}{I}\, dM$, while from § 240 $dM = Jdx$;

$$\therefore Xy''dx = \frac{Jdx}{I} \int_{z''}^{e} zdF \therefore X = \frac{J}{Iy''} \int_{z''}^{e} zdF \quad . \; . \; . \; . \quad (3)$$

as the required intensity *per unit of area* of the shearing force in a surface parallel to the neutral surface and at a distance z'' from it. It is seen to depend on the "shear" J and the moment of inertia I of the *whole* vertical section; upon the horizontal thickness* y'' of the beam at the surface in question; and upon the integral $\int_{z''}^{e} zdF$, which (from § 23) is the *product of the area of that part of the vertical section extending from the surface in question to the outer fibre, by the distance of the centre of gravity of that part from the neutral surface.*

---

* Thickness of *actual substance.*

It now follows, from § 209, that the intensity (per unit area) of the shear on an elementary area of the *vertical cross section* of a bent beam, and this intensity we may call Z, is equal to that X, just found, in the horizontal section which is at the same distance (z'') from the neutral axis.

254. Mode of Distribution of J, the Total Shear, over the Vertical Cross Section.—The intensity of this shear, Z (lbs. per sq. inch, for instance) has just been proved to be

$$Z=X=\frac{J}{Iy''}\int_{z''}^{e} zdF \qquad (4)$$

To illustrate this, required the value of Z two inches above the neutral axis, in a cross section close to the abutment, in Ex. 5, § 252. Fig. 251 shows this section. From it we have for the *shaded portion*, lying above the locality in question, $y'' =$ 4 inches, and $\int_{z''=2}^{e=5.2} zdF =$ (area of shaded portion) × (distance of its centre of gravity from NA) = (12.8 sq. in.) × (3.6 in.) = 46.08 cubic inches.

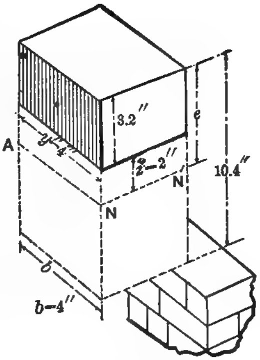

FIG. 251.

The total shear $J =$ the abutment reaction = 600 lbs., while $I = \frac{1}{12}bh^3 = \frac{1}{12} \times 4 \times (10.4)^3 = 375$ biquad. inches. Both J and I refer to the *whole section*.

$$\therefore Z=\frac{600\times 46.08}{375\times 4}=18.42 \text{ lbs. per sq. in.},$$

quite insignificant. In the neighborhood of the neutral axis, where $z'' = 0$, we have $y'' = 4$ and

$$\int_{z''=0}^{e} zdF=\int_{0}^{e} zdF=20.8 \times 2.6=54.8,$$

while J and I of course are the same as before. Hence for $z'' =0$

$$Z = Z_0 = 21.62 \text{ lbs. per sq. in.}$$

At the outer fibre since $\int_{e}^{e} zdF = 0$, z'' being $= e$, Z is $= 0$ for a beam of any shape.

For a solid rectangular section like the above, Z and z'' bear the same relation to each other as the co-ordinates of the parabola in Fig. 252 (axis horizontal).

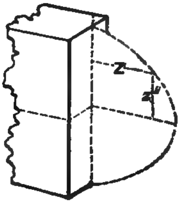

FIG. 252

Since in equation (4) the horizontal thickness, y'', from side to side ef the section of the locality where Z is desired, occurs in the denominator, and since $\int_{z''}^{e} zdF$ increases as z'' grows *numerically* smaller, the following may be stated, as to the distribution of J, the shear, in any vertical section, viz.:

The intensity (lbs. per sq. in.) of the shear is zero at the outer elements of the section, and for beams of ordinary shapes is greatest where the section crosses the neutral surface. For forms of cross section having thin webs its value may be so great as to require special investigation for safe design.

Denoting by Z_0 the value of Z at the neutral axis, (which $= X_0$ in the neutral surface where it crosses the vertica section in question) and putting the thickness of the substance of the beam $= b_0$ at the neutral axis, we have,

$$Z_0 = X_0 = \frac{J}{Ib_0} \times \left\{ \begin{array}{c} \text{area above} \\ \text{neutral axis} \\ \text{(or below)} \end{array} \right\} \times \left\{ \begin{array}{c} \text{the dist. of its cent.} \\ \text{grav. from that axis} \end{array} \right\} \quad (5)$$

255. Values of Z_0 for Special Forms of Cross Section.—From the last equation it is plain that for a prismatic beam the value of Z_0 is proportional to J, the total shear, and hence to the ordinate of the shear diagram for any particular case of loading. The utility of such a diagram, as obtained in Figs. 234–237 inclusive, is therefore evident, for by locating the greatest shearing stress in the beam it enables us to provide proper relations between the loading and the form and material of the beam to secure safety against rupture by shearing.

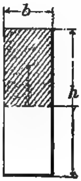

FIG. 253.

The table in § 210 gives safe values which the maximum Z_0 in any case should not exceed. It is only in the case of beams with thin webs (see Figs. 238 and 240) however, that Z_0 is likely to need attention.

For a **Rectangle** we have, Fig. 253, (see eq. 5, § 254) $b_0=b$, $I={}^1/_{12}bh^3$, and $\int_0^e zdF=\frac{1}{2}bh\,.\,\frac{1}{4}h=\frac{1}{8}bh^2$

$$\therefore Z_0=X_0=\tfrac{3}{2}\,\tfrac{J}{bh}\text{ i.e., }=\tfrac{3}{2}\,(\text{total shear})\div(\text{whole area})$$

Hence the greatest intensity of shear in the cross-section is $\frac{3}{2}$ as great per unit of area as if the total shear were uniformly distributed over the section.

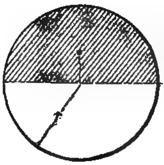

FIG. 254.

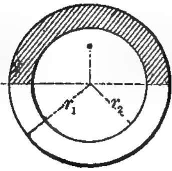

FIG. 255.

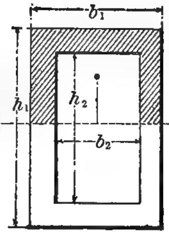

FIG. 256.

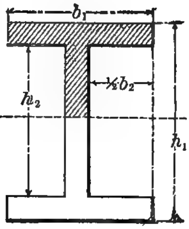

FIG. 257.

For a **Solid Circular** section Fig. 254

$$Z_0=\frac{J}{Ib_0}\int_0^e zdF=\frac{J}{\frac{1}{4}\pi r^4\,.\,2r}\,.\,\frac{\pi r^2}{2}\,.\,\frac{4r}{3\pi}=\frac{4}{3}\,.\,\frac{J}{\pi r^2}$$

[See § 26 Prob. 3].

For a **Hollow Circular** section (concentric circles) Fig. 255, we have similarly,

$$Z_0 = \frac{J}{\frac{1}{4}\pi(r_1^4 - r_2^4)2(r_1 - r_2)}\left[\frac{\pi r_1^2}{2}\cdot\frac{4r_1}{3\pi} - \frac{\pi r_2^2}{2}\cdot\frac{4r_2}{3\pi}\right]$$

$$= \frac{4}{3}\cdot\frac{J(r_1^3 - r_2^3)}{\pi(r_1^4 - r_2^4)(r_1 - r_2)}$$

Applying this formula to Example 2 § 252, we first have as the max. shear $J_m = \frac{1}{2}P = 1{,}735$ lbs., this being the abutment reaction, and hence (putting $\pi = (22 \div 7)$)

$$Z_0 \text{ max.} = \frac{4\times 7\times 1735[64-42.8]}{3\times 22[256-150](4-3.5)} = 294 \text{ lbs. per sq. in.}$$

which cast iron is abundantly able to withstand in shearing.

For a **Hollow Rectangular Beam**, symmetrical about its neutral surface, Fig. 256 (box girder)

$$Z_0 = \frac{J\,\frac{1}{8}(b_1h_1^2 - b_2h_2^2)}{\frac{1}{12}(b_1h_1^3 - b_2h_2^3)(b_1 - b_2)} = \frac{3}{2}\cdot\frac{J[b_1h_1^2 - b_2h_2^2]}{[b_1h_1^3 - b_2h_2^3][b_1 - b_2]}$$

The same equation holds good for Fig. 257 (I-beam with square corners) but then b_2 denotes the sum of the widths of the hollow spaces.

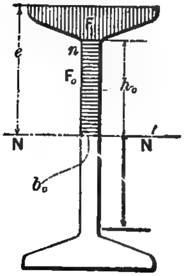

FIG. 258.

256. Shearing Stress in the Web of an I-Beam.—It is usual to consider that, with I-beams (and box-beams) with the web vertical the shear J, in any vertical section, is borne exclusively by the web and is uniformly distributed over its section. That this is nearly true may be proved as follows, the flange area being comparatively large. Fig. 258. Let F_1 be the area of one flange, and F_0 that of the half web. Then since

$$I = \tfrac{1}{12}b_0h_0^3 + 2F_1\left(\frac{h_0}{2}\right)^2,$$

(the last term approximate, $\frac{1}{2}h_0$ being taken for the radius of gyration of F_1,) while

$\int_0^e zdF = F_1\frac{h_0}{2}+F_0\frac{h_0}{4}$, (the first term approx.) we have

$$Z_0 = \frac{J\int_0^e zdF}{Ib_0} = \frac{J\frac{1}{4}h_0(2F_1+F_0)}{\frac{1}{12}h_0^2b_0(6F_1+2F_0)}\text{, which} = \frac{J}{b_0h_0},$$

if we write $(2F_1+F_0) \div (6F_1+2F_0) = \frac{1}{3}$. But b_0h_0 is the area of the whole web, $\therefore$ the shear per unit area at the neutral axis is nearly the same as if J were uniformly distributed over the web. E. g., with $F_1 = 2$ sq. in., and $F_0 = 1$ sq. in. we obtain $Z_0 = 1.07\ (J \div b_0h_0)$.

Similarly, the shearing stress per unit area at n, the upper edge of the web, is also nearly equal to $J \div b_0h_0$ (see eq., 4 (§254) for then $\left[\int_{z'=\frac{1}{2}h_0}^{e}(zdF)\right] = F_1.\frac{1}{2}h_0$ nearly, while I remains as before.

The shear per unit area, then, in an ordinary I-beam is obtained by dividing the total shear J by the area of the web section.*

Example.—It is required to determine the proper thickness to be given to the web of the 15-inch structural steel rolled I-beam of Example 3 of p. 282, the height of web being 13 in., and the maximum safe shearing stress being taken as 8750 lbs./in.^{2} (as prescribed by the Philadelphia building laws for mild steel). The web is vertical.

The greatest total shear, J_m, which occurs at either support, is equal to half the load, i.e., to 15,715 lbs.; and hence, with b_0 denoting the thickness of web, we have

$$Z_0\text{ max.} = \frac{J_m}{b_0h_0};\ \text{i.e., } 8750 = \frac{15,715}{b_0\times 13};\ \therefore b_0 = 0.138\text{ in.}$$

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* That is, for the vertical, or horizontal, section of web. The shear on some oblique plane may be somewhat larger than this. See §§ 270a and 314.

(Units, inch and pound). The 15-inch I-beam in question of the Cambria Steel Co.), weighing 42 lbs. to the linear foot, has a web 0.41 in. thick, which provides a very ample resistance to shearing stress.

In the middle of the span, $Z_0=0$, since $J=0$.

257. Designing of Riveting for Built Beams.—The latter are generally of the I-beam and box forms, made by riveting together a number of continuous shapes, most of the material being thrown into the flange members. *E. g.*, in fig. 259, an I-beam* is formed by riveting together, in the manner shown in the figure, a "vertical stem plate" or web, four "angle-bars," and two "flange-plates," each of these seven pieces being continuous through the whole length of the beam. Fig 260 shows a box-girder. If the riveting is well done, the combination forms a single rigid beam whose safe load for a given span may be found by foregoing rules; in computing the moment of inertia, however, the portion of cross section cut out by the rivet holes must not be included. (This will be illustrated in a subsequent paragraph.) The safe load having been computed from a consideration of normal stresses only, and the web being made thick enough to take up the max. total shear, J_m, with safety, it still remains to design the riveting, through whose agency the web and flanges are caused to act together as a single continuous rigid mass. It will be on the side of safety to consider that at a given

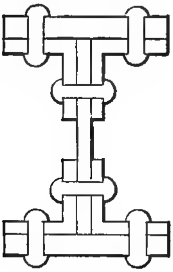
FIG. 259.

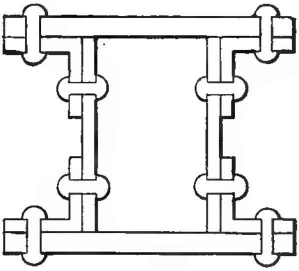
FIG. 260.

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* Such a built I-beam is usually designated a "*plate-girder.*" See handbook of the Cambria Steel Co.

locality in the beam the shear carried by the rivets connecting the angles and flanges, per unit of length of beam, is the same as that carried by those connecting the angles and the web (" vertical stem-plate "). The amount of this shear may be computed from the fact that it is equal to that occurring in the surface (parallel to the neutral surface) in which the web joins the flange, in case the web and flange were of continuous substance, as in a solid I-beam. But this shear must be of the same amount per horizontal unit of length as it is per vertical linear unit in the web itself, where it joins the flange ; (for from § 254 $Z = X$.) But the shear in the vertical section of the web, being uniformly distributed, is the same per vertical linear unit at the junction with the flange as at any other part of the web section (§ 256,) and the whole shear on the vertical section of web $= J$, the " total shear " of that section of the beam.

Hence we may state the following :

The riveting connecting the angles with the flanges, (or the web with the angles) in any locality of a built beam, must safely sustain *a shear equal to J on a horizontal length equal to the height of web.*

The strength of the riveting may be limited by the resistance of the rivet to being sheared (and this brings into account its cross section) or upon the crushing resistance of the side of the rivet hole in the plate (and this involves both the diameter of the rivet and the thickness of the metal in the web, flange, or angle. In its hand-book, the Cambria Steel Co. gives tables for the safe strength of rivets, and compressive resistance of plates; based on unit shearing stresses from 6,000 to 10,000 lbs./sq. in. for shearing stress in the rivets, and 12,000 to 20,000 lbs./sq. in. compressive resistance in the side of the rivet hole, the axial plane section of the hole being the area of reference.

In fig. 259 the rivets connecting the web with the angles are in *double shear*, which should be taken into account in considering their shearing strength, which is then double ; those connecting the angles and the flange plates are in *single shear.* In fig. 260 (box-beam) where the beam is built of two webs, four angles, and two flange plates, all the rivets are in single shear. If the web plate is very high compared with its thickness, vertical stiffeners in the form of "angles" may need to be riveted upon them laterally [see § 314].

Example.—A built I-beam of structural steel (fig. 259) is to support a uniformly distributed load of 40 tons, its extremities resting on supports 20 feet apart, and the height and thickness of web being 20 ins. and $\frac{1}{2}$ in. respectively. How shall the rivets, which are $\frac{7}{8}$ in. in diameter, be spaced between the web and the angles which are also $\frac{1}{2}$ in. in thickness? Let the unit stresses taken be 7500 for shearing, and 12,500 for side compression (lbs./in.^{2}). Referring to fig. 235 we find that $J=\frac{1}{2}W=20$ tons at each support and diminishes regularly to zero at the middle, where no riveting will therefore be required. Each rivet, having a sectional area of $\frac{1}{4}\pi(\frac{7}{8})^2 = 0.60$ sq. inches, can bear a safe shear of $0.60\times7500=4500$ lbs. in single shear, and hence of 9000 lbs. in double shear, which is the present case. But the safe compressive resistance of the side of the rivet hole in either the web or the angle is only $\frac{7}{8}$ in. $\times\frac{1}{2}$ in. $\times 12500=5470$ lbs., and thus determines the spacing of the rivets as follows:

Near a support the riveting must sustain a shear equal to 40,000 lbs. on a horizontal length equal to the height of web, i.e., to 20 ins., and the safe compression for each rivet is 5470 lbs. Hence $40{,}000\div5470$, or 7.2, rivets will be needed for the 20-inch length. In other words, they must be spaced $20\div7.2=2.7$ in. apart, center to center, near the supports; 5.4 in. apart at $\frac{1}{4}$ the span from a support; none at all in the middle. By the Cambria handbook, this distance apart should never be less than 3 diameters of the rivet; and, in connecting plates in compression, should not exceed 16 times the thickness of the plate.

As for the rivets connecting the angles and flange plates, being in two rows and opposite (in pairs) the safe shearing resistance of a pair (each in single shear) is 9,000 lbs., while the safe compressive resistance of the sides of the two rivet holes in the angle bars (the flange plate being much thicker) is =10,940 lbs. Hence the former figure (9,000) divided into 40,000, gives 4.44 as the number of pairs of rivets for 20 in. of length of the beam; i.e., the rivets in one row should be 20÷4.44=4.5 in. apart, centre to centre, near a support; the interval to be increased in inverse ratio to the distance from the middle of span, (bearing in mind the practical limitation just given).

If the load is concentrated in the middle of the span, instead of uniformly distributed, J is constant along each half-span, (see fig. 234) and the rivet spacing must accordingly be made the same at all localities of the beam.

SPECIAL PROBLEMS IN FLEXURE.

258. Designing Cross Sections of Built Beams.—The last paragraph dealt with the riveting of the various plates; we now consider the design of the plates themselves. Take for instance a plate-girder, fig. 261; one vertical stem-

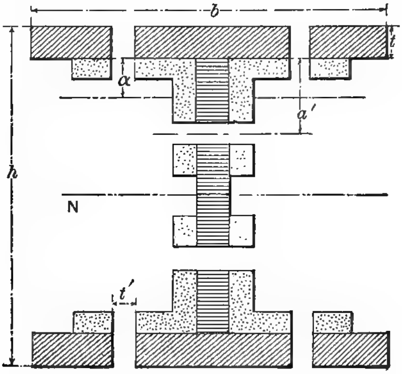

FIG. 261.

plate, four angle bars, (each of sectional area $= A$, remaining after the holes are punched, with a gravity axis parallel to, and at a distance $= a$ from its base), and two flange plates of width $= b$, and thickness $= t$. Let the whole depth of girder $= h$, and the diameter of a rivet hole $= t'$. To safely resist the tensile and compressive forces induced in this section by M_m inch-lbs. (M_m being the greatest moment in the beam which is prismatic) we have from § 239,

$$M_m = \frac{R'I}{e} \qquad (1)$$

R' for mild steel = 15,000 lbs. per sq. inch, e is $= \frac{1}{2} h$ while I, the moment of inertia of the compound section, is obtained as follows, taking into account the fact that the rivet holes cut out part of the material. In dealing with the sections of the angles and flanges, we consider them concentrated at their centres of gravity (an approximation, of course,) and treat their moments of inertia about N as single terms in the series $\int dF\, z^2$

(see § 85). The subtractive moments of inertia for the rivet holes in the web are similarly expressed; let b_0 = thickness of web.

$$\therefore \begin{cases} I_N \text{ for web} = \frac{1}{12} b_0 \, (h-2t)^3 - 2b_0 t' \left[\frac{h}{2} - t - a'\right]^2 \\ I_N \text{ for four angles} = 4A \left[\frac{h}{2} - t - a\right]^2 \\ I_N \text{ for two flanges} = 2(b-2t') \, t \left(\frac{h-t}{2}\right)^2 \end{cases}$$

the sum of which makes the I_N of the girder. Eq. (1) may now be written

$$\frac{M_m h}{2R'} = I_N \qquad (2)$$

which is available for computing any one unknown quantity. The quantities concerned in I_N are so numerous and they are combined in so complex a manner that in any numerical example it is best to adjust the dimensions of the section to each other by successive assumptions and trials. (The hand-book of the Cambria Steel Co. gives tables of safe loads of beam box-girders and plate-girders for a large variety of sizes and distances between supports; but attention is called to the fact that the loads given in the tables are based on the assumption that the girder is *supported laterally*, and that otherwise a proper reduction in the allowable safe load must be made, as explained elsewhere in the hand-book. The value of 15,000 lbs./sq. in. for R' has been used in computing these tables.)

EXAMPLE.—(Units, inch and pound). A plate-girder with end supports, of span = 20 ft. = 240 inches, is to support a uniformly distributed load of 45 tons = 90,000 lbs. If $\frac{3}{4}$ inch rivets are used, angle bars $3'' \times 3'' \times \frac{1}{2}''$, vertical web $\frac{1}{2}''$ in thickness, and plates 1 inch thick for flanges, how wide ($b = ?$) must these flange-plates be ? taking $h = 22$ inches = total height of girder.

Solution.—From the table in § 250 we find that the max. M for this case is $\frac{1}{8}$ Wl, where $W =$ the total distributed load (including the weight of the girder) and $l =$ span. Hence the left hand member of eq. (2) reduces to

$$\frac{Wl}{16} \cdot \frac{h}{R'} = \frac{90000 \times 240 \times 22}{16 \times 15000} = 1980.$$

That is, the total moment of inertia of the section must be = 1,980 biquad. inches, of which the web and angles supply a known amount, since $b_0 = \frac{1}{2}''$, $t = 1''$, $t' = \frac{3}{4}''$, $a' = 1\frac{3}{4}''$, $A = 2.0$ sq. in., $a = 0.9''$, and $h = 22''$, are known, while the remainder must be furnished by the flanges, thus determining their width b, the unknown quantity.

The *effective* area, A, of an angle bar is found thus: The full sectional area for the size given, $= 3 \times \frac{1}{2} + 2\frac{1}{2} \times \frac{1}{2} = 2.75$ sq. inches, from which deducting for two rivet holes we have

$$A = 2.75 - 2 \times \frac{3}{4} \times \frac{1}{2} = 2.0 \text{ sq. in.}$$

The value $a = 0.90''$ is found by cutting out the shape of two angles from sheet iron, thus: and balancing it on a knife edge.* (The gaps left by the rivet holes may be ignored, without great error, in finding a). Hence, substituting we have

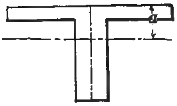

FIG. 261 *a*.

$$I_N \text{ for web } = \tfrac{1}{12} \cdot \tfrac{1}{2} \times 20^3 - 2 \times \tfrac{1}{2} \cdot \tfrac{3}{4}\ [8\tfrac{1}{4}]^2 = 282.3$$
$$I_N \text{ for four angles } = 4 \times 2 \times [9.10]^2 = 662.5$$
$$I_N \text{ for two flanges} = 2(b - \tfrac{6}{4}) \times 1 \times (10\tfrac{1}{2})^2 = 220.4\,(b - 1.5)$$
$$\therefore\ 1980 = 282.3 + 662.5 + (b - 1.5)220.4$$

whence $b = 4.6 + 1.5 = 6.1$ inches

the required total width of each of the 1 in. flange plates. This might be increased to 6.5 in. so as to equal the united width of the two angles and web.

The rivet spacing can now be designed by § 257, and the assumed thickness of web, ½ in., tested for the max. total shear by § 256. The latter test results as follows: The max. shear J_m occurs near either support and $= \frac{1}{2} W = 45{,}000$ lbs. ∴, calling b'_0 the least allowable thickness of web in order to keep the shearing stress as low as 8,000 lbs. per sq. inch,

$$b'_0 \times 20'' \times 8000 = 45000 \quad \therefore\ b'_0 = 0.28 \text{ in.}$$

showing that the assumed width of ½ in. is safe.

This girder will need vertical stiffeners near the ends, as explained subsequently, and is understood to be supported laterally.† Built beams of double web, or box-form, (see Fig. 260) do not need this lateral support.

259. Set of Moving Loads.—When a locomotive passes over a number of parallel prismatic girders, each one of which experiences certain detached pressures corresponding to the different wheels, by selecting any definite position of the wheels on the span, we may easily compute the reactions of the supports, then form the shear diagram, and finally as in § 243 obtain the max. moment, M_m, and the max. shear J_m, for this particular position of the wheels. But the values of M_m and J_m for some other position may be greater than those just found. We therefore inquire which will be the greatest moment among the infinite number of (M_m)'s (one for each possible position of the wheels on the span). It is evident from Fig. 236 from the nature of the moment diagram, that when the pressures or loads are detached, the M_m for any position of the loads, which of course are in this case at fixed distances apart, must occur under one of the loads (i.e. under a wheel). We begin ∴ by asking: What is the position of the set of moving loads when the moment under a given wheel is greater than will occur under that wheel in any other position? For example, in Fig. 262, in what position of the loads P_1, P_2, etc. on the span will the moment M_2, i.e., under P_2, be a maximum as compared with its value under P_2 in any other position on the span. Let R be the resultant of the *loads which are now on the span*, its *variable* distance from O be $= \bar{x}$, and its *fixed* distance from $P_2 = a'$; while a, b, c, etc., are the fixed distances between the loads (wheels). For any values of $\bar{x}$, as the loading moves through the range of motion within which no wheel of the set under consideration goes off the span, and no new wheel comes on it, we have $R_1 = \frac{\bar{x}}{l} R$, and the moment under P_2

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* The Cambria hand-book gives values of I and a for sections of angle-bars.

† See § 314.

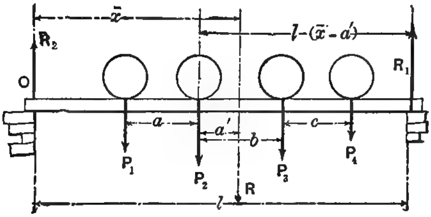

FIG 262.

$$=M_2=R_1[l-(\bar{x}-a')]-P_3b-P_4(b+c)$$

i.e. $$M_2=\frac{R}{l}(l\bar{x}-\bar{x}^2+a'\bar{x})-P_3b-P_4(b+c) \quad \ldots\ldots\ldots \quad (1)$$

In (1) we have M_2 as a function of $\bar{x}$, all the other quantities in the right hand member remaining constant as the loading moves; $\bar{x}$ may vary from $\bar{x}=a+a'$ to $\bar{x}=l-(c+b-a')$. For a max. M_2, we put $dM_2 \div d\bar{x}=0$, *i. e.*

$$\frac{R}{l}(l-2\bar{x}+a')=0 \;\therefore\; \bar{x}\,(\text{for Max } M_2)=\tfrac{1}{2}l+\tfrac{1}{2}a'$$

(For this, or any other value of $\bar{x}$, $d^2M_2 \div d\bar{x}^2$ is negative, hence a maximum is indicated). For a max. M_2, then, R must be as as far ($\frac{1}{2}a'$) on one side of the middle of the span as P_2 is on the other; *i.e.*, as the loading moves, the moment under a given wheel becomes a max. when that wheel and the centre of gravity of all the loads (*then on the span*) are *equi-distant from the middle of the span.*

In this way in any particular case we may find the respective max. moments occurring under each of the wheels during the passage, and the greatest of these is the M_m to be used in the equation $M_m=R'I \div e$ for safe loading.*

As to the shear J, for a given position of the wheels this will be the greatest at one or the other support, and equals the reaction at that support. When the load moves toward either support the shear at that end of the beam evidently increases so long as no wheel rolls completely over and beyond it. To find J max., then, dealing with each support in turn, we compute the successive reactions at the support when the loading is successively so placed that consecutive wheels, in turn, are on the point of rolling off the girder at that end; the greatest of these is the max. shear, J_m. As the max. moment is apt to come under the heaviest load it may not be necessary to deal with more than one or two wheels in finding M_m.

EXAMPLE.—Given the following wheel pressures,

$A< \,.\,.\, 8' \,.\,.\, >B< \,.\,.\, 5' \,.\,.\, >C< \,.\,.\, 4 \,.\,.\, <D$

4 tons. 6 tons. 6 tons. 5 tons.

on one rail which is continuous over a girder of 20 ft. span, under a locomotive.

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* Since this may be regarded as a case of "sudden application" of a load, it is customary to make R' much smaller than for a dead load; from one-third to one-half smaller.

1. Required the position of the resultant of A, B, and C;
2. " " " " A, B, C, and D;
3. " " " " B, C, and D.
4. In what position of the wheels on the span will the moment under B be a max.? Ditto for wheel C? Required the value of these moments and which is M_m?
5. Required the value of J_m, (max. shear), its location and the position of loads.

Results.—(1.) 7.8′ to right of A. (2.) 10′ to right of A. (3.) 4.4′ to right of B. (4.) Max. M_B = 1,273,000 inch lbs. with all the wheels on; Max. M_C = 1,440,000 inch-lbs. with wheels B, C, and D on. (5.) J_m = 13.6 tons at right support with wheel D close to this support.

260. Single Eccentric Load.—In the following special cases of prismatic beams, peculiar in the distribution of the loads, or mode of support, or both, the main objects sought are the values of the max. moment M_m, for use in the equation

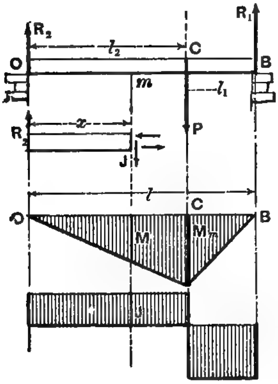

FIG. 263.

$$M_m = \frac{R'I}{e} \text{ (see §239)};$$

and of the max. shear J_m, from which to design the web riveting in the case of an I or box-girder. The modes of support will be such that the reactions are independent of the form and material of the beam (the weight of beam being neglected). As before, the flexure is to be slight, and the forces are all perpendicular to the beam.

The present problem is that in fig. 263, the beam being prismatic, supported at the ends, with a single eccentric load, P. We shall first disregard the weight of the beam itself. Let the span $= l_1 + l_2$. First considering the whole beam free we have the reactions $R_1 = Pl_2 \div l$ and $R_2 = Pl_1 \div l$.

Making a section at m and having Om free, x being $< l_2$, Σ (vert. compons.) = 0 gives

$$R_2 - J = 0, \text{ i.e., } J = R_2;$$

while from $\Sigma\,(\text{mom.})_m = 0$ we have

$$\frac{pI}{e} - R_2 x = 0 \therefore M = R_2 x = \frac{Pl_1}{l} x$$

These values of J and M hold good between O and C, J being constant, while M is proportional to x. Hence for OC the shear diagram is a rectangle and the moment diagram a triangle. By inspection the greatest M for OC is for $x = l_2$, and $= Pl_1 l_2 \div l$. This is the max. M for the beam, since between C and B, M is proportional to the distance of the section from B.

$$\therefore M_m = \frac{Pl_1 l_2}{l} \text{ and } \frac{R'I}{e} = \frac{Pl_1 l_2}{l} \quad . \quad . \quad . \qquad (1)$$

is the equation for safe loading.

$J = R_1$ in any section along CB, and is opposite in sign to what it is on OC; i.e., practically, if a dove-tail joint existed anywhere on OC the portion of the beam on the right of such section would slide downward relatively to the left hand portion; but vice versâ on CB.

Evidently the max. shear $J_m = R_1$ or R_2, as l_2 or l_1 is the greater segment.

It is also evident that for a given span and given beam the safe load P′, as computed from eq. (1) above, becomes very large as its point of application approaches a support; this would naturally be expected but not without limit, as the shear for sections between the load and the support is equal to the reaction at the near support and may thus soon reach a limiting value, when the safety of the web or the spacing of the rivets, if any, is considered.

Secondly, considering the *weight of the beam*, or *any uniformly distributed loading*, weighing w lbs. per unit of length of beam, in addition to P, Fig. 264, we have the reactions

$$R_1 = \frac{Pl_2}{l} + \frac{W}{2}; \text{ and } R_2 = \frac{Pl_1}{l} + \frac{W}{2}$$

Let l_2 be $> l_1$; then for a portion Om of length $x < l_2$ moments about m give

$$\frac{pI}{e} - R_2 x + wx.\frac{x}{2} = 0$$

i.e., *on* OC, $M = R_2 x - \frac{1}{2} wx^2$ . . . . . (2)

Evidently for $x = 0$ (i.e. at O) $M = 0$, while for $x = l_2$ (i.e. at C) we have, putting $w = W \div l$

$$M_C = R_2 l_2 - \tfrac{1}{2} w l_2^2 = \frac{Pl_1 l_2}{l} + \frac{Wl_2}{2} - \tfrac{1}{2}\frac{Wl_2^2}{l} \quad . \qquad (3)$$

It remains to be seen whether a value of M may not exist in some section between O and C, (i.e., for a value of $x < l_2$ in eq. (2)), still greater than M_C. Since (2) gives M as a continuous function of x between O and C, we put $dM \div dx = 0$, and obtain, substituting the value of the constants R_2 and w,

$$R_2 - wx = 0 \;\therefore\; x_n \left\{ \text{for } M \begin{matrix}\text{max.}\\ \text{or}\\ \text{min.}\end{matrix} \right\} = \frac{Pl_1}{W} + \tfrac{1}{2} l. \qquad (4)$$

This must be for M max., since $d^2M \div dx^2$ is negative when this value of x is substituted. If the particular value of x given by (4) is $< l_2$, the corresponding value of M (call it M_n) from eq. (2) will occur on OC and will be greater than M_C (Diagrams II. in fig. 264 show this case); but if x_n is $> l_2$, we are not concerned with the corresponding value of M, and the greatest M on OC would be M_C.

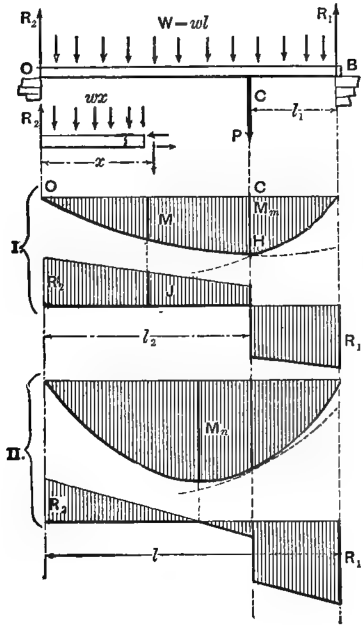

FIG. 264.

For the short portion BC, which has moment and shear diagrams of its own not continuous with those for OC, it may easily be shown that M_C is the greatest moment of any section. Hence the M

max., or M_m, of the whole beam is either M_C or M_n, according as x_n is $>$ or $<$ l_2. This latter criterion may be expressed thus, [with $l_2 - \frac{1}{2}l$ denoted by l_3, the distance of P from the middle of the span]:

From (eq. 4) $\left[\left(\frac{Pl_1}{W}+\frac{1}{2}l\right) \gtrless l_2\right.$ is equivalent to $\left.\left[\left(\frac{P}{W}\right) \gtrless \left(\frac{l_3}{l_1}\right)\right]\right.$

and since from (4) and (2)

$$M_n = \left[\frac{Pl_1}{l}+\frac{1}{2}W\right]\left[\frac{Pl_1}{W}+\frac{1}{2}\ l\right] - \frac{1}{2}\frac{W}{l}\left[\frac{Pl_1}{W}+\frac{1}{2}l\right]^2 \qquad (5)$$

The equation for safe loading is

$$\left.\begin{aligned} &\frac{R'I}{e} = M_c, \text{ when } \frac{P}{W} \text{ is } > \frac{l_3}{l_1} \\ \text{and} \quad &\\ &\frac{R'I}{e} = M_n, \text{ when } \frac{P}{W} \text{ is } < \frac{l_3}{l_1} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

See eqs. (3) and (5) for M_c and M_n

If either P, W, l_3, or l_1 is the unknown quantity sought, the criterion of (6) cannot be applied, and we ∴ use both equations in (6) and then discriminate between the two results.

The greatest shear is $J_m = R_1$, in Fig. 264, where l_2 is $> l_1$.

261. Two Equal Terminal Loads, Two Symmetrical Supports Fig. 265. [Same case as in Fig. 231, § 238]. Neglect weight of beam. The reaction at each support being $=P$, (from symmetry), we have for a free body Om with $x < l_1$

$$Px - \frac{pI}{e} = 0 \quad \therefore \quad M = Px \quad . \quad . \quad . \quad (1)$$

while where $x > l_1$ and $< l_1 + l_2$

$$Px - P(x - l_1) - \frac{pI}{e} = 0 \quad \therefore \quad M = Pl_1 \quad . \quad (2)$$

That is, see (1), M varies directly with x between O and C, while between C and D it is constant. Hence for safe loading

$$\frac{R'I}{e} = M_m \quad i.e., \frac{R'I}{e} = Pl_1 \quad . \quad . \quad (3)$$

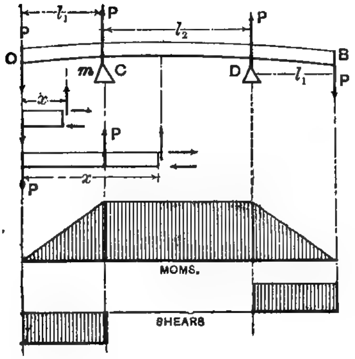

Fig. 265

The construction of the moment diagram is evident from equations (1) and (2).

As for J, the shear, the same free bodies give, from Σ (vert. forces)$=0$.

On OC . $J=P$ . . . (4)
On CD . $J=P-P=zero$ (5)

(4) and (5) might also be obtained from (1) and (2) by writing $J=d\,M\div dx$, but the former method is to be preferred in most cases, since the latter requires M to be expressed as a function of x while the former is applicable for examining separate sections without making use of a variable.

If the beam is an I-beam, the fact that J is zero anywhere on $C\,D$ would indicate that we may dispense with a web along $C\,D$ to unite the two flanges; but the lower flange being in compression and forming a "long column" would tend to buckle out of a straight line if not stayed by a web connection with the other, or some equivalent bracing.

262. Uniform Load over Part of the Span. Two End Supports. Fig. 266. Let the load $=W$, extending from one support over a portion $=c$, of the span, (on the left, say,) so that $W=wc$, w being the load per unit of length. Neglect weight of beam. For a free body Om of any length $x<\overline{O\,B}$ (*i.e.* $<c$), $\Sigma\ moms_m=0$ gives

$$\frac{pI}{e}+\frac{wx^2}{2}-R_1x=0\ \therefore M=R_1x-\frac{wx^2}{2}\quad . \; . \; . \quad (1)$$

which holds good for any section on $O\,B$. As for sections on $B\,C$ it is more simple to deal with the free body $m'C$, of length

$$x'<C\,B \text{ from which we have } M=R_2\,x' \quad . \; (2)$$

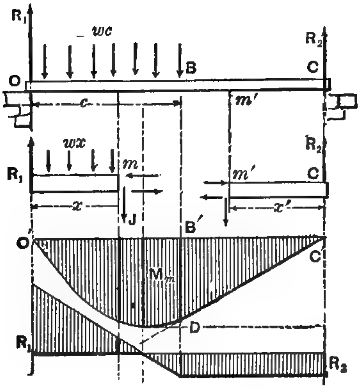

FIG. 266.

which shows the moment curve for $B\,C$ to be a straight line DC, tangent at D to the parabola $O'\,D$ representing eq. (1.) (If there were a concentrated load at B, $C\,D$ would meet the tangent at D at an angle instead of coinciding with it; let the student show why, from the shear diagram).

The shear for any value of x on $O\,B$ is:

$$\text{On } O\,B \quad . \quad . \quad . \quad J = R_1 - wx \quad . \quad . \quad . \quad (3)$$

$$\text{while on } B\,C \quad . \quad . \quad . \quad J = R_2 = \text{constant} \quad . \quad (4)$$

The shear diagram is constructed accordingly.

To find the position of the max. ordinate of the parabola, (and this from previous statements concerning the tangent at the point D must occur on $O\ B$, as will be seen and will ∴ be the M_m for the whole beam) we put $J=0$ in eq (3) whence

$$x \text{ (for } M_m) = \frac{R_1}{w} = \frac{W}{w}\frac{[l - \frac{c}{2}]}{l} = c - \frac{c^2}{2l} \quad . \quad . \quad (5)$$

and is less than c, as expected. [The value of $R_1 = \frac{W}{l}(l - \frac{c}{2})$,

$= (wc \div l)(l - \frac{c}{2})$, (the whole beam free) has been substituted]. This value of x substituted in eq. (1) gives

$$_m = (1 - \tfrac{1}{2}\frac{c}{l})^2 \cdot \tfrac{1}{2} \cdot Wc \therefore \frac{R'I}{e} = \tfrac{1}{2}[1 - \tfrac{1}{2} \cdot \frac{c}{l}]^2 Wc \ldots . \ (6)$$

is the equation for safe loading.

The max. shear J_m is found at O and is $= R_1$, which is evidently $> R_2$, at C.

263. Uniform Load Over Whole Length With Two Symmetric Supports. Fig. 267.—With the notation expressed in the figure, the following results may be obtained, after having divided the length of the beam into three parts for separate treatment as necessitated by the external forces, which are the distributed load *W*, and and the two reactions, each = ½ *W*. The moment curve is made up of parts of three distinct parabolas, each with its axis vertical. The central parabola may sink below the horizontal axis of reference if the supports are far enough apart, in which case (see Fig.) the elastic curve of the beam itself becomes concave upward between the points *E* and *F* of "contrary flexure." At each of these points the moment must be zero, since the radius of curvature is ∞ and $M = EI \div \rho$ (see § 231) at any section; that is, at these points the moment curve crosses its horizontal axis.

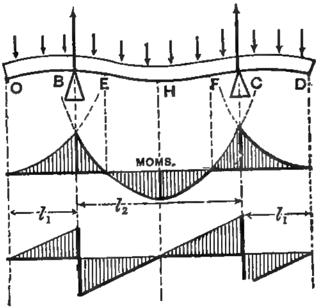

FIG. 267.

As to the location and amount of the max. moment M_m, inspecting the diagram we see that it will be either at *H*, the middle, or at both of the supports *B* and *C* (which from symmetry have equal moments), i.e., (with l = total length,)

$$M_m\left[\text{and} \therefore \frac{R'I}{e}\right] = \begin{cases} \text{either } \dfrac{W}{2l}\left[\tfrac{1}{4}l_2^2 - l_1^2\right] \;\ldots\ldots\; \text{at } H \\ \text{or } \quad \dfrac{W}{2l}l_1^2 \;\ldots\ldots\; \text{at } B \text{ and } C \end{cases}$$

according to which is the greater in any given case; i.e. according as l_2 is $>$ or $< l_1\sqrt{8}$.

The shear close on the left of $B = wl_1$, while close to the right of B it $= \frac{1}{2}W - wl_1$. (It will be noticed that in this case since the beam *overhangs*, beyond the support, the shear near the support is not equal to the reaction there, as it was in some preceding cases.)

$$\text{Hence } J_m = \left\{ \begin{array}{c} wl_1 \\ \tfrac{1}{2} W - wl_1 \end{array} \right\} \text{according as } l_1 \gtrless \tfrac{1}{4} l$$

264. Hydrostatic Pressure Against a Vertical Plank.—From elementary hydrostatics we know that the pressure, per unit area, of quiescent water against the vertical side of a tank, varies directly with the depth, x, below the surface, and equals the weight of a prism of water whose altitude $= x$, and whose sectional area is unity. See Fig. 268.

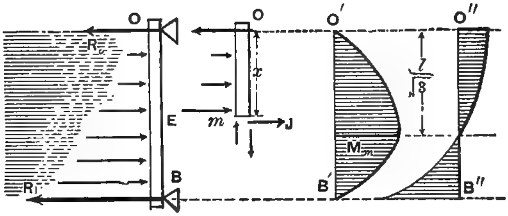

FIG. 268.

The plank is of rectangular cross section, its constant breadth, $= b$, being ⊥ to the paper, and receives no support except at its two extremities, O and B, *O being level with the water surface.* The loading, or pressure, *per unit of length of the beam,* is here *variable* and, by above definition, is $= w = \gamma x b$, where γ = weight of a cubic unit (i.e. the *heaviness,* see § 7) of water, and $x = Om$ = depth of any section m below the surface. The hydrostatic pressure on $dx = w dx$. These pressures for equal dx's, vary as the ordinates of a triangle OR_1B.

Consider Om free. Besides the elastic forces of the exposed section m, the forces acting are the reaction R_0, and the triangle of pressure OEm. The total of the latter is

$$W_x = \int_0^x w dx = \gamma b \int_0^x x dx = \gamma b \frac{x^2}{2} \quad \ldots\ldots \quad (1)$$

and the sum of the moments of these pressures about m is equal to that of their resultant ($=$ their sum, since they are parallel) about m, and $\therefore = \gamma b \frac{x^2}{2} \cdot \frac{x}{3}$.

[From (1) when $x = l$, we have for the total water pressure on the beam $W_l = \gamma b \frac{l^2}{2}$ and since one-third of this will be borne at O we have $R_0 = \frac{1}{6} \gamma b l^2$.]

Now putting Σ(moms. about the neutral axis of m)$=0$, for Om free, we have

$$R_0 x - W_x \cdot \frac{x}{3} - \frac{pI}{e} = 0 \therefore M = \frac{1}{6}\gamma b\,(l^2 x - x^3) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

(which holds good from $x = 0$ to $x = l$). From Σ (horiz. forces) $= 0$ we have also the shear

$$J = R_0 - W_x = \frac{1}{6}\gamma b\,(l^2 - 3x^2) \quad . \; . \; . \; . \quad (3)$$

as might also have been obtained by differentiating (2), since $J = dM \div dx$. By putting $J = 0$ (§ 240, corollary) we have for a max. M, $x = l \div \sqrt{3}$, which is less than l and hence is applicable to the problem. Substitute this in eq. 2, and reduce, and we have

$$\frac{R'I}{e} = M_m, \text{ i.e. } \frac{R'I}{e} = \frac{1}{9} \cdot \frac{1}{\sqrt{3}} \cdot \gamma b l^3 \quad . \; . \; . \; . \quad (4)$$

as the equation for safe loading.

265. Example.—If the thickness of the plank is h, required $h = ?$, if R' is taken $=$ 1,000 lbs. per sq. in. for timber (§ 251), and $l = 6$ feet. For the *inch-pound-second* system of units, we must substitute $R' = 1{,}000$; $l = 72$ *inches*; $\gamma = 0.036$ lbs. per cubic inch (heaviness of water in this system of units); while $I = bh^3 \div 12$, (§ 247), and $e = \frac{1}{2} h$. Hence from (4) we have

$$\frac{1000\,bh^3}{12 \times \frac{1}{2}h} = \frac{0.036 b \times 72^3}{9\sqrt{3}}, \therefore h^2 = 5.16 \therefore h = 2.27 \text{ in.}$$

It will be noticed that since x for $M_m = l \div \sqrt{3}$, and not $\frac{2}{3}\, l$, M_m does not occur in the section opposite the resultant of the water pressure; see Fig. 268. The shear curve is a parabola here; eq. (3).

266. Flat Circular Plate, Homogeneous and of Uniform Thickness, Supported all Round its Edge and Subjected to Uniform Fluid Pressure of w lbs. per sq. in. A strict treatment of this case being very complicated, an approximate method, due to Prof. C. Bach, will be presented.* Fig. 269 shows a top view of the circular plate, in a horizontal position and covering a circular opening, its edge being supported continuously on the edge of the opening (but not clamped to it). Let the radius of the plate be r and its thickness h. Under the plate is the atmosphere, while on its upper surface, acting uniformly over the whole of that surface, is a fluid pressure whose *excess* over that of the atmosphere is w lbs./sq. in. The particles near the upper surface are under compressive stress, which is obviously greater near the center of the circle; those near the lower surface are in tension.

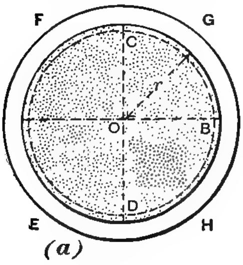

FIG. 269.

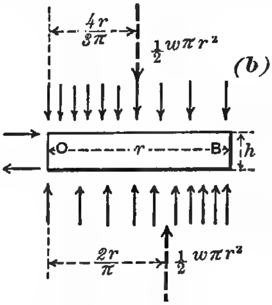

FIG. 269a.

Let now the half-plate, $CODB$, (cutting along the diameter CD) be considered as a "free body" in Fig. 269a; the tensile and compressive stresses in the section COD being assumed to form a stress-couple, as in previous case of flexure, the unit-stress varying as the distance from the middle of the thickness, with the stress in the outermost fiber denoted by p. Then the moment of this couple will be written $pI \div e$, as before, where $e = \frac{1}{2}h$ and $I = 2rh^3 \div 12$. On the upper surface of the free body we find a total pressure of $\frac{1}{2}w\pi r^2$ lbs., covering a semicircular *surface;* so that (p. 22) the distance of the resultant from O is $4r \div 3\pi$. The upward reaction from the supporting *edge* is also $\frac{1}{2}w\pi r^2$ lbs., but its resultant acts $2r/\pi$ in. from O (center of gravity of a semicircular "wire," p. 20). Taking moments, then, about O we have

$$\frac{w\pi r^2}{2}\left[\frac{2r}{\pi}-\frac{4r}{3\pi}\right]=\frac{prh^2}{3};\ \text{or,}\ w=\frac{h^2}{r^2}p \quad . \quad . \quad . \quad . \quad (0)$$

---

* Elasticitaet und Festigkeit, by C. Bach; Berlin, 1898.

Notwithstanding the imperfections of this analysis, the experimental work of Prof. Bach shows that a modification of eq. (0), viz.:—

$$w = \frac{5}{6} \cdot \frac{h^2}{r^2} \cdot R', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

may be used with safety for the design of a plate under these circumstances; R', a safe unit working stress for the material, having been substituted for p.

For example, let the plate (e.g., cylinder-head of a locomotive) be of mild steel with $h = 1$ in. and $r = 8$ in. Putting $R' = 16,000$ lbs. / sq. in., we have from eq. (1) a safe $w = \frac{5}{6}$ (16000) × $(1 \div 8)^2 = 208$ lbs. per sq. in.

[N.B. If the plate is *clamped* all round the edge, we may write $\frac{5}{4}$ instead of the $\frac{5}{6}$. (Bach.)]

266a. Homogeneous Circular Plate of Uniform Thickness, h, Supported all Round the Edge, with Concentrated Load (P lbs.) in Center. By the same method as before we may here derive $P = \frac{1}{3}\pi h^2 p$; but from his experiments in this case Prof. Bach concludes that the formula for safe design should be written

$$P = \frac{1}{5}\pi h^2 R'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It is seen from eq. (2) that the value of P is *independent of the radius of the plate;* depending only on the material and the thickness, h.

266b. Homogeneous Elliptical and Rectangular Plates of Constant Thickness, h, Supported all Round the Periphery. According to Prof. Bach's approximate analysis, as supplemented by his experimental researches, we may use the following formulæ for

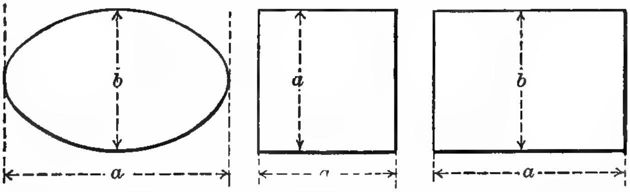

FIG. 269*b*.

safe design of elliptical and rectangular plates, supported (not clamped) around the whole periphery. See Fig. 269*b* for notation of dimensions; h being the uniform thickness in each case, and a being $>b$. R' = max. safe unit stress for the material.

For the **elliptical plate** under **uniformly distributed pressure** (over whole area) of w lbs. / sq. in., denoting the ratio $b \div a$ by m, we have

$$w = \tfrac{5}{3}\,(1 + m^2)\cdot\frac{h^2}{b^2}\cdot R'; \quad \ldots\ldots \quad (3)$$

and under a *central concentrated load* of P lbs.,

$$P = \frac{3\pi}{8m}\cdot\frac{3 + 2m^2 + 3m^4}{8 + 4m^2 + 3m^4}\cdot h^2R'. \quad \ldots\ldots \quad (4)$$

(N.B. If the edge is clamped down all round we may use values of w and P about 50 per cent. greater than the above.)

With **rectangular plates** under a **uniformly distributed pressure**, denoting the ratio $b \div a$ by m, we have

$$w = \frac{9}{5}(1 + m^2)\frac{h^2}{b^2}\cdot R' \quad \ldots\ldots \quad (5)$$

and for a concentrated central load P, with n denoting the ratio $a \div b$,

$$P = \frac{1}{3n}(1 + n^2)\cdot h^2R' \quad \ldots\ldots \quad (6)$$

In the particular case of the **square plate**, the side of the square being a, eqs. (5) and (6) reduce to

(uniform pressure) $$w = 3.6\frac{h^2}{a^2}\cdot R'; \quad \ldots\ldots \quad (7)$$

(central load) $$P = \frac{2}{3}h^2R' \quad \ldots\ldots \quad (8)$$

266c. Homogeneous Flat Circular Plate, of Uniform Thickness, used as Piston of an Engine. In such a case we have fluid pressures on both sides of the plate or disc, neither of which is necessarily one atmosphere; while at the center we have acting the concentrated pull or thrust, P lbs., of the piston rod. (Frictional forces around the edge may be disregarded.) If w denote the greatest difference between the (uniform) fluid pressures (per sq. in.) on the two sides, we may write (according to Grashof's analysis, as quoted by Unwin), for safe design in this case: —

$$w = \frac{6}{5}\cdot\frac{h^2}{r^2}\cdot R'. \quad \ldots\ldots \quad (9)$$

(R', h, and r, have the same meaning as before.)

267. Resilience of Beam With End Supports.—Fig. 270. If a mass whose weight is G (G large compared with that of beam) be allowed to fall freely through a height $= h$ upon the centre of a beam supported at its extremities, the pressure P felt by the beam increases from zero at the first instant of contact up to a maximum P_m, as already stated in §233*a*, in which the equation was derived, d_m being small compared with h,

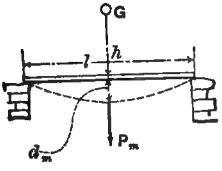

FIG. 270.

$$Gh = \frac{1}{96} \cdot \frac{P_m^2 l^3}{EI} \quad . \quad . \quad . \quad (a)$$

The elastic limit is supposed not passed. In order that the maximum normal stress in any outer fibre shall at most be $= R'$, a safe value, (see table §251) we must put $\frac{R'I}{e} = \frac{P_m l}{4}$ [according to eq. (2) §241,] i.e. in equation (a) above, substitute $P_m = [4\ R'I] \div le$, which gives

$$Gh = \tfrac{1}{6} \cdot \frac{R'^2 Il}{Ee^2} = \tfrac{1}{6} \cdot \frac{R'^2}{E} \cdot \frac{k^2}{e^2} \cdot Fl = \tfrac{1}{6} \cdot \frac{R'^2}{E} \cdot \frac{k^2}{e^2} V \quad (b)$$

having put $I = Fk^2$ (k being the radius of gyration §85) and $Fl = V$ the volume of the (prismatic) beam. From equation (b) we have the *energy*, Gh, (in ft.-lbs., or inch-lbs.) of the vertical blow at the middle which the beam of Fig. 270 will safely bear, and any one unknown quantity can be computed from it, (but the mass of G should not be small compared with that of the beam.)

The energy of this safe impact, for two beams of the same material and similar cross-sections (similarly placed), is seen to be proportional to their *volumes;* while if furthermore their cross-sections are the *same* and similarly placed, the safe Gh is proportional to their *lengths.* (These same relations hold good, approximately, beyond the elastic limit.)

It will be noticed that the last statement is just the reverse of what was found in §245 for static loads, (the pressure at the centre of the beam being then equal to the weight of the safe load); for there the longer the beam (and ∴ the span) the less the safe load, in inverse ratio. As appropriate in this connection, a quotation will be given from p. 186 of "The Strength of Materials and Structures," by Sir John Anderson, London, 1884, viz.:

"It appears from the published experiments and statements of the Railway Commissioners, that a beam 12 feet long will only support ½ of the load that a beam 6 feet long of the same breadth and depth will support, but that it will bear double the weight suddenly applied, as in the case of a weight falling upon it," (from the *same height*, should be added); "or if the same weights are used, the longer beam will not break by the weight falling upon it unless it falls through twice the distance required to fracture the shorter beam."

268. Combined Flexure and Torsion. Crank Shafts. Fig. 271. Let O_1B be the crank, and NO_1 the portion *projecting* beyond the nearest bearing N. P is the pressure of the connecting-rod against the crank-pin at a definite instant, the rotary motion being uniform. Let $a =$ the perpendicular dropped from the axis OO_1 of the shaft upon P, and $l =$ the distance of P, along the axis OO_1 from the cross-section NmN' of the shaft, close to the bearing. Let NN' be a diameter of this section, and parallel to a. In considering the portion NO_1B free, and thus exposing the circular section NmN', we may assume that the stresses to be put in on the elements of this surface are the tensions (above NN') and the compressions (below NN') and shears ┐ to NN', due to the bending action of P; and the shearing stress tan-

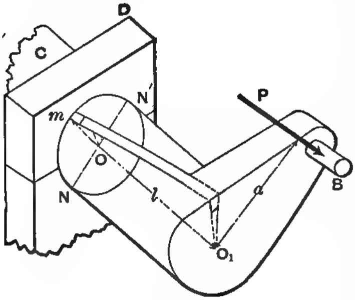

Fig. 271.

gent to the circles which have O as a common centre, and pass through the respective dF's or elementary areas, these latter stresses being due to the twisting action of P.

In the former set of elastic forces let $p =$ the tensile stress per unit of area in the small parallelopipedical element m of the helix which is furthest from NN' (the neutral axis) and $I=$ the moment of inertia of the circle about NN'; then taking moments about NN' for the free body, (disregarding the motion) we have as in cases of flexure (see §239)

$$\frac{pI}{r} = Pl\,; \text{ i.e., } p = \frac{Plr}{I} \quad . \quad . \quad (a)$$

[None of the shears has a moment about NN'.] Next taking moments about OO_1, (the flexure elastic forces, both normal and shearing, having no moments about OO_1) we have as in torsion (§216)

$$\frac{p_s I_p}{r} = Pa\,; \text{ i.e., } p_s = \frac{Par}{I_p} \quad . \quad . \quad (b)$$

in which p_s is the shearing stress per unit of area, in the torsional elastic forces, on any outermost dF, as at m; and I_p the polar moment of inertia of the circle about its centre O.

Next consider free, in Fig. 272, a small parallelopiped taken from the helix at m (of Fig. 271.) The stresses [see §209] acting on the four faces ┌ to the paper in Fig. 272 are there represented, the dimensions (infinitesimal) being $n \parallel$ to NN', $b \parallel$ to OO_1, and d ┐ to the paper in Fig. 272.

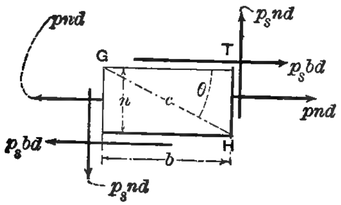

FIG. 272.

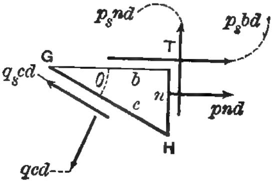

Fig. 273.

By altering the ratio of b to n we may make the angle θ what we please. It is now proposed to consider free the triangular prism, GHT, to find the intensity of normal stress q, per unit of area, on the diagonal plane GH, (of length $=c$,) which is a bounding face of that triangular prism. See Fig. 273. By writing Σ (compons. in direction of normal to GH) $=0$, we shall have, transposing,

$qcd = pnd \sin\theta + p_s bd \sin\theta + p_s nd \cos\theta$; and solving for q

$$q = p\frac{n}{c}\sin\theta + p_s\left[\frac{b}{c}\sin\theta + \frac{n}{c}\cdot\cos\theta\right]; \quad . \quad (1)$$

but $n : c = \sin\theta$ and $b : c = \cos\theta$ $\therefore$

$$q = p\sin^2\theta + p_s 2\sin\theta\cos\theta \quad . \quad . \quad (2)$$

This may be written (see eqs. 63 and 60, O. W. J. Trigonometry)

$$q = \tfrac{1}{2}p(1-\cos 2\theta) + p_s\sin 2\theta \quad . \quad . \quad (3)$$

As the diagonal plane GH is taken in different positions (i.e., as θ varies), this tensile stress q (lbs. per sq. in. for instance) also varies, being a function of θ, and its max. value may be $> p$. To find θ for q max. we put

$$\frac{dq}{d\theta}, = p\sin 2\theta + 2p_s\cos 2\theta, \quad . \quad . \quad (4)$$

$=0$, and obtain: $\tan[2(\theta \text{ for } q \text{ max})] = -\dfrac{2p_s}{p}$ . . . (5)

Call this value of θ, θ'. Since $\tan 2\theta'$ is negative, $2\theta'$ lies either in the second or fourth quadrant, and hence

$$\sin 2\theta' = \pm\frac{2p_s}{\sqrt{p^2+4p_s^2}} \quad \text{and} \quad \cos 2\theta' = \mp\frac{p}{\sqrt{p^2+4p_s^2}} \quad (6)$$

[See equations 28 and 29 Trigonometry, O. W. J.] **The**

upper signs refer to the second quadrant, the lower to the fourth. If we now differentiate (4), obtaining

$$\frac{d^2q}{d\theta^2} = 2p\cos 2\theta - 4p_s \sin 2\theta \quad . \quad . \quad . \quad (7)$$

we note that if the sine and cosine of the $[2\theta']$ of the 2nd quadrant [upper signs in (6)] are substituted in (7) the result is *negative*, indicating a maximum; that is, q is a maximum for $\theta =$ the θ' of eq. (6) *when the upper signs are taken* (2nd quadrant). To find q max., then, put θ' for θ in (3) substituting from (6) (upper signs). We thus find*

$$q \text{ max} = \tfrac{1}{2}[p + \sqrt{p^2 + 4p_s^2}] \quad . \quad . \quad (8)$$

A similar process, taking components *parallel* to GH, Fig. 273, will yield q_s max., i.e., the max. shear per unit of area, which for a given p and p_s exists on the diagonal plane GH in any of its possible positions, as θ varies. This max. shearing stress is

$$q_s \text{ max} = \tfrac{1}{2}\sqrt{p^2 + 4p_s^2} \quad . \quad . \quad (9)$$

In the element diametrically opposite to m in Fig. 271, p is compression instead of tension; q maximum will also be compression but is numerically the same as the q max. of eq. 8.

269. Example.—In Fig. 271 suppose $P=2$ tons $= 4{,}000$ lbs., $a=6$ in., $l=5$ in., and that the shaft is of wrought iron. Required its radius that the max. tension or compression may not exceed $R'=12{,}000$ lbs. per sq. in.; nor the max. shear exceed $S'=7{,}000$ lbs. per sq. in. That is, we put $q=12{,}000$ in eq. (8) and solve for r: also $q_s=7{,}000$ in (9) and solve for r. The greater value of r should be taken. From equations (a) and (b) we have (see §§ 219 and 247 for I_p and I)

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* According to the conceptions of § 405*b*, safe design would require that we put the max. "*strain*" in this case equal to a safe value, as determined by simple tensile or compressive tests. Here the max. strain (tensile) is $\varepsilon = [\tfrac{3}{8}p + \tfrac{5}{8}\sqrt{p^2 + 4p_s^2}] \div E$. (Grashof's method.)

$$p=\frac{4Pl}{\pi r^3} \text{ and } p_s=\frac{2Pa}{\pi r^3}$$

which in (8) and (9) give

$$\text{max. } q=\tfrac{1}{2}\frac{P}{\pi r^3}\left[4l+\sqrt{(4l)^2+4(2a)^2}\right] \quad . \quad . \quad (8a)$$

and *

$$\text{max. } q_s=\tfrac{1}{2}\frac{P}{\pi r^3}\sqrt{(4l)^2+4(2a)^2} \quad . \quad . \quad . \quad (9a)$$

With max. $q=12{,}000$, and the values of P, a, and l, already given, (units, inch and pound) we have from (8a), $r^3=2.72$ cubic inches $\therefore$ $r=1.39$ inches.

Next, with max. $q_s=7{,}000$; P, a, and l as before; from (9a), $r^3=2.84$ cubic inches $\therefore$ $r=1.41$ inches.

The latter value of r, 1.41 inches, should be adopted. It is here supposed that the crank-pin is in such a position (when $P=4{,}000$ lbs., and $a=6$ in.) that q max. (and q_s max.) are greater than for any other position; a number of trials may be necessary to decide this, since P and a are different with each new position of the connecting rod. If the shaft and its connections are exposed to shocks, R' and S' should be taken much smaller.

270. Another Example of combined torsion and flexure is shown in Fig. 274. The work of the working force P_1(vertical cog-pressure) is expended in overcoming the resistance (another vertical cog-pressure) Q_1.

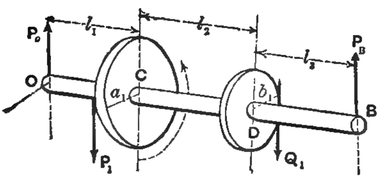

FIG. 274.

That is, the rigid body consisting of the two wheels and shaft is employed to transmit power, at a uniform angular velocity, and since it is symmetrical about its axis of rotation the forces acting on it, considered free, form a balanced system. (See § 114). Hence given P_1 and the various geometrical quantities a_1, b_1, etc., we may obtain Q_1, and the reactions P_0 and P_B, in terms of P_1. The greatest moment of flexure in the shaft will be either P_0l_1, at C; or P_Bl_3, at D. The portion CD is under torsion, of a moment of torsion $= P_1a_1 = Q_1b_1$. Hence we proceed as in the example of § 269, simply putting P_0l_1 (or P_Bl_3, whichever is the greater) in place of Pl, and P_1a_1 in place of Pa. We have here neglected the weight of the shaft and wheels. If Q_1 were an *upward* vertical force and hence on the same side of the shaft as P_1, the reactions P_0 and P_B would be less than before, and one or both of them might be reversed in direction.

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* See *Engineering Record* of Oct. 1909, p. 399, where Prof. Hancock states that experiments indicate that the use of eqs. (9) and (9a), involving the *maximum shear*, is to be preferred to that of (8) and (8a).

270a. Web of I-Beam. Maximum Stresses on an Oblique Plane.—The analysis of pp. 315, 316, etc., also covers the case of an element of the web of a horizontal I-beam under stress, when this element is taken near the point of junction with the flange. Supposing that the thickness of web has already been designed such that the shearing stress on the vertical (and therefore also on the horizontal) edges of such an element is at rate of 8000 lbs. per sq. inch; and that the horizontal tension at each end of this element (since it is not far from the outer fibre of the whole section) is at rate of 14,000 lbs. per sq. in.; we note that Fig. 272 gives us a side view of this element, with $p_s = 8000$, and $p = 14{,}000$, lbs. per sq. inch. GT is the lower edge of the upper flange, corresponding (in an end view) to the point n in Fig. 258 on p. 290. (We here suppose the upper flange to be in tension; of course, an illustration taken from the compression side would do as well.)

Substitution in equations (8) and (9) of p. 317 results in giving as maximum stresses on internal oblique planes:

$$q \text{ max.} = 17{,}630 \text{ lbs. per sq. in. tension;}$$

and $$q_s \text{ max.} = 10{,}630 \text{ " " " " shearing.}$$

These two values are seen to be considerably in excess of the respective safe values for shearing and tensile stresses in the case of structural steel, and the necessity is therefore emphasized of adopting values for shearing stress in webs somewhat lower than the 8000 lbs./in.^{2} used above; to avoid the occurrence of excessive stress on internal oblique planes. See p. 291.

CHAPTER IV.

FLEXURE, CONTINUED.

CONTINUOUS GIRDERS.

271. Definition.—A continuous girder, for present purposes, may be defined to be a loaded straight beam supported in *more than two points,* in which case we can no longer, as heretofore, determine the reactions at the supports from simple Statics alone, but must have recourse to the equations of the several elastic curves formed by its neutral line, which equations involve directly or indirectly the reactions sought; the latter may then be found as if they were constants of integration. Practically this amounts to saying that the reactions depend on the manner in which the beam bends; whereas in previous cases, with only two supports, the reactions were independent of the forms of the elastic curves (the flexure being slight, however).

As an **Illustration,** if the straight beam of Fig. 275 is placed on three supports O, B, and C, at the same level, the reactions of these supports seem at first sight indeterminate; for on considering the whole beam free, we have three unknown quantities and only two equations, viz: Σ (vert. compons.) $= 0$ and Σ (moms. about some point) $= 0$. If now O be gradually lowered, it receives less and less pres-

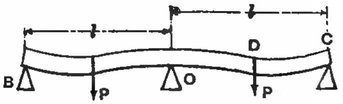

FIG. 275.

sure, until it finally reaches a position where the beam barely touches it; and then O's reaction is zero, and B and C support the beam as if O were not there. As to how low O must sink to obtain this position, depends on the stiffness and load of the beam. Again, if O be raised above the level of B and C it receives greater and greater pressure, until the beam fails to touch one of the other supports. Still another consideration is that if the beam were tapering in form, being stiffest at O, and pointed at B and C, the three reactions would be different from their values for a prismatic beam. It is therefore evident that for more than two supports the values of the reactions depend on the relative heights of the supports and upon the form and elasticity of the beam, as well as upon the load. The circumstance that the beam is made *continuous* over the support O, instead of being cut apart at O into two independent beams, each covering its own span and having its own two supports, shows the significance of the term "continuous girder."

All the cases here considered will be comparatively simple, from the symmetry of their conditions. The beams will all be prismatic, and all external forces (i.e. loads and reactions) perpendicular to the beam and in the same plane. All supports at the same level.

272. Two Equal Spans; Two Concentrated Loads, One in the Middle of Each Span. Prismatic Beam.—Fig. 275. Let each half-span $= \frac{1}{2} l$. Neglect the weight of the beam. Required the reactions of the three supports. Call them P_B, P_O and P_C. From symmetry $P_B = P_C$, and the tangent to the elastic curve at O is horizontal; and since the supports are *on a level* the deflection of C (and B) below O's tangent is zero. The *separate* elastic curves OD and DC have a common slope and a common ordinate at D.

For the equation of OD, make a section n anywhere between O and D, considering nC a free body. Fig. 276 (a)

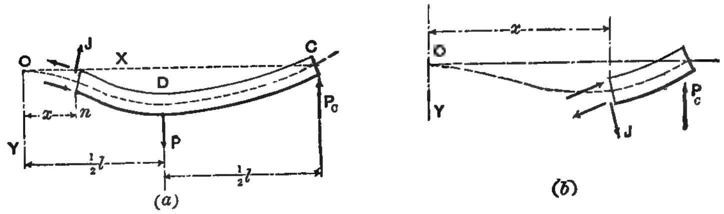

FIG. 276.

with origin and axes as there indicated.* From Σ (moms about neutral axis of n) = 0 we have (see § 231)

$$EI\frac{d^2y}{dx^2}=P(\tfrac{1}{2}l-x)-P_c(l-x) \quad . \quad . \quad . \qquad (1$$

$$\therefore \qquad EI\frac{dy}{dx}=P\left(\tfrac{1}{2}lx-\frac{x^2}{2}\right)-P_c\left(l\,x-\frac{x^2}{2}\right)+(C=0) \quad . \quad (2)$$

The constant = 0, for at O both x, and $dy \div dx$, = 0.

Taking the x-anti-derivative of (2) we have

$$EIy=P\left(\frac{lx^2}{4}-\frac{x^3}{6}\right)-P_c\left[\frac{lx^2}{2}-\frac{x^3}{6}\right] \quad . \quad . \quad (3)$$

Here again the constant is zero since at O,x and y both =0. (3) is the equation of OD, and allows no value of $x<0$ or $>\frac{l}{2}$. It contains the unknown force P_c.

For the equation of DC, let the variable section n be made anywhere between D and C, and we have (Fig. 276 (b); x may now range between $\tfrac{1}{2}l$ and l)

$$EI\frac{d^2y}{dx^2}=-P_c(l-x) \quad . \quad . \quad . \quad . \quad , \qquad (4)$$

$$\therefore \qquad EI\frac{dy}{dx}=-P_c\left(lx-\frac{x^2}{2}\right)+C' \qquad (5)'$$

To determine C', put $x=\tfrac{1}{2}l$ both in (5)′ and (2), and equate the results (for the two curves have a common tangent line at D) whence $C'=\tfrac{1}{8}Pl^2$

$$\therefore \qquad EI\frac{dy}{dx}=\tfrac{1}{8}Pl^2-P_c\left(lx-\frac{x^2}{2}\right) \quad . \quad . \quad . \qquad (5)$$

---

* These are such that XOY is our "first quadrant"; in which, for points in a part of a curve convex toward the axis of X, d^2y/dx^2 is essentially positive; and *vice versa*. It will be seen that both eqs. (1) and (4) are on this basis. They must be on the *same basis*; otherwise, later comparisons of equations would result in error.

Hence $$EIy = \tfrac{1}{8}\,Pl^2x - P_c\left[\frac{lx^2}{2}-\frac{x^3}{6}\right]+C'' \quad . \quad . \quad (6)'$$

At D the curves have the same y, hence put $x = \frac{l}{2}$ in the right hand member both of (3) and (6)$'$, equating results, and we derive $C'' = -\frac{1}{48}\,Pl^3$

$$\therefore \qquad EIy = \tfrac{1}{8}Pl^2x - P_c\left[\frac{lx^2}{2}-\frac{x^3}{6}\right] - \tfrac{1}{48}Pl^3 \quad . \quad . \quad (6)$$

which is the equation of DC, but contains the unknown reaction P_c. To determine P_c we employ the fact that O's tangent passes through C, (supports on *same level*) and hence when $x = l$ in (6), y is known to be zero. Making these substitutions in (6) we have

$$0 = \tfrac{1}{8}Pl^3 - \tfrac{1}{3}P_cl^3 - \tfrac{1}{48}Pl^3 \;\therefore\; P_c = \tfrac{5}{16}P$$

From symmetry P_B also $= \frac{5}{16}P$, while P_0 must $= \frac{22}{16}P$, since $P_B + P_O + P_C = 2\,P$ (whole beam free). [NOTE.—If the supports were not on a level, but if, (for instance) the middle support O were a small distance $= h_0$ below the level line joining the others, we should put $x = l$ and $y = -h_0$ in eq. (6), and thus obtain $P_B = P_C = \frac{5}{16}\;P + 3EI\frac{h_0}{l^3}$, which depends on the material and form of the prismatic beam and upon the length of one span, (whereas with supports *all on a level*, $P_B = P_C = \frac{5}{16}\;P$ is independent of the material and form of the beam so long as it is homogeneous and prismatic.) If P_0, which would then $= \frac{22}{16}\;P - 6\;EI\,(h_0 \div l^3)$, is found to be *negative*, it shows that O requires a support from above, instead of below, to cause it to occupy a position h_0 below the other supports, i.e. the beam must be "latched down" at O.]

The *moment diagram* of this case can now be easily constructed; Fig. 277. For any free body nC, n lying in DC, we have

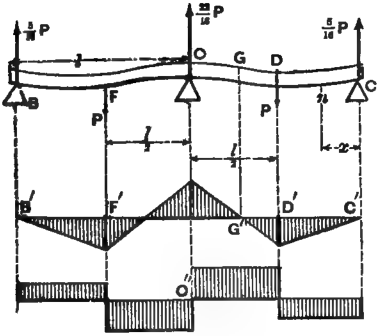

Fig. 277.

$$M={}^{5}/_{16}Px,$$

i.e., varies directly as x, until x passes D when, for any point on DO,

$$M={}^{5}/_{16}Px-P(x-\frac{l}{2})$$

which is $=0$, (point of inflection of elastic curve) for $x={}^{8}/_{11}l$ (note that x is measured from C in this figure) and at O, where $x=l$, becomes $-\frac{6}{32}Pl$

$$\therefore\ M_o=-\frac{6}{32}Pl\,;\ M_G=0\,;\ M_D=\frac{5}{32}Pl\,;\ \text{and}\ M_c=0$$

Hence, since M max. $=\frac{6}{32}Pl$, the equation for safe loading is

$$\frac{R'I}{e}=\frac{6}{32}Pl \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The shear at C and anywhere on $CD=\frac{5}{16}P$, while on DO it $=\frac{11}{16}P$ in the opposite direction

$$\therefore\ J_m=\frac{11}{16}P \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The moment and shear diagrams are easily constructed, as shown in Fig. 277, the former being symmetrical about a vertical line through O, the latter about the *point* O'' Both are bounded by right lines.

273. Two Equal Spans. Uniformly Distributed Load Over Whole Length. Prismatic Beam.—Fig. 278. *Supports B, O, C, on a level.* Total load $=2W=2wl$ and may include that of the beam; w is constant. As before, from symmetry $P_B=P_c$, the unknown reactions at the extremities.

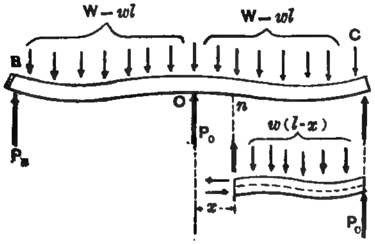

Fig. 278.

Let $On = x$; then with nC free, Σ moms. about $n = 0$ gives

$$EI\frac{d^2y}{dx^2} = w(l-x)\left(\frac{l-x}{2}\right) - P_0(l-x) - \frac{w}{2}[l^2 - 2lx + x^2] - P_c(l-x) \quad (1)$$

$$\therefore EI\frac{dy}{dx} = \frac{w}{2}\left[l^2x - lx^2 + \frac{x^3}{3}\right] - P_c\left[lx - \frac{x^2}{2}\right] + [\text{Const} = 0] \quad (2)$$

[Const. $=0$ for at O both $dy \div dx$ the slope, and x, are $=0$]

$$\therefore EIy = \frac{w}{2}[\tfrac{1}{2}l^2x^2 - \tfrac{1}{3}lx^3 + \tfrac{1}{12}x^4] - P_c[\tfrac{1}{2}lx^2 - \tfrac{1}{6}x^3] + (C=0) \quad (3)$$

[Const. $=0$ for at O both x and y are $=0$]. Equations (1), (2), and (3) admit of any value of x from 0 to l, i.e., hold good for any point of the elastic curve OC, the loading on which follows a continuous law (viz.: $w =$ constant). But when $x = l$, i.e., at C, y is known to be equal to zero, since O, B and C are on the axis of X, (tangent at O). With these values of x and y in eq. (3) we have

$$0 = \frac{w}{2} \cdot \frac{l^4}{4} - \tfrac{1}{3}P_cl^3 \quad \therefore P_c = \tfrac{3}{8}wl = \tfrac{3}{8}W$$

$$\therefore P_B = \tfrac{3}{8}W \text{ and } P_0 = 2W - 2P_c = \tfrac{10}{8}W$$

The Moment and Shear Diagrams can now be formed since all the external forces are known. In Fig. 279 measure x from C. Then in any section n the moment of the "stress-couple" is

$$M = \tfrac{3}{8}Wx - \frac{wx^2}{2} \quad . \quad (1)$$

which holds good for any value of x on CO, i.e., from $x=0$ up to $x=l$. By inspection it is seen that for $x=0$, $M=0$; and also for $x = \tfrac{3}{4}l$, $M=0$, at the *inflection point* G, beyond which, toward O, the upper fibres are in tension

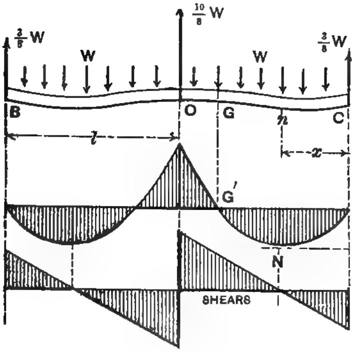

Fig. 279.

the lower in compression, whereas between C and G they are vice versâ. As to the greatest moment to be found on CG, put $dM \div dx = 0$ and solve for x. This gives

$$\tfrac{3}{8}W - wx = 0 \therefore [x \text{ for } M \text{ max.}] = \tfrac{3}{8}l \quad . \quad (2)$$

which in eq. (1) gives

$$M_N \text{ (at } N\text{, see figure)} = +\tfrac{9}{128}Wl \quad . \quad . \quad (2)$$

But this is numerically less than $M_0 (= -\tfrac{1}{8}Wl)$ hence the stress in the outer fibre at O being

$$p_0 = \tfrac{1}{8}\frac{Wle}{I}, \quad . \quad . \quad . \quad (3)$$

the equation for safe loading is

$$\frac{R'I}{e} = \tfrac{1}{8}Wl \quad . \quad . \quad . \quad . \quad (4)$$

the same as if the beam were cut through at O, each half, of length l, retaining the same load as before [see § 242 eq. (2)]. Hence making the girder continuous over the middle support does not make it any stronger under a uniformly distributed load; but it does make it considerably *stiffer*.

As for the shear, J, we obtain it for any section by taking the x-derivative of M in eq. (1), or by putting Σ(vertical forces) $=0$ for the free body nC, and thus have for any section on CO

$$J = \tfrac{3}{8}W - wx \quad . \quad . \quad . \quad (5)$$

J is zero for $x = \tfrac{3}{8}l$ (where M reaches its calculus maximum M_N; see above) and for $x = l$ it $= -\tfrac{5}{8}W$ which is numerically greater than $\tfrac{3}{8}W$, its value at C. Hence

$$J_m = \tfrac{5}{8}W \quad . \quad . \quad . \quad . \quad (6)$$

The moment curve is a parabola (a separate one for each span), the shear curve a straight line, inclined to the horizontal, for each span.

Problem.—How would the reactions in Fig. 278 be changed if the support O were lowered a (small) distance h_0 below the level of the other two?

274. Prismatic Beam Fixed Horizontally at Both Ends (at **Same Level). Single Load at Middle.**—Fig. 280. [As usual the beam is understood to be homogeneous so that E is the same at all sections]. The building in, or fixing, of the two ends is supposed to be of such a nature as to cause *no horizontal constraint;* i.e., the beam does not act as a cord or chain, in any manner, and hence the sum of the horizontal components of the stresses in any section is zero, as in all preceding cases of flexure. In other words the neutral axis still contains the centre of gravity of the section and the tensions and compressions are equivalent to a couple (the stress-couple) whose moment is the " moment of flexure."

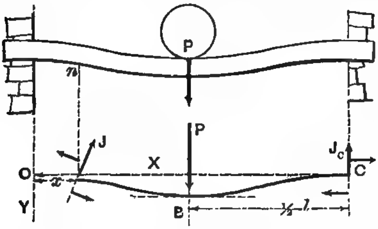

FIG. 280.

If the beam is conceived cut through close to both wall faces, and this portion of length$=l$, considered free, the forces holding it in equilibrium consist of the downward force P (the load); two upward shears J_o and J_c (one at each section); and two " stress-couples " one in each section, whose moments are M_o and M_c. From symmetry we know that $J_o = J_c$, and that $M_o = M_c$. From $\Sigma Y = 0$ for the free body just mentioned, (but not shown in the figure), and from symmetry, we have $J_o = \frac{1}{2} P$ and $J_c = \frac{1}{2} P$; but to determine M_o and M_c, the form of the elastic curves $O\,B$ and $B\,C$ must be taken into account as follows:

Equation of OB, Fig. 280. Σ [mom. about neutral axis of *any* section n on $O\,B$] $= 0$ (for the free body nC which has a section exposed at each end, n being the variable section) will give

$$EI\frac{d^2y}{dx^2} = P(\tfrac{1}{2}\,l-x) + M_c - \tfrac{1}{2}\,P\,(l-x) \quad . \quad . \ (1)$$

[**Note.** In forming this moment equation, notice that M_c is the sum of the moments of the tensions and compressions at C about the neutral axis at n, just as much as about the neutral axis of C; for those tensions and compressions are equivalent to a couple, and hence the sum of their moments is the same taken about any axis whatever ┐ to the plane of the couple (§32).]

Taking the x-anti-derivative of each member of (1),

$$EI\frac{dy}{dx} = P(\tfrac{1}{2}\,l\,x - \tfrac{1}{2}\,x^2) + M_c\,x - \tfrac{1}{2}\,P(l\,x - \tfrac{1}{2}\,x^2) \quad . \quad (2)$$

(The constant is not expressed, as it is zero). Now from symmetry we know that the tangent-line to the curve $O\,B$ at B is horizontal, *i.e.*, for $x = \tfrac{1}{2}l$, $dy \div dx = 0$, and these values in eq. (2) give us

$$0 = \tfrac{1}{8}\,Pl^2 + \tfrac{1}{2}M_c l - \tfrac{3}{16}Pl^2; \text{ whence } M_c = M_0 = \tfrac{1}{8}\,Pl \quad . \quad (3)$$

Safe Loading. Fig. 281. Having now all the forces which *act as external forces* in straining the beam OC, we are ready to draw the moment diagram and find M_m. For convenience measure x from C. For the free body nC, we have [see eq. (3)]

$$\tfrac{1}{2}\,Px - M_c + \frac{pI}{e} = 0 \ \therefore\ M = \tfrac{1}{8}\,Pl - \tfrac{1}{2}Px \quad . \quad . \quad . \quad (4)$$

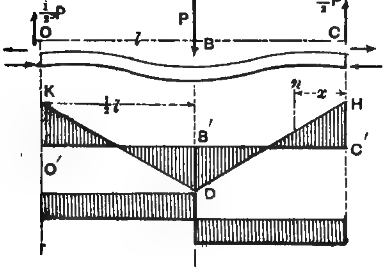

FIG. 281.

Eq. (4) holds good for any section on CB. By putting $x=0$ we have $M = M_c = \tfrac{1}{8}\,Pl$; lay off $HC' = M_c$ to scale (so many inch-pounds moment to the inch of paper). At B, for $x = \tfrac{1}{2}\,l$, $M_B = -\tfrac{1}{8}\ Pl$; hence lay off $B'D = \tfrac{1}{8}\,Pl$ on the opposite side of the axis $O'C'$

from HC', and join DH. DK, symmetrical with DH about $B'D$, completes the moment curves, viz.: two right lines. The max. M is evidently $= \frac{1}{8} Pl$ and the equation of safe loading

$$\frac{R'I}{e} = \frac{1}{8} Pl \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence the beam is twice as strong as if simply supported at the ends, under this load; it may also be proved to be four times as stiff.

The points of inflection of the elastic curve are in the middles of the half-spans, while the max. shear is

$$J_m = \frac{1}{2}P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

275. Prismatic Beam Fixed Horizontally at Both Ends [at Same Level]. Uniformly Distributed Load Over the Whole Length. Fig. 282. As in the preceding problem, we know from symmetry that $J_0 = J_c = \frac{1}{2} W = \frac{1}{2} wl$, and that $M_0 = M_c$, and determine the latter quantities by the equation of the curve OC, there being but one curve in the present instance, instead of two, as there is no change in the law of loading between O and C. With nO free, Σ (mom$_n$)=0 gives

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2} Wx + M_0 + \frac{wx^2}{2} \quad . \quad . \quad . \quad . \quad (1)$$

and $\therefore$ $$EI\frac{dy}{dx} = -\frac{1}{2} W\frac{x^2}{2} + \boldsymbol{M}_0 x + \frac{wx^3}{6} + [C=0] \quad . \quad . \quad (2)$$

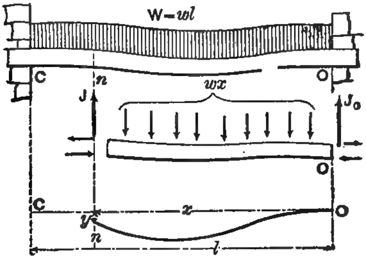

Fig. 282.

The tangent line at O being horizontal we have for $x=0, \frac{dy}{dx}=0$, $\therefore$ $C=0$. But since the tangent line at C is also horizontal, we may for $x=l$ put $dy \div dx=0$, and obtain

$$0=-\tfrac{1}{4}Wl^2+M_0l+\tfrac{1}{6}wl^3;\text{ whence } M_0=\tfrac{1}{12}Wl \quad . \quad (3)$$

as the moment of the stress-couple close to the wall at O and at C.

Hence, Fig. 283, the equation of the moment curve (a single continuous curve in this case) is found by putting Σ (mom$_n$)$=0$ for the free body nO, of length x, thus obtaining

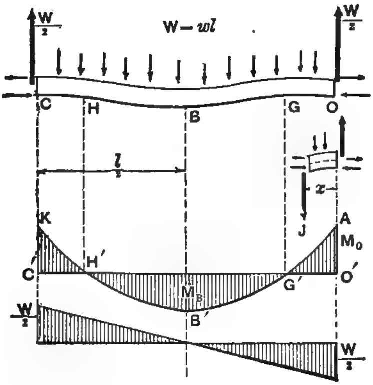

FIG. 283.

$$\frac{pI}{e}+\tfrac{1}{2}Wx-M_0-\frac{wx^2}{2}=0$$

i.e.,
$$M=\tfrac{1}{12}Wl+\frac{wx^2}{2}-\tfrac{1}{2}Wx \quad . \quad . \quad . \quad . \quad . \quad (4)$$

an equation of the second degree, indicating a conic. At O, $M=M_0$ of course, $=\frac{1}{12}Wl$; at B by putting $x=\frac{1}{2}l$ in (4), we have $M_B=-\frac{1}{24}Wl$, which is less than M_0, although M_B is the calculus max. (negative) for M, as may be shown by writing the expression for the shear ($J=\frac{1}{2}W-wx$) equal to zero, etc.

Hence $M_m = \frac{1}{12} Wl$, and the equation for *safe loading* is

$$\frac{R'I}{e} = \frac{1}{12} Wl \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Since (with this form of loading) if the beam were not built in but simply rested on two end supports, the equation for safe loading would be $[R'I \div e] = \frac{1}{8} Wl$, (see §242), it is evident that with the present mode of support it is 50 per cent. stronger as compared with the other ; i.e., as regards normal stresses in the outer elements. As regards shearing stresses in the web if it has one, it is no stronger, since $J_m = \frac{1}{2} W$ in both cases.

As to *stiffness* under the uniform load, the max. deflection in the present case may be shown to be only $\frac{1}{5}$ of that in the case of the simple end supports.

It is noteworthy that the shear diagram in Fig. 283 is identical with that for simple end supports §242, under uniform load ; while the moment diagrams differ as follows: The parabola $KB'A$, Fig. 283, is identical with that in Fig. 235, but the horizontal axis from which the ordinates of the former are measured, instead of joining the extremities of the curve, cuts it in such a way as to have equal areas between it and the curve, on opposite sides

i.e., areas $[KC'H' + AG'O'] = \text{area } H'G'B'$

In other words, the effect of fixing the ends horizontally is to shift the moment parabola upward a distance $= M_c$ (to scale), $= \frac{1}{12} Wl$, with regard to the axis of reference, $O'B'$, in Fig. 235.

276. Remarks.—The foregoing very simple cases of continuous girders illustrate the means employed for determining the reactions of supports and eventually the max. moment and the equations for safe loading and for deflections When there are more than three supports, with spans of unequal length, and loading of any description, the analysis leading to the above results is much more complicated and tedious, but is considerably simplified and systematized by the use of the remarkable **theorem of three moments**, the discovery of Clapeyron, in 1857. By this theorem, given the spans, the loading, and the vertical heights of the supports, we are enabled to write out a relation between the moments of each three consecutive supports, and thus obtain a sufficient number of equations to determine the moments at all the supports [p. 641 Rankine's Applied Mechanics.] From these moments the shears close to each side of each support are found, then the reactions, and from these and the given loads the moment at any section can be determined; and hence finally the max. moment M_m, and the max. shear J_m.

The treatment of the general case of continuous girders by *algebraic methods founded on the properties of familiar geometrical figures*, however, is comparatively simple; and will be developed and applied in another part of this book. (See Chap. XII, pp. 485, etc.)

THE DANGEROUS SECTION OF NON-PRISMATIC BEAMS.

277. Remarks. By "*dangerous section*" is meant that section (in a given beam under given loading with given mode of support) where p, the normal stress in the outer fibre, at distance e from its neutral axis, is greater than in the outer fibre of any other section. Hence the elasticity of the material will be first impaired in the outer fibre of this section, if the load is gradually increased in amount (but not altered in distribution).

In all preceding problems, the beam being prismatic, I, the moment of inertia, and e were the same in all sections, hence when the equation $\frac{pI}{e}=M$ [§239] was solved for p, giving

$$p=\frac{Me}{I} \quad . \quad . \quad . \quad . \quad (1)$$

we found that p was a max., $= p_m$, for that section whose M was a maximum, since p varied as M, or the moment of the stress-couple, as successive sections along the beam were examined.

But for a non-prismatic beam I and e change, from section to section, as well as M, and the ordinate of the moment diagram no longer shows the variation of p, nor is p a max. where M is a max. To find the dangerous section, then, for a non-prismatic beam, we express the M, the I, and the e of any section in terms of x, thus obtaining $p =$ func. (x), then writing $dp \div dx = 0$, and solving for x.

278. Dangerous Section in a Double Truncated Wedge. Two End Supports. Single Load in Middle.—The form is shown in Fig. 284. Neglect weight of beam; measure x from one support O. The reaction at each support is ½ P. The width of the beam $= b$ at all sections, while its height, v, varies, being $= h$ at O. To express the $e = \frac{1}{2} v$, and the $I = \frac{1}{12} bv^3$ (§247) of any section on OC, in terms of x, conceive the sloping faces of the truncated wedge to be prolonged to their intersection A, at a known distance $= c$ from the face at O. We then have from similar triangles

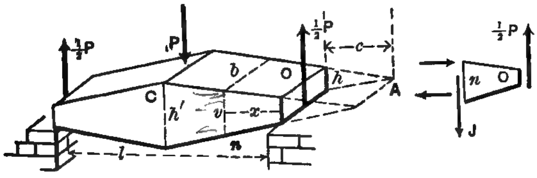

Fig. 284.

$$v : x + c :: h : c, \therefore v = \frac{n}{c}(x + c) \quad . \quad . \quad (1)$$

$$\text{and} \therefore e = \tfrac{1}{2}\frac{h}{c}(x+c) \text{ and } I = \tfrac{1}{12}\, b\, \frac{h^3}{c^3}[x+c]^3 \quad . \quad (2)$$

For the free body nO, Σ (moms.$_n$) $= 0$ gives

$$\tfrac{1}{2}Px - \frac{pI}{e} = 0 \;\therefore p = \frac{Pxe}{2I} \quad . \quad . \quad . \quad (3)$$

[That is, the $M = \frac{1}{2}Px$.] But from (2), (3) becomes

$$p = 3P\frac{c^2}{bh^2} \cdot \frac{x}{(x+c)^2}\text{; and } \frac{dp}{dx} = 3P\frac{c^2}{bh^2} \cdot \frac{(x+c)^2 - 2x(x+c)}{(x+c)^4} \quad (4)$$

By putting $dp \div dx = 0$ we find $x = +c$; showing a maximum for p (since it will be found to give a negative result on substitution in $d^2p \div dx^2$).

Hence the dangerous section is as far from the support O, as the imaginary edge, A, of the completed wedge, but of course on the opposite side. This supposes that the half-span, $\frac{1}{2}l$, is $> c$; if not, the dangerous section will be at the middle of the beam, as if the beam were prismatic.

Hence, with $\frac{1}{2}l < c$ } the equation for safe loading is: (h'=height at middle) {

$$\frac{R'bh'^2}{6} = \frac{1}{4}Pl \tag{5}$$

while with $\frac{1}{2}l > c$ } the equation for safe loading is: (put $x=c$ and $p=R'$ in [3]) {

$$\frac{R'b\,[2h]^2}{6} = \frac{1}{2}Pc \tag{6}$$

(see §239.)

279. Double Truncated Pyramid and Cone. Fig. 285. For

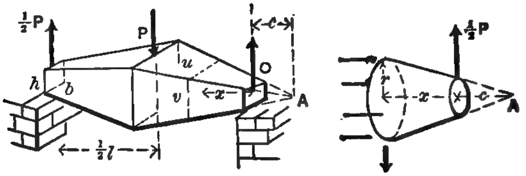

FIG. 285.

the truncated pyramid both width $= u$, and height $= v$, are variable, and if b and h are the dimensions at O, and $c = \overline{OA}$ = distance from O to the imaginary vertex A, we shall have from similar triangles $u = \frac{b}{c}(x+c)$ and $v = \frac{h}{c}(x+c)$. Hence, substituting $e = \frac{1}{2}v$ and $I = \frac{1}{12}uv^3$, in the moment equation

$$\frac{pI}{e} - \frac{Px}{2} = 0, \text{ we have } p = 3P\frac{c^3}{bh^2}\cdot\frac{x}{(x+c)^3} \quad . \quad . \tag{7}$$

$$\therefore \frac{dp}{dx} = 3P\frac{c^3}{bh^2}\cdot\frac{(x+c)^3 - 3x\,(x+c)^2}{(x+c)^6} \quad . \quad . \tag{8}$$

Putting the derivative $=0$, for a maximum p, we have $x = +\frac{1}{2}c$, hence the dangerous section is at a distance $x = \frac{1}{2}c$ from O, and the equation for safe loading is

$$\text{either } \frac{R'b'h'^2}{6} = \tfrac{1}{4}Pl \ldots . \text{ if } \tfrac{1}{2}l \text{ is } < \tfrac{1}{2}c \quad . \ . \ . \ . \quad (9)$$

(in which b' and h' are the dimensions at mid-span)

$$\text{or } \frac{R\,(^3/_2\,b)\,(^3/_2\,h)^2}{6} = \tfrac{1}{4}Pc \text{ if } \tfrac{1}{2}l \text{ is } > \tfrac{1}{2}c \quad . \ . \ . \quad (10)$$

For the **truncated cone** (see Fig. 285 also, on right) where $e =$ the variable radius r, and $I = \frac{1}{4}\pi r^4$, we also have

$$p = [\text{Const.}] \cdot \frac{x}{(x+c)^3}. \quad . \ . \ . \ . \quad (11)$$

and hence p is a max. for $x = \frac{1}{2}c$, and the equation for safe loading

$$\text{either } \frac{\pi R'r'^3}{4} = \tfrac{1}{4}Pl, \text{ for } \tfrac{1}{2}l < \tfrac{1}{2}c \quad . \ . \ . \ . \ . \quad (12)$$

(where $r' =$ radius of mid-span section);

$$\text{or } \frac{\pi R'\,(\frac{3}{2}r_0)^3}{4} = \tfrac{1}{4}Pc, \text{ for } \tfrac{1}{2}l > \tfrac{1}{2}c \quad . \ . \ . \ . \ . \quad (13)$$

(where $r_0 =$ radius of extremity.)

NON-PRISMATIC BEAMS OF "UNIFORM STRENGTH."

280. Remarks. A beam is said to be of "*uniform strength*" when its form, its mode of support, and the distribution of loading, are such that the normal stress p has the same value in all the outer fibres, and thus one element of economy is secured, viz.: that all the outer fibres may be made to do full duty, since under the safe loading, p will be $=$ to R' in all of them. [Of course, in all cases of flexure, the elements between the neutral surface and the outer fibres being under tensions and compressions less than R' per sq. inch, are not doing full duty, as regards economy of material, unless perhaps with respect to shearing stresses.] In Fig. 265, §261, we have already had an instance of a body of uniform strength in flexure, viz.: the middle segment, *CD*, of that figure; for the moment is the same for all sections of *CD* [eq. (2) of that §], and hence the normal stress p in the outer fibres (the beam being prismatic in that instance).

In the following problems the weight of the beam itself is neglected. The general method pursued will be to find an expression for the outer-fibre-stress p, at a *definite* section of the beam, where the dimensions of the section are known or assumed, then an expression for p in the variable section, and equate the two. For clearness the figures are exaggerated, vertically.

281. Parabolic Working Beam. Unsymmetrical. Fig. 286

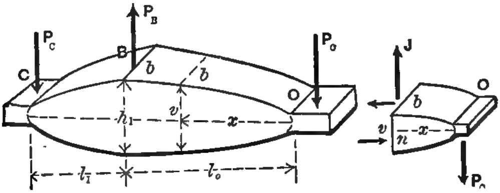

FIG. 286.

CBO is a working beam or lever, *B* being the fixed fulcrum or bearing. The force P_0 being given we may compute P_c from the mom. equation $P_0 l_0 = P_c l_1$, while the fulcrum reaction is $P_B = P_0 + P_C$. All the forces are ⏋ to the beam. The beam is to have the same width b at all points, and is to be *rectangular* in section.

Required **first**, the proper height h_1, at B, for safety. From the free body BO, of length $= l_0$, we have $\Sigma\,(\text{moms}_B) = 0$; i.e.,

$$\frac{p_B I}{e} = P_0 l_0; \text{ or } p_B = \frac{6\,P_0 l_0}{b h_1^2} \quad . \quad . \quad . \quad (1)$$

Hence, putting $p_B = R'$, h_1 becomes known from (1).

Required, **secondly**, the relation between the variable height v (at any section n) and the distance x of n from O. For the free body nO, we have (Σ moms$_n = 0$)

$$\frac{p_n I}{e} = P_0 x\,;\ \text{ or } \frac{p_n \frac{1}{12} bv^3}{\frac{1}{2}v} = P_0 x \text{ and } \therefore\ p_n = \frac{6P_0 x}{bv^2} \qquad (2)$$

But for "uniform strength" p_n must $= p_B$; hence equate their values from (1) and (2) and we have

$$\frac{x}{v^2} = \frac{l_0}{h_1^{\,2}},\ \text{which may be written } (\tfrac{1}{2}\,v)^2 = \frac{(\frac{1}{2}h_1)^2}{l_0}\,x \qquad (3)$$

so as to make the relation between the abscissa x and the ordinate $\frac{1}{2}\,v$ more marked; it is the equation of a parabola, whose vertex is at O.

The parabolic outline for the portion BC is found similarly. The local stresses at C, B, and O must be properly provided for by evident means. The shear $J = P_0$, at O, also requires special attention.

This shape of beam is often adopted in practice for the working beams of engines, etc.

The parabolic outlines just found may be replaced by trapezoidal forms, Fig. 287, without using much more material, and by making the sloping plane faces tangent to the parabolic outline at points T_0 and T_1, *half-way* between O and B, and C and B, respectively. It can be proved that they contain *minimum volumes*, among all trapezoidal forms capable of circumscribing the given parabolic bodies. The dangerous sections of these trapezoidal bodies are at the tangent points T_0 and T_1. This is as it should be, (see § 278), remembering that the subtangent of a parabola is bisected by the vertex.

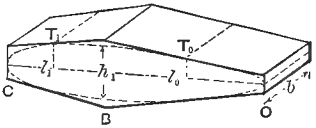

FIG. 287.

283. Rectang. Section. Height Constant. Two Supports (at Extremities). Single Eccentric Load.—Fig. 289. b and h are the dimensions of the section at B. With BO free we have

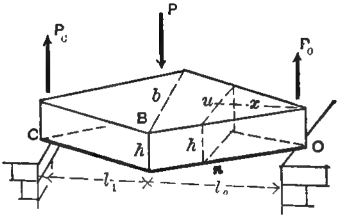

FIG. 289.

$$\frac{p_B I_B}{e_B} - P_0 l_0 = 0 \therefore p_B = \frac{6P_0 l_0}{bh^2} \quad (1)$$

At any other section on BO, as n, where the width is u, the variable whose relation to x is required, we have for nO free

$$\frac{p_n I_n}{e_n} = P_0 x; \text{ or } \frac{p_n \frac{1}{12} uh^3}{\frac{1}{2}h} = P_0 x \therefore p_n = \frac{6P_0 x}{uh^2} \quad . \quad . \quad . \quad (2)$$

Equating p_B and p_n we have $u : b :: x : l_0$ . . (3)

That is, BO must be wedge-shaped; edge at O, vertical.

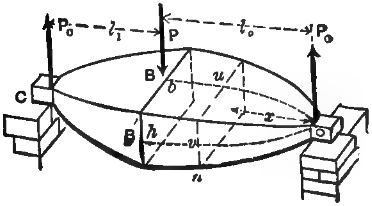

FIG. 289 *a*.

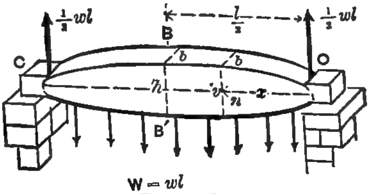

FIG. 289 *b*.

283a. Sections Rectangular and Similar. Otherwise as Before.—Fig. 289a. The dimensions at B are b and h; at any other section n, on BO, the height v, and width u, are the variables whose relation to x is desired, and by hypothesis are connected by the relation $u : v :: b : h$ (since the section at n is a rectangle *similar* to that at B). By the same method as before, putting $p_B = p_n$, we obtain $l_0 \div bh^2 = x \div uv^2$; in which placing $u = bv \div h$, we have finally

$$v^3 = (h^3 \div l_0)x; \quad \text{and similarly,} \quad u^3 = (b^3 \div l_0)x; \quad . \quad . \quad . \quad (4)$$

i.e., the width u, and height v, of the different sections are each proportional to the *cube* root of the distance x from the support. (The same relation would hold for the radii, in case all sections were circular.)

283b. Beam of Uniform Strength under Uniform Load. Two End Supports. Sections Rectangular with Constant Width.—Fig. 289*b*. Weight of beam neglected. How should the height v vary, (the height and width at middle being h and b)? As before, we equate p_B and p_n; whence finally

$$(\tfrac{1}{2}v)^2 = [h^2 \div l^2](lx - x^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This relation between the half-height $\frac{1}{2}v$ (as ordinate) and the abscissa x is seen to be the equation to an ellipse with origin at vertex.

CHAPTER V.

FLEXURE OF REINFORCED CONCRETE BEAMS.

284. Concrete and "Concrete-Steel" Beams. Concrete is an artificial stone composed of broken stone or gravel (sometimes cinders), cement and sand, properly mixed and wet beforehand and then rammed into moulds or "forms" and left to harden or "set." This material, after thorough hardening or "setting," though fairly strong in resisting compressive stress is comparatively weak in tension. When it is used in the form of beams to bear transverse loads (i. e., under "transverse stress") the side of the beam subjected to tensile stress is frequently "reinforced" by the imbedding of steel rods on that side. In this way a composite beam may be formed which is cheaper than a beam of equal strength composed entirely of concrete or one composed entirely of steel.

Of course the steel rods are placed in the mixture when wet, and previous to the ramming and compacting, and their aggregate sectional area may not need to be more than about one per cent. of that of the concrete.

No reliance being placed on the tensile resistance of the concrete (on the tension side of the beam) it is extremely important that there should be a good adhesion, and consequent resistance to shearing, between the sides of the steel rods and the adjacent concrete, for without this adhesion the rods and the concrete would not act together as a beam of continuous substance.*

In some specifications, for instance, it is required that the shearing stress, or tendency to slide, between the steel rods and the concrete shall not exceed 64 lbs. per sq. in. Sometimes the steel rods are provided with projecting shoulders, or ridges, or corrugations, along their sides, to secure greater resistance to sliding.

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* For an account of tests of this adhesion see *Engineering News*, Aug. 15, 1907, p. 169, and also p. 120 of the *Engineering Record* for Aug. 3, 1907.

Fig. 290 gives a perspective view of a concrete-steel beam of rectangular section, placed in a horizontal position on two supports at its extremities, and thus fitted to sustain vertical loads or weights; while Fig. 291 shows a concrete-steel beam of T-section, in which the flange is intended to resist compression, while the steel rods in the lower part of the "stem" are to take care of the tension. These two shapes of beam will be the only ones to be considered here, in a theoretical treatment.

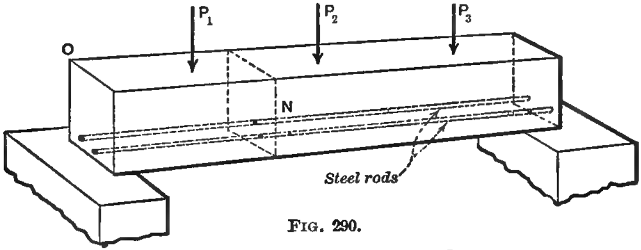


FIG. 290.

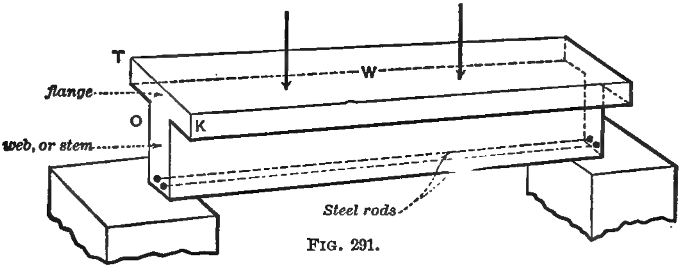


FIG. 291.

The ratio of the Modulus of Elasticity of steel (viz. — about 30,000,000 lbs. per sq. in.) to that of concrete (say, from 1,000,000 to 4,000,000 lbs. per sq. in., according to the proportions of ingredients used) is of great importance in the theory, since in general the stresses induced in two materials for a given percentage change of length are directly proportional to the modulus of elasticity (for same sectional area).

Generally the diameter of a steel rod is so small compared with the full height of the beam that the stress in the rod is taken as uniform over the whole of its section.

285. Concrete-Steel Beam of Rectangular Section. Flexure Stresses. — As in the common theory of flexure of homogeneous beams, it will be assumed that cross-sections plane before flexure are still plane when the beam is slightly bent, so that changes of length occurring in the various fibers are proportional to the distances of those fibers from a certain *neutral axis* of the cross-section, and upon the amount of any such change of length (relative elongation) can be based an expression for the accompanying stress. Now in the case of concrete it is not strictly true that stresses are proportional to changes of length ("strains" or deformations); in other words its modulus of elasticity, E, is not constant for different degrees of shortening under compressive stress. Nevertheless, since this modulus does not vary much, within the limits of stress to which the concrete is subjected in safe design, it will be considered constant, the resulting equations being sufficiently accurate for practical purposes.

Let us now take as a "free body" any portion, ON, of the beam in Fig. 290, extending from the left-hand support to any section, at any distance x from that support. In the plane section terminating this body on the right, DNS (see now Fig. 292, in which we have also, at the right-hand, an end-view of

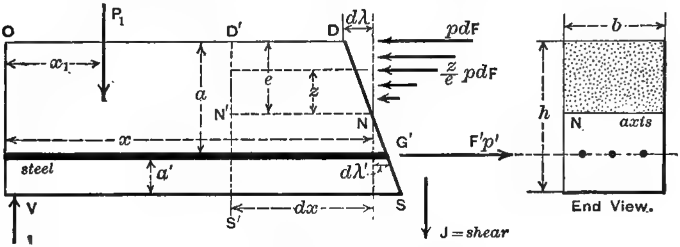


FIG. 292.

the body), we note that the fibers of concrete from D down to a neutral axis N are in a state of compression, while below N the steel rod alone is considered as under stress, viz., a total tensile stress of $F'p'$, where F' is the aggregate sectional area of all the steel rods, these rods being at a common distance a' above the lower edge of the section, and p' is the unit (tensile) stress in the steel rods.

The distance DN, or "e," of the neutral axis N below the "outer fiber" D, is to be determined. Let p denote the unit compressive stress in the fiber at D (outer fiber) of the concrete; then the unit stress in any fiber of the concrete at distance z from N will be $\frac{z}{e}p$, lbs. per sq. in., and the total stress on any such fiber is $\frac{z}{e}\ pdF$, lbs. (where dF is the sectional area of the fiber). All the (horizontal) fibers between the two consecutive cross sections DS and $D'S'$ were originally dx inches long, but now (during stress), we find that the fiber at D has been shortened an amount $d\lambda$ and the steel rod "fibers" elongated an amount $d\lambda'$, so that we have the proportion $d\lambda : d\lambda' :: e : a - e$;

or, $$\frac{d\lambda}{d\lambda'} = \frac{e}{a - e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (0)$$

For the free body in Fig. 292 we have, for equilibrium, the sum of horizontal components of forces $= 0$ (the shear J has no horizontal component); that is, remembering that below N no tensile forces are considered as acting on the concrete, but simply the total tensile stress $F'p'$ in the steel rods,

$$\int_0^e \frac{z}{e}\ pdF - F'p' = 0.$$

But here $\frac{p}{e}$ is a constant; and for the rectangular cross-section, $dF = b\,.\,dz$, and

$$\frac{pb}{e}\int_0^e zdz = \frac{pb}{e}\,.\,\frac{e^2}{2};\ \therefore\ \frac{pbe}{2} = F'p' \quad . \quad . \quad . \quad (1)$$

But from the definition of modulus of elasticity (E for the concrete and E' for the steel), we have (§ 191)

$E = p \div$ (relat. elongation), or $E = p \div (d\lambda/dx)$; and similarly, $E' = p' \div (d\lambda'/dx)$; whence

$$\frac{d\lambda}{d\lambda'} = \frac{p}{p'}\,.\,\frac{E'}{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But, from eq. (1), $p \div p' = 2F' \div be$, combining which with eqs. (0) and (2), and denoting the ratio $E' \div E$ by n,

we find $$\frac{e}{a - e} = \frac{2F'n}{be} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The ratio n may have a value from 10 to 25 for "rock-concrete," and still higher for "cinder-concrete;" see § 284. Now solve eq. (3) for the distance e, obtaining

$$e = \frac{F'n}{b}\left(\sqrt{\frac{2ab}{F'n} + 1} - 1\right) \quad . \quad . \quad . \quad . \quad (4)$$

This locates the neutral axis, N. [See, later, eq. (29), § 291.]

Returning to the free body ON in Fig. 292 above, we note that the resultant compression in the concrete between N and D, viz., $\frac{1}{2}\,p\,.\,be$, lbs. [see eq. (1)], is equal in value to the total tension $F'p'$, lbs., in the steel rods at G', and that they are parallel. Consequently they form a couple (the "stress-couple" of the section) whose moment is equal to the product of one of these forces, say $F'p'$, by the perpendicular distance $= a''$, between G' and a point G (see now Fig. 293) whose distance from the "outer fiber" D is one-third of e. The "arm" of this couple is $a'' = a - \dfrac{e}{3}$. For equilibrium of the free body ON in Fig. 292 the shear J and the two forces V (reaction) and P_1 (load) must be equivalent to a couple of opposite and equal moment to that of the stress couple. Call this moment M [in this case it has a value of $Vx - P_1(x - x_1)$]; it is the "bending moment" of the section at DS. We may therefore write (see Fig. 293):

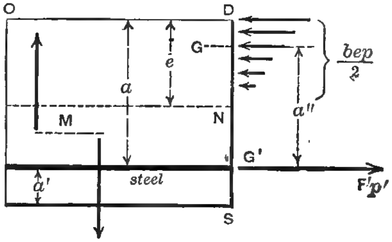


FIG. 293.

$$M = F'p'\,[a - \tfrac{1}{3}e]; \text{ and } \therefore p' = \frac{M}{F'(a - \frac{1}{3}e)}; \quad . \quad (5)$$

which will give the unit-stress p', induced in the steel rods at section DS. It is seen to depend on the position of the neutral axis N (i.e., upon e); upon the bending moment, M, at that section; upon the sectional area F' of the steel rods (aggregate); and on the distance, a, at which they are placed from the compression edge, D, of the beam.

But since the resultant compression, $\frac{1}{2}\,p\,.\,be$, is equal to the resultant tension, $F'p'$, we may also write

$$M = \tfrac{1}{2}\,p\,.\,be\,[a - \tfrac{1}{3}e] \text{ and } \therefore p = \frac{2M}{be\,(a - \frac{1}{3}e)}; \quad . \quad (6)$$

which gives the unit-stress (compression) in the outer "fiber" at D, of the concrete, for this section DS.

286. Horizontal Shear in the Foregoing Case (Rectangular Section). The shear per sq. in. along the sides of the steel rods, and also along the horizontal "*Neutral Surface*," NN'' (see Fig. 294), may be obtained as follows: — Let dx be the length of a small portion of the beam (of Fig. 290) situated between two vertical sections DS and $D''S''$. Fig. 294 shows this portion as a "free body." The forces acting consist of the tension $F'p'$ on the left-hand end of the steel; the tension on the right-hand end of these rods [being something greater (say) and expressed by $F'(p' + dp')$ in which dp' is the difference between the unit-tensions at the two ends of the steel "re-inforcement"]; the resultant compression, $\frac{1}{2}\, be \,.\, p$, in the concrete on the left; and that, $\frac{1}{2}\, be \,.\, (p + dp)$, on the right; and, finally, the two vertical shears, J and J''. Here p is the unit compressive stress (lbs. per sq. in.) in outer fiber of concrete at the left-hand extremity of the same, while $p + dp$ expresses the unit compressive stress in the same outer fiber at the right-hand extremity.

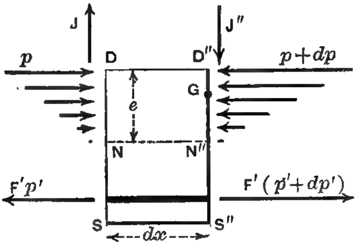


FIG. 294.

Evidently the difference between the total tensile stresses at the extremities of the steel rods will give the total horizontal shearing stress on the sides of those rods and this may be written $p_s' l_0 dx$ (lbs.), where p_s' = unit shearing stress between the steel and concrete and l_0 = aggregate perimeter of the steel rods (so that $l_0 dx$ = total area of the outside surface of rods in Fig. 294);

hence $$p_s' l_0 dx = F'(p' + dp') - F'p' \quad . \; . \; . \; . \quad (7)$$

But if, for the free body of Fig. 294, we put Σ moms. = 0 about the point G (a distance $\frac{1}{3}\, e$ from upper fiber)

we find $$Jdx = [F'(p' + dp') - F'p'](a - \tfrac{1}{3}\, e); \quad . \; . \quad (8)$$

and hence see (7), $$p_s' = \frac{J}{l_0\,(a - \frac{1}{3}\, e)} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

Also, if we let p_s denote the unit shearing stress (or tendency to slide) along the horizontal surface NN'' or neutral surface, the total amount is $p_s bdx$ (lbs.).

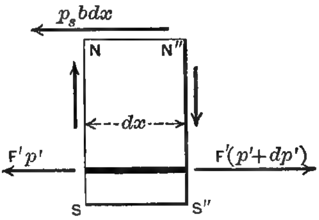


FIG. 295.

In Fig. 295, which shows as a free body the portion $NN''S''S$ of Fig. 294, we see this horizontal force (of concrete on concrete) acting toward the left. The other forces acting on the free body are as shown in Fig. 295 and, by putting Σ horiz. compons. = 0,

we find
$$F'(p' + dp') - F'p' = p_s bdx; \quad . \quad . \quad . \quad . \quad (9a)$$

and finally, see eq. (8),
$$p_s = \frac{J}{b\left(a - \frac{1}{3}e\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This (unit) shearing stress in the concrete along NN'', the "neutral surface," should nowhere exceed a certain value (e.g., 64 lbs. per sq. in.). For horizontal planes above NN'' it is smaller than along NN''. Similarly, the unit stress p_s' should not exceed a proper limit.

287. Numerical Example of a Concrete-Steel Beam of Rectangular Section. (See foregoing equations.)

Fig. 296 shows the section [8 by 11 inches] of the beam. Four round steel rods are imbedded near the under (tension) side, their centers being 10 in. from the top of section ($a = 10$ in.). This beam is to be placed on two supports at the same level and 8 feet apart, and is to support a concentrated load P, lbs., at the middle of the span as well as its own weight, which is $W = 600$ lbs.

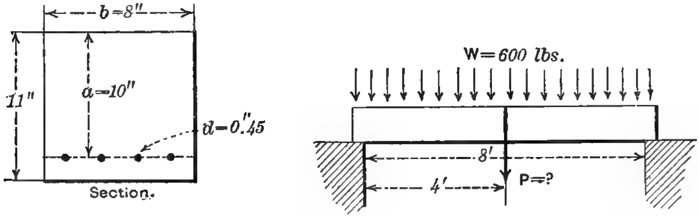


FIG. 296.

P is to be determined of such a safe value that the greatest stress in the steel rods shall not exceed 16,000 lbs. per sq. in. The compressive stress in concrete is not to exceed 700 lbs. per sq. in., nor the greatest shear either in the concrete or between the steel and the concrete, 64 lbs. per sq. in.

Each steel rod is continuous throughout the whole span and has a diameter of 0.45 in., from which we easily compute the aggregate perimeter of the rods to be 5.65 inches ($= l_0$), and the aggregate sectional area to be **0.64** sq. in ($= F'$).

The ratio of the modulus of elasticity for steel to that of the concrete will be taken as 15 to 1; i.e., $n = 15$.

The first step is to locate the neutral axis by finding the value of e from eq. (4), thus:—

$$e = \frac{64}{100} \cdot \frac{15}{8} \left(\sqrt{\frac{2 \times 10 \times 8}{.64 \times 15} + 1} - 1 \right) = 3.84 \text{ in.}$$

Next, if for p' we write 16,000 (using inch and pound) and substitute in eq. (5), solving for M, we obtain the greatest bending moment to which any section of the beam should be exposed, so far as the steel is concerned, viz:—

$$M = p'F'\left(a - \frac{e}{3} \right) = 16,000 \times 0.64\,(10 - 1.28) = \left\{ \begin{matrix} 89,300 \\ \text{in.-lbs.} \end{matrix} \right.$$

i.e., max. moment is to be 89,300 in.-lbs.

For the mode of loading of the present beam the max. moment occurs at the section at the middle of the span and has a value (with l denoting the span, or 96 in.) of $\frac{Pl}{4} + \frac{Wl}{8}$. We therefore write

$$\frac{P \times 96}{4} + \frac{600 \times 96}{8} = 89,300. \quad \text{Hence } P = 3,420 \text{ lbs.}$$

To find the accompanying maximum compressive stress in the concrete, eq. (6) gives (for outer "fiber")

$$p = \frac{2M}{be\,(a - \frac{1}{3}e)} = \frac{2 \times 89,300}{8 \times 3.84 \times 8.72} = 666 \text{ lbs. per sq. in.,}$$

which is within the limit set (700 lbs. per sq. in.).

As for the max. *shearing* unit stresses p_s' and p_s, they are greatest where the vertical shear, J, is a max., which is close to one of the supports. Here we note that J is equal to $\frac{1}{2}$ of 3,420 + $\frac{1}{2}$ of 600 = 2,010 lbs. Hence, from eq. (9),

$$p_s' = \frac{2,010}{5.65 \times (10 - 1.28)} = \frac{2,010}{5.65 \times 8.72} = 40.8 \text{ lbs. per sq. in.,}$$

$$\left. \begin{matrix} \text{while} \\ \text{from (10)} \end{matrix} \right\} p_s = \frac{2,010}{8 \times 8.72} = 28.8 \text{ lbs. per sq. in.,}$$

These shearing stresses are seen to be well within the limit set, of 64 lbs. per sq. in. As to compressive stress, the building laws of most cities put 500 lbs. per sq. in. as max. safe limit for p, the compressive stress in concrete.

288. Concrete-Steel Beam of T-Form Section. See Fig. 297. In this form of beam, to secure simplicity in treatment, it will be considered that the flange (TK) alone is subjected to compressive stress [although strictly a small portion of "stem" between the flange and the neutral axis of a section is under that kind of stress]. The part of stem below the neutral axis (as before) is not considered to offer any tensile resistance, all tension being borne by the steel rods or "re-inforcement." Fig. 297 shows a side view and also an end-view of a portion of the beam in Fig. 291 extending from the left-hand support up to any section DS (or up to W in Fig. 291). As before, sections plane before flexure are considered to be still plane during flexure, so that the elongations or shortenings of any horizontal "fiber," whether steel or concrete, are proportional to the distances from a neutral axis N, at some distance e from the top fiber of the flange, where the unit compressive stress has some value p.

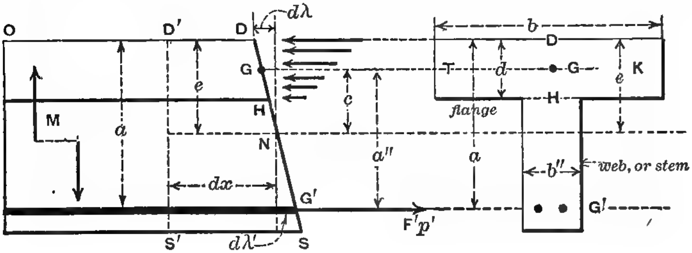


FIG. 297.

Also, since the E for concrete in compression is to be taken as constant the stresses in the concrete will also be proportional to the distances of the "fibers" from N the neutral axis. Let p'' denote the unit-stress in the concrete at H, the bottom fiber of the flange; then, by proportion, $p : p'' :: e : e - d$, where d is the thickness of the flange. Since the compressive stresses in the concrete between H and D are distributed over a rectangle their average unit-stress is $(p + p'')/2$, and their resultant, which acts horizontally through some point G, has a value of $bd \,.\, (p + p'')/2$; or, as it may be written (see above for p''), $(p\,(2\,e - d)\,.\,bd) \div (2\,e)$.

The total tensile stress in the steel rods will be $F'p'$, as before, where F' is the aggregate sectional area of the rods and p' the unit stress in them at section DS. Besides the stresses just mentioned the other forces acting on the free body in Fig. 297 are all vertical; viz., the shear J and the pier reaction and certain loads between O and D; hence by summing the horizontal components we note that the compressive stress in the concrete is equal to the tensile stress $F'p'$ in the steel, so that this tensile force $F'p'$ and the resultant compressive stress form a couple ("*stress-couple*" of the section; with an arm $= \overline{GG'}$, $= a''$), and we have

$$\frac{p\,(2\,e - d)\,bd}{2\,e} = F'p' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Consequently the shear and the other vertical forces acting on the free body form a couple also, and the moment of this couple (equal to that of the "stress-couple") will be called M. In the figure these vertical forces are not shown, but simply an equivalent couple (on the left).

If at this part of the beam a length dx of the steel has stretched an amount $d\lambda'$ and an equal length, dx, of the outer fiber at D has shortened an amount $d\lambda$, we have from eq. (2) of previous work

$$\frac{d\lambda}{d\lambda'} = \frac{p}{p'} \cdot \frac{E'}{E}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where E' and E are the moduli of elasticity of the steel and concrete, respectively. But, from (11),

$$\frac{p}{p'} = \frac{F'e}{(e - \frac{1}{2}\,d)\,bd}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and from similar triangles $d\lambda : d\lambda' :: e : (a - e)$ . . . . . (14)

Eqs. (13), (14), and (12), with $E' \div E = n$, give

$$e = \frac{F' \,.\, n \,.\, a + \dfrac{bd^2}{2}}{bd + F'n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and thus the neutral axis, N, is located.

It will now be necessary to locate the point of application, between H and D, of the resultant compressive stress on HD; that is, the point G in Fig. 298 which gives a side view of these stresses alone, forming, as they do, a trapezoidal figure whose center of gravity, U, projected horizontally on HD gives the desired point, G. The lower base HC'' of this trapezoid represents the unit stress p''; the upper, DC''', represents the unit stress p. The distance, call it c, of G from N, is to be determined. Let the trapezoid be divided into a rectangle $DD'''C''H$ and a triangle $D''C'''C''$. The center of gravity of the latter is at a vertical distance of $\frac{1}{6}d$ from a line $H''H'''$ drawn horizontally at distance $\frac{1}{2}d$ from D. $H''H'''$ passes through the center of gravity of the rectangle. Let us now find the distance $\overline{GH''}$ by writing the moment of the resultant stress about point H'' equal to the sum of those of its two parts, or components, represented by the rectangle and the triangle; whence we have

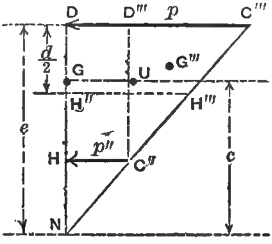


Fig. 298.

$$\tfrac{1}{2}(p+p'').bd \times \overline{GH''} = 0 + \frac{(p-p'')bd}{2}.\frac{d}{6} \quad . \quad (16)$$

Noting that $p'' = \left(\frac{e-d}{e}\right)p$, we have, solving,

$\overline{GH''} = \frac{1}{6}.\frac{d^2}{2e-d}$, and therefore, measuring from N,

$$\overline{NG}, \text{ i.e., } c, = e - \frac{d}{2} + \frac{1}{6}.\frac{d^2}{2e-d} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Now that both e and c have been determined in any given case it remains to find expressions for the unit stresses p' and p (in steel and fiber D of concrete).

Since G is the point of application of the resultant compression in flange, the arm of the stress-couple, a'' (Fig. 297), is the distance from G to G' (see Fig. 297); that is,

$a'' = c + (a - e)$; and hence we may write

$$F'p'(c+a-e) = M; \therefore p' = \frac{M}{F'(c+a-e)} \quad . \quad (18)$$

Also, by eliminating the ratio $d\lambda : d\lambda'$ from eqs. (12) and (14) we have, solving for p,

$$p = \frac{p'e}{n(a-e)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

289. Shearing Stresses in T-Form Concrete-Steel Beams. As regards the unit shearing stress, p'_s induced on the sides of the steel rods, in this case of the concrete steel beam of T-form section, an analysis similar to the corresponding one in the case of the beam of rectangular section leads to the result

$$p'_s = \frac{J}{l_0\,(c+a-e)} \quad \left\{ \begin{array}{l} \text{where } J \text{ is the total vertical shear at} \\ \text{the section } BS\text{, and } l_0 \text{ the aggregate} \end{array} \right\} \quad (20)$$

perimeter of the steel rods.

And, similarly, for the unit shearing stress on the horizontal surface separating the flange from the "web" or "stem" (see Fig. 297) at H, where the width of the web is b'', we find for this unit horizontal shear, p_s,

$$p_s = \frac{J}{b''(c+a-e)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

290. Deflections of Concrete-Steel Beams. The deflection of a loaded *prismatic* concrete-steel beam resting on two supports at its extremities, may be obtained for the cases dealt with in §§ 233–236 inclusive, in connection with homogeneous beams; provided the product EI occurring in the expressions for these deflections be replaced by $\frac{Ebe^2}{2}\left(a-\frac{e}{3}\right)$, for concrete-steel beams of rectangular section; and by $E'F'\,(a-e)\,(c+a-e)$, for those of T-form section.

291. Practical Formulæ and Diagrams for use with Concrete-Steel Beams of Rectangular Section. The equations of the foregoing theory will now be put into convenient form for practical use in designing these beams. Let us denote the ratio of p' (stress in steel at section of max. moment) to p (stress in outer fiber of concrete) by r; i.e., $r = p'/p$; while $n = E'/E$, as before. Also let $m, = M \div b$, denote the max. bending moment *per inch* of width (b) of beam; and let F' (area of steel) $\div\, b$ be called f', i.e., steel area *per inch of width* (b). In other words, we have the notation

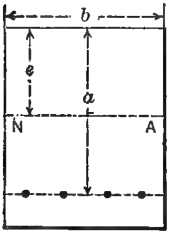


FIG. 299.

$$r = \frac{p'}{p};\ n = \frac{E'}{E};\ m = \frac{M}{b};\ \text{and } f' = \frac{F'}{b};\quad . \quad . \quad . \quad . \quad (22)$$

If we now substitute $e = 2\,rf'$, from eq. (1), in eq. (3),

we have $$2\,rf'\,(r+n) = an \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Now $e = 2\,rf'$, which from (23) $= an \div (r+n)$; Hence eq. (5)

will give $$3\,m(r+n) = af'p'\,(3\,r + 2\,n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

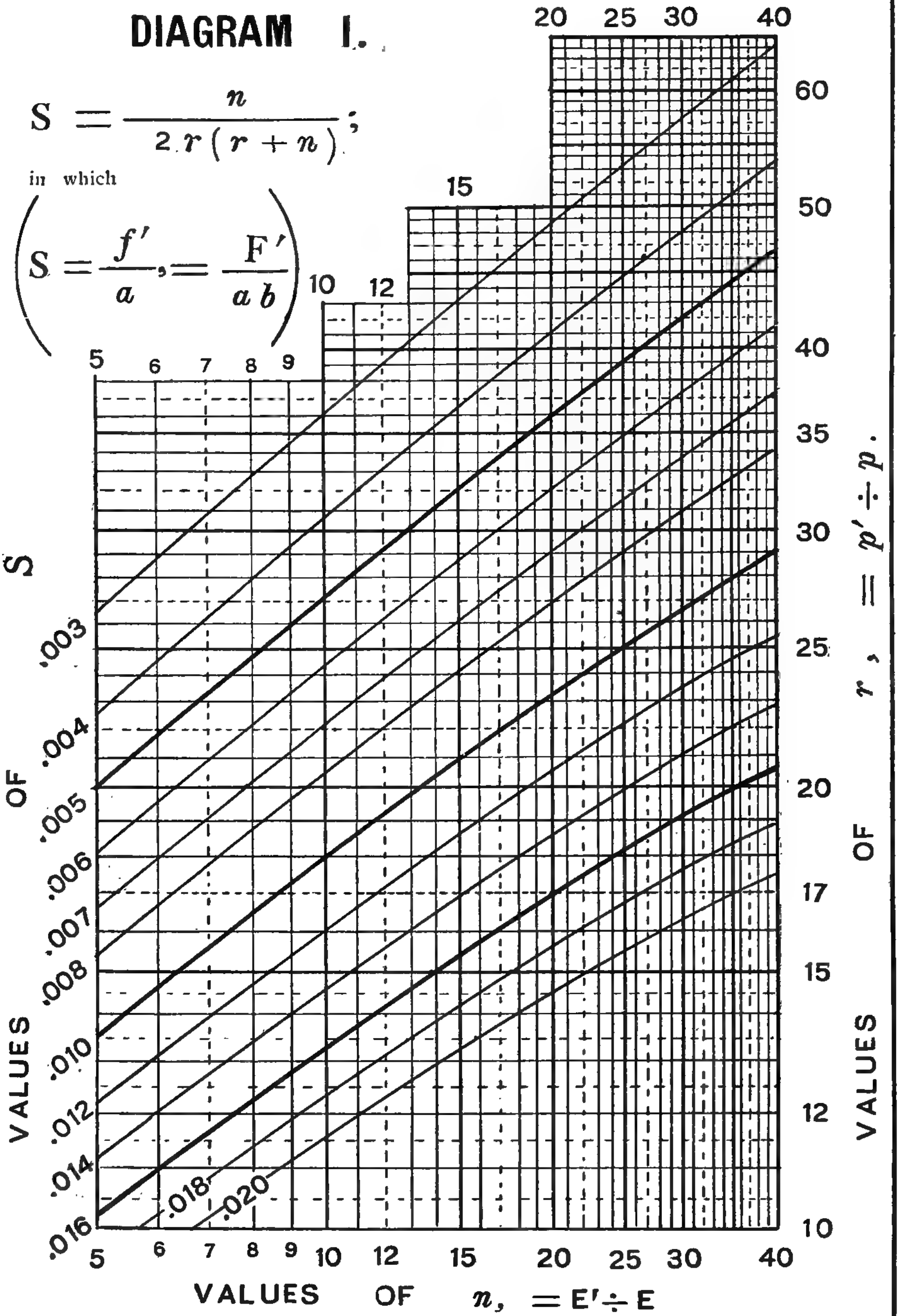
REINFORCED CONCRETE BEAMS (rectangular).

DIAGRAM I.

$$S = \frac{n}{2\,r\,(r+n)};$$

in which

$$\left(S = \frac{f'}{a}, = \frac{F'}{a\,b}\right)$$



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REINFORCED CONCRETE BEAMS (rectangular).

DIAGRAM II.

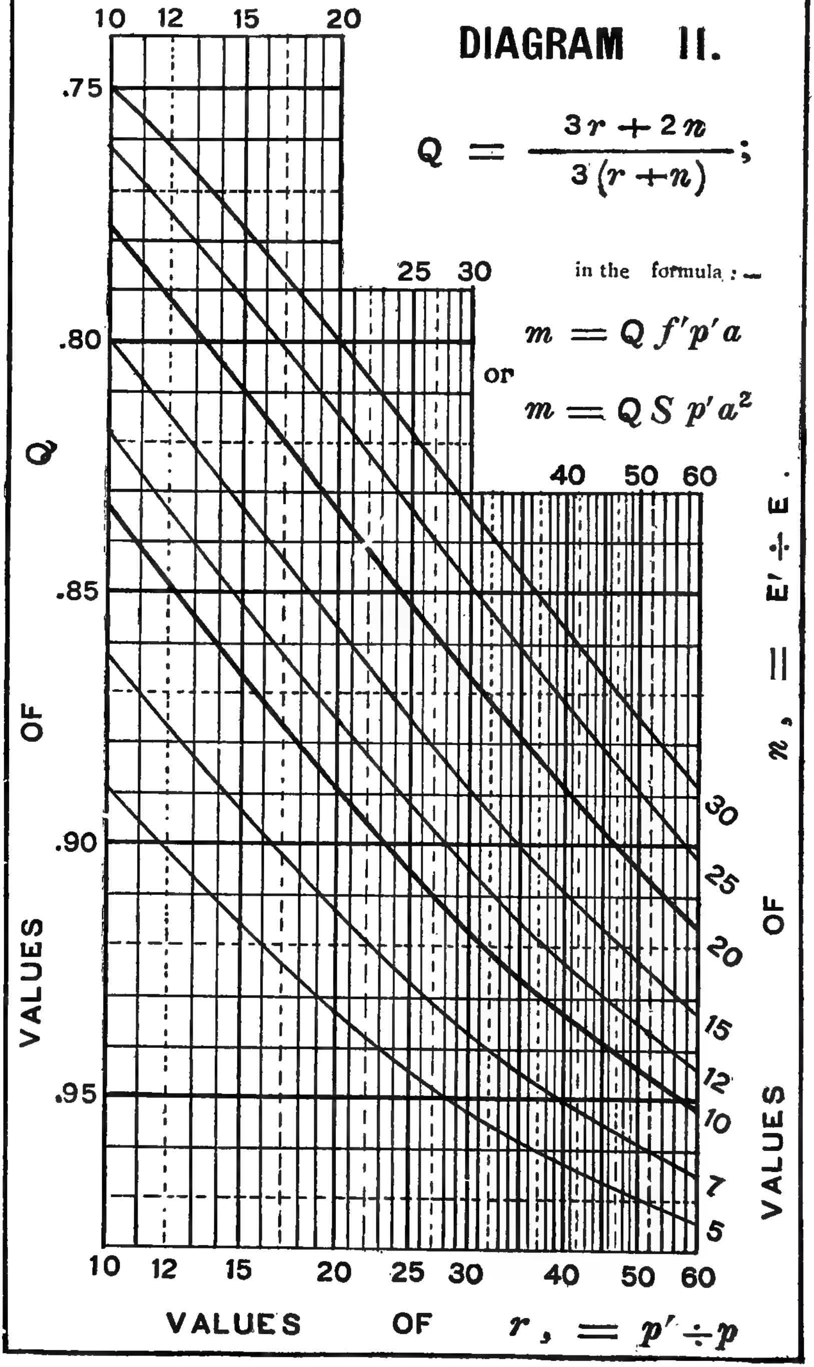
$$Q = \frac{3r + 2n}{3(r + n)};$$

in the formula :—

$$m = Q f'p'a$$

or

$$m = Q S p'a^2$$



[To face page 351.

If now S denote the quotient $f' \div a$ (i.e., S = area of steel section ***per unit area of concrete section above the steel***), (23) may be written in the form

$$S = n \div [2\,r\,(r+n)] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Again, let $\dfrac{3\,r+2\,n}{3\,(r+n)}$ be denoted by Q . . . . (26)

From (24), (25) and (26) we may then derive

$M = QSp'a^2b$ . . . . (27); and $m = QSp'a^2$ . . . . (28

for practical use in design; in connection with Diagrams I and II (opp. pp 350 and 351) from which we may obtain values of Q and S, respectively, fo any given values of r and n.

As to unit *shearing stresses*, which should not exceed 64 lbs. per sq. in. (say), use is made of eqs. (9) and (10), § 286, in which the maximum total vertical shear J_m should be substituted. In computing "e" for use in (9) and (10), it is simplest to employ the relation [derived from eqs. (1), (23) and (25)]

$$e = 2\,Sra \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

292. Numerical Examples. Rectangular Section. Let us suppose that in the cross-section of maximum moment the stresses in *both* materials are to attain their greatest safe values; viz., 16,000 lbs. per sq. in. tension for the steel, and 500 for the compressive stress in outer fiber of the (rock) concrete. Also suppose that $E' = 30{,}000{,}000$ and $E = 2{,}000{,}000$ lbs. per sq. in. That is, we have $n = 15$ and $r = 16{,}000 \div 500, = 32$; and from Diagram I find $S = 0.005$. In other words the necessary area of steel section is $\frac{1}{2}$ of one per cent. of the area of the concrete (above steel rods). We also find, from Diagram II, that $Q = 0.894$.

If now "a" be taken as 10 in., eq. (28) gives:—

$m = 0.894 \times 0.005 \times 16{,}000 \times 100, = 7{,}152$ inch-lbs. bending moment that could safely be withstood by each inch in the width b; so that if "b" were 8 inches we should have $M = mb = 7{,}152 \times 8 = 57{,}216$ inch-lbs., safe bending moment for the section of the beam.

Again, with r and n still equal to 32 and 15, respectively, and hence with Q and S as before, viz., 0.894 and 0.005, if b is assumed as 10 in., and the max. bending moment to be sustained is $M = 80{,}000$ inch-lbs. (so that $m = 8{,}000$), we find from eq. (28)

$$a = \sqrt{\frac{8{,}000}{0.894 \times 0.005 \times 16{,}000}} = 10.58 \text{ inches},$$

as necessary value of a; while the total area of steel section needed is

$$F', = S\,.\,a\,b = 0.005 \times 10.58 \times 10 = 0.5290 \text{ sq. in.}$$

which is seen to be one half of one per cent. of the area $[10 \times 10.58]$ of the concrete above steel (i.e., $S = 0.005$, as found from Diagram I originally).

293. Cost of Beams of Rectangular Section. While in a general sense economy in cost is favored by having the width "b" of the rectangular section small compared with the height "a," a limit to narrowness of width is set by the unit shearing stress in the neutral surface which would be found to exceed a safe limit if the beam were too narrow. The thickness "a'" of concrete below the steel rods (see Fig. 292) might be made $\frac{1}{10}$ of "a."

CHAPTER VI.

FLEXURE. COLUMNS AND HOOKS. OBLIQUE LOADS.

294. Oblique Prismatic Cantilever. In Fig. 301, at (*a*), (on p. 354) we have a prismatic beam built in at *K*, projecting out obliquely, and carrying a vertical load *P* at upper end; the line of action of *P* passing through the center of gravity of the upper base of the prism. In such a case the fibers of the beam where they cross any transverse plane *mg* will evidently be subjected to compressive stress (called a "*thrust*") due to the component of *P* parallel to the axis *OK* of the prism; to a *shear* *J* due to the component of *P* at right angles to that axis; and also to additional stresses, both tensile and compressive, forming a "*stress-couple*" due to the moment of *P* (i.e., *Pu*) about *g*, the center of gravity of the cross-section *m′m*. (See (*b*), Fig. 301.)

More in detail, consider in Fig. 300 a portion *AB* of the prism, being the part lying above a cross-section *mm′* near the top, so that the portion *gO* of axis is practically perpendicular to the section *mm′* which is a plane both before and after flexure, *g* being the center of gravity of the plane figure formed by the cross-section.

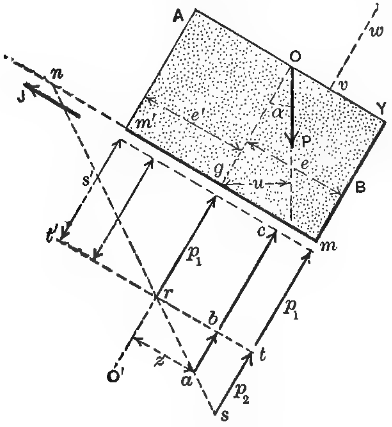


FIG. 300.

Let the unit stress on the end of the extreme fiber at *m* be represented by the length *sm* and that [also compression (say)] on the other extreme fiber, at *m′*, by *s′m′*. Draw the straight line *ss′*; then by the common theory of flexure the stress on any intermediate fiber, at *c*, would be the intercept, or ordinate, *ac* to this line. Now the unit stress p_1, on the fiber *g* at the center of gravity of cross-section, being $\overline{gr}$, draw through *r* a line *t′rt*

parallel to *m′m*, and we now have the stress on any fiber as *c* divided into two parts $\overline{bc}$, or p_1, the same for all the fibers; and $\overline{ab}$, different for the different fibers but *proportional to the distance z* of the fiber from *g*. Hence we have:
the unit stress on any fiber *c* is

$$p = p_1 + \frac{z}{e} p_2 \quad (lbs.\ per\ sq.\ in.) \quad . \quad . \quad . \quad . \quad (1)$$

where p_2 is $\overline{st}$ and *e* the distance of the extreme fiber *m* from *g*; and hence the total stress on fiber *c* is $p dF = p_1 dF + \frac{z}{e} p_2 dF$, *lbs.*; where dF is the area (sq. in.) of section of fiber, or element of area of the cross-section, *F* being the total area of the cross-section, *mm′*. Geometrically, we note that while the system of normal stresses on all the fibers forms a trapezoid, *m′s′sm* in this side-view, and that they are all compressive, they are equivalent to a rectangle, *m′t′tm*, of stress of *uniform* compressive unit-stress p_1; and two *triangles*, one, *rst*, of compressive stress, and the other, *rs′t′*, of *tensile* stress.* It will now be shown that the sum of the moments of the stresses of the rectangle about center *g* is *zero*, and that the two triangles of stress form a *couple*.

Σ(moms.) of stresses in rectang. $= \int_{z=-e'}^{z=+e} (p_1 dF) z = p_1 \int dF z$
$= p_1 F\bar{z} =$ *zero;* since $\bar{z} =$ zero, the *z*'s being measured from the center of gravity, *g*, of section *mm′* [§ 23, eq. (4)].

Again, if we sum (algebraically) the stresses of the two triangles,

we have $$\int_{z=-e'}^{z=+e} \frac{z}{e}\, p_2 dF = \frac{p_2}{e} \int z dF = \frac{p_2}{e} F\bar{z}, = \text{zero}$$

that is, the resultant of the compressive stresses in *rts* equals that of the tensile stresses in *rs′t′*; *hence they form a couple.*

If, therefore, we have occasion to sum the moments about *g*, of all the stresses acting on the fibers in section *mm′* we are to note that this moment-sum involves the stresses of *the triangles alone* (that is, of the couple), and is

$$\int_{-e'}^{e} \left(\frac{z}{e}\, p_2 dF\right) z, = \frac{p_2}{e} \int dF z^2 = \frac{p_2 I_g}{e} \quad . \quad . \quad . \quad (2)$$

in.-lbs.; where I_g is the "moment of inertia" of the cross-section referred to an axis through g (its center of gravity) and perpendicular to the "force plane" (plane of paper here).

---

* These plane figures are the side views of geometric solids.

If, again, we sum the components of all the stresses (on plane mm') parallel to the axis gO we note that this sum is zero for the couple and also for the shear J and hence reduces simply to

$$\int p_1 dF = p_1 \int dF = p_1 F, \text{ lbs. (the } \textit{Thrust}\text{)} \quad . \quad . \ (3)$$

(corresponding to the rectangle, $t'm$).

The sum of components perpendicular to axis gO is of course simply the *shear*, J, *lbs.*

Evidently the unit stress (normal) in fiber at m is expressed as $p_m = p_1 + p_2$; and that at m' as $p_{m'} = p_1 - \frac{e'}{e} p_2$. If in any case the latter is negative it indicates that the actual stress in this fiber is tension.

295. Oblique Cantilever. Fig. 301, (*a*) and (*b*). At (*b*) is shown as a "free body," a portion [of the cantilever at (*a*)] of any length x from top. The forces acting are the vertical load P at O, and the stresses on the ends of the fibers in the section $m'm$; and these stresses are now indicated as consisting of a *thrust*, T, of uniform intensity p_1, the total thrust being p_1F, lbs., (where F is the total area of section); of a *stress-couple*, C, whose moment is $\frac{p_2 I}{e}$ in.-lbs., in which $p_2 = p_m - p_1$ and I is the "moment of inertia" of the cross-section (about an axis through its center of gravity g at right angles to the plane ("force plane") containing Og and force P; *the same I that has been used in previous cases of flexure*); and the *total shear*, J, lbs., parallel to force plane and perpendicular to gO. The lever arm of P about g is u which practically $= x \sin a$ (unless the beam is considerably bent or is nearly vertical).

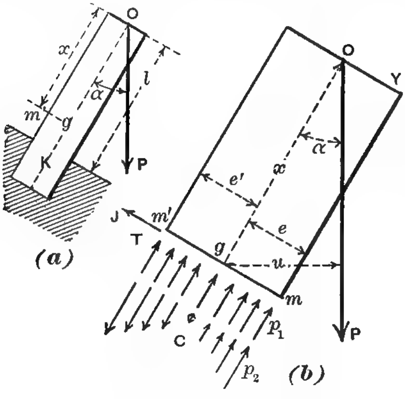


FIG. 301.

For this free body (in order to find p_1, p_2 and J)

$$\sum X = 0 \text{ gives: } P\cos a - p_1 F = 0;\ \therefore\ p_1 = \frac{P\cos a}{F} \quad . \quad (4)$$

$$\sum(\text{moms.})_g = 0\ .\ .\ \frac{p_2 I}{e} - Pu = 0;\ \therefore\ p_2 = \frac{Pue}{I} \quad .\ .\ (5)$$

and $$\sum Y = 0\ .\ .\ P\sin a - J = 0;\ \therefore J = P\sin a, \quad (6)$$

As x varies, from 0 to l, we note that p_1 and J remain unchanged but that p_2 *increases with* u; so that the maximum value of the unit stress p_m, which $= p_1 + p_2$, will be found in the section at K, where $x = l$; and if this stress is not to exceed a safe value, R', for the material, we put p_2 (at K) $+ p_1 = R'$, (as the *equation of safe loading*);

or, $$P\left[\frac{\cos a}{F} + \frac{le\sin a}{I}\right] = R' \quad .\ .\ .\ .\quad (7)$$

(N. B. For a cross-section of unusual shape the stress $p_{m'} = p_1 - \frac{e'}{e}\, p_2$, at K, might happen to be numerically greater than p_m, and thus govern the design).

296. Experimental Proof of Foregoing. A stick or test piece of straight-grained pine wood, 12 inches in length and of square cross-section (one inch square), originally straight and planed smooth and with bases perpendicular to the length, was placed in a testing machine; steel shoes, with (outside) spherical bearing surfaces, being centered on the ends. See Fig. 302, where ab is the stick and S, S', the two steel shoes. The stick was gradually compressed between the two horizontal plates B, B', of the machine and bent progressively in a smooth curve under increasing force. From the nature of the "end conditions," as the stick changed form, the line of action of the two end pressures P, P, always passed through the centers of gravity, a and b, of the respective bases.

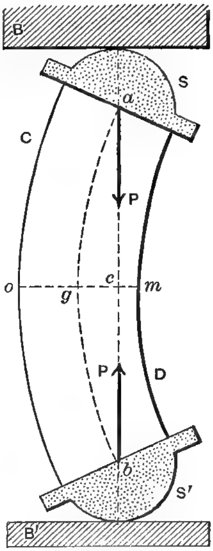


Fig. 302.

When the force P had reached the value 4500 lbs. a fine wrinkle was observed to be forming on the right-hand surface of the stick at the outside fiber m of the middle section gm. The other fibers of this section were evidently uninjured. At m then, the unit-stress must have been about 8000 lbs. per sq. in., the crushing stress (as known from previous experiments with sticks of similar material and equal section but only three or four inches long; these were too short to bend, and wrinkles formed around the *whole perimeter*, showing incipient crushing in *all* the fibers). The distance $\overline{gc}$ at this time was found to be $\frac{1}{8}$ in.; i.e., the lever arm, u, of the force P about g, the center of gravity of the section. In this case, then, it is to be noted that the value of u was *entirely due to the bending of the piece.*

Substituting, in eqs. (4) and (5) of § 295, the values $u=\frac{1}{8}$ in., $\alpha=0$, $\cos\alpha=1$, $e=e'$, $=\frac{1}{2}$ inch, $F=1$ sq. in.,

$$\text{and } I, = \frac{bh^3}{12}, = \frac{1\times 1^3}{12} = \frac{1}{12}\text{in.}^4, \text{ we find } p_1=4500$$

lbs. per sq. in. and $p_2=3375$ lbs. per sq. in.
Hence stress at m, $=p_1+p_2$, $=7875$ lbs. per sq. in., which is about 8000, as should be expected. On the fiber at o, however, we find a stress of p_1-p_2, or of only 1125 lbs. per sq. in. *compression.*

We find, then, that in the section om, when P reached the value of 4500 lbs., there was a *total-thrust* (p_1F) of 4500 lbs.; a *unit-thrust* (p_1) of 4500 lbs. per sq. in.; and a stress-couple having a moment of $Pu, = \frac{p_2I}{e}$, $=562.5$ in.-lbs., (implying a separate stress of $p_2=3375$ lbs. per sq. in. in the outer fibers, to be combined with that due to the thrust). Also that J, the shear, was zero.

297. Crane-Hooks. First (Imperfect) Theory. Fig. 303 shows a common crane-hook of iron or steel. Early writers (Brix and others) treated this problem as follows: —

The load being P, if we make a horizontal section at AB, about whose gravity axis, g, P has its greatest moment, and consider the lower portion C as a free body, in Fig. 304), we find, using the notation and subdivision of stresses already set forth in § 294 for an oblique prism, that the uniformly distributed pull (or "negative thrust") on the fibers is $p_1F=P$, lbs.;

while the moment of the stress-couple is $\frac{p_2 I}{e} = Pa$ ft.-lbs.; and that the shear, J, is zero.

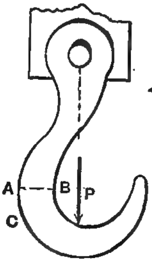


FIG. 303.

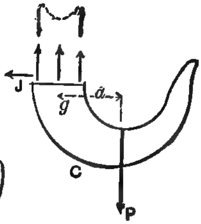


FIG. 304.

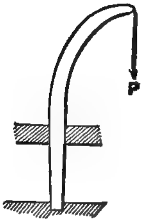


FIG. 305.

Hence on the extreme fiber at B we have a total unit tensile stress of $p_1 + p_2 = \frac{P}{F} + \frac{Pae}{I}$, which for safe design must not exceed the safe unit-stress for the material, R' lbs. per sq. in.; whence we should have

$$P\left[\frac{1}{F} + \frac{ae}{I}\right] = R' \quad \ldots\ldots \quad (8)$$

as the equation of safe loading.*

Example: Safe $P = ?$, if section AB is a circle of radius 2 in., while $a = 4$ in.; the material being mild steel for which (in view of the imperfection of the theory) a low value, say 6000 lbs. per sq. in., should be taken for R'. With these data we obtain:—

$$P = 6000 \div \left[\frac{1}{12.56} + \frac{4 \times 2}{12.56}\right] = 8376 \text{ lbs.}$$

The simple crane in Fig. 305, being practically an inverted hook, may be treated in the same manner.

298. Crane-Hooks. Later, More Exact, Theories. The most exact and refined theory of hooks yet produced is that of Andrews and Pearson,† but it is very complicated in practical application and far too elaborate and extended to be given here.

The next best (and fairly satisfactory) treatment is that of Winkler and Bach, of which the principal practical features and results will now be presented.

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* See experiments by Prof. Goodman, in *Engineering*, vol. 72, p. 537. Results are irregular, due probably to the use of this imperfect theory.

† Drapers' Company Research Memoirs. Technical Series I. London, 1904.

In *AB*, Fig. 306, we have again the free body of Fig. 304, but the vertical stresses acting on the cross-section *m'm* are proportional to the ordinates of a *curve* instead of a straight line. The imperfection of the early theories lies in the fact that the sides of a hook are curved, and not straight and parallel as in the prismatic body of Fig. 301; and the variation of stress from fiber to fiber on the cross-section must follow a different law, as may thus be illustrated:

As preliminary, the student should note, from the expression $\frac{P}{F} = \frac{E\lambda}{l}$ of p. 209, that in the case of two fibers under tension, with the same sectional area F, the unit-stress $P \div F$ (or p) is not proportional to the elongation λ of the fiber unless the two lengths l are equal. In Fig. 306 the center of gravity of the cross-section is g, and O is the center of curvature of the curved axis gk of this part of the hook (or other curved body). The two consecutive radial sections *m'm* and *t't* are assumed to remain plane during stress, and hence the changes of length, due to stress, of the (vertical) fiber lengths between them are proportional to the ordinates of a straight line; and if these fiber lengths were equal in length (as would be the case for a prismatic beam) the unit-stresses acting would also be proportional to the ordinates of a straight line (this is the case in Fig. 301).

But in the present case these fiber-lengths are *unequal*, so that the unit-stresses in action are (in general) proportional to the ordinates of a *curved* line. Such a curved line we note in *ve*, Fig. 306, the ordinates between which and the horizontal line *hi* represent the unit-stresses, p, acting on the upper ends

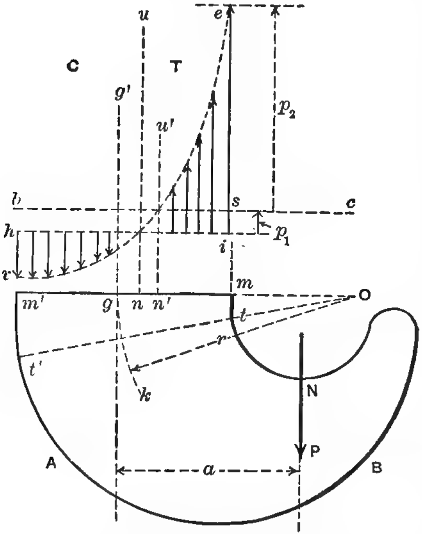

FIG. 306.

of the vertical fibers from m' to m. Thus, the stress on the fiber mt is $p_m = \overline{ei}$ (tension); and that on the other extreme fiber, (at m') is $p_{m'} = \overline{hv}$ (compression).

If now we compute the average unit-stress $p_1 = P \div F$ and lay it off, $= is$, upward from hi, and draw the horizontal bs, we thereby re-arrange the stresses into a uniformly distributed pull (or "negative thrust") p_1F lbs., represented by the rectangle $bsih$, and a stress-couple formed by the ordinates lying between the curve and the axis bs.

It will be noted in Fig. 306 that there is a fiber at some point n (on right of g) where the stress is zero; i.e., the "*neutral axis*" of the section is at n, ⏋ to paper. Also, at some point n', the actual stress is equal to the average, p_1, and an axis ⏋ to paper through this point would be the neutral axis if the forces acting on this free body, other than the fiber stresses, consisted, not of a single force P, but of a *couple*, with a moment $= Pa$. This axis through n' might be called the neutral axis for "pure bending", since then the whole system of fiber stresses would reduce to a couple and the stresses would be measured by the ordinates between bs and the curve.

299. Crane-Hooks. Winkler-Bach Theory. Formula for Stress. In Fig. 306, let F be the area of the plane figure formed by the section $m'm$, dF an element of this area, and z its distance (reckoned *positive toward the right*) from the gravity axis, g, of the section. The radius of curvature of gk is r, and a is the lever arm of P, the load, about g. Let $\overline{gm} = e$ and $\overline{gm'} = e'$ (distances of extreme fibers) and let

$$S \text{ denote the quantity } \left(\left[\frac{r}{F}\int_{z=-e'}^{z=+e}\left(\frac{dF}{r-z}\right)\right]-1\right); \quad . \quad (1)$$

an abstract number depending on the area, shape, and position, of the cross-section $m'm$; and upon the radius of curvature r. Its value may be obtained by the calculus (or Simpson's Rule) for ordinary cases. For instance, if the section is a rectangle of width b, and altitude $= h$, $= \overline{m'm}$, we find

$$S = \frac{r}{h}\left(\log_e \frac{2r+h}{2r-h}\right) - 1 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From the Winkler-Bach theory it results that the unit-stress on any fiber between m and m', at a distance z from the gravity axis g (on the right, *toward* the center of curvature, O; if on the left, z is negative) is

$$p = \frac{P}{F}\left[1 - \frac{a}{r}\left(1 - \frac{z}{r-z}\cdot\frac{1}{S}\right)\right] \quad . \quad . \quad . \quad . \quad (3)$$

lbs. per sq. inch. A positive result from (3) indicates tension; a negative, compressive stress. Of course, for $P \div F$ we might write the symbol p_1, or "average stress." If p were set = zero, a solution of (3) for z would locate the neutral axis, n, of Fig. 306; while by placing $p - p_1 = 0$, a solution for z would locate the point n', or neutral axis for "pure bending."

300. Numerical Example. Let the cross-section be a trapezoid, of base $b = 3$ in. at m, and upper base $b' = 1$ in. at m', both ⏋ to paper; the altitude $h, = \overline{m'm}$, being 4 in. This brings g $\frac{5}{3}$ in. ($= e$) from m and $\frac{7}{3}$ in. ($= e'$) from m'. Let N be in the same vertical as O and $\overline{Om} = 2$ in. Hence $r = a = 2 + \frac{5}{3} = \frac{11}{3}$ in. The material is mild steel and the load P is 8 tons; find p_m and p_m'.

From above dimensions we find area $F = 8$ sq. in., while from eq. (1), (using the calculus), $S = 0.0974$. For p_m we put $z = + \frac{5}{3}$ in. in eq. (3); and for p_m', $z = - \frac{7}{3}$ in.; obtaining, finally, $p_m = 17{,}120$ lbs. per sq. in. (tension); and $p_m' = - 7{,}980$ (compression). Evidently the elastic limit is not passed.

Using the imperfect theory of § 297, we should have obtained $p_m = 12{,}000$ lbs. per sq. in., only; which is seen to be about 30 per cent. in error, compared with the above value of 17,120. The reason for taking a low value for the safe unit-stress, R', in the example of § 297 is now apparent, an additional reason being the fact that loads are sometimes "suddenly applied" on hooks.

301. By **"column"** or **"long column"** is meant a straight beam, usually prismatic, which is acted on by two compressive forces, one at each extremity, and whose length is so great compared with its diameter that it gives way (or "fails") by buckling sideways, i.e. by flexure, instead of by crushing or splitting like a short block (see § 200). The pillars or columns used in buildings, the compression members of bridge-trusses and roofs, the "bents" of a trestle work, and the piston-rods and connecting-rods of steam-engines, are the principal practical examples of long columns. That they should be weaker than short blocks of the same material and cross-section is quite evident, but their theoretical treatment is much less satisfactory than in other cases of flexure, experiment being very largely relied on not only to determine the physical constants which theory introduces in the formulae referring to them, but even to modify the algebraic form of those formulae, thus rendering them to a certain extent empirical.

302. End Conditions.—The strength of a column is largely dependent on whether the ends are free to turn, or are fixed and thus incapable of turning. The former condition is attained by rounding the ends, or providing them with hinges or ball-and-socket-joints; the latter by facing off each end to an accurate plane surface, the bearing on which it rests being plane also, and incapable of turning. In the former condition the column is spoken of as having

round ends;* Fig. 311, (*a*); in the latter as having **fixed ends**, (**or flat bases; or square ends**), Fig. 311, (*b*).

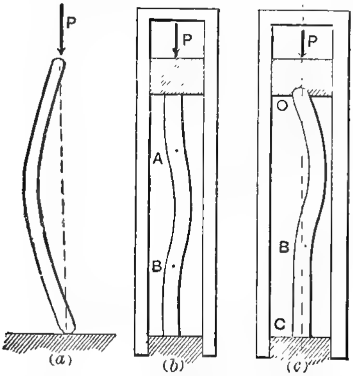

FIG. 311.

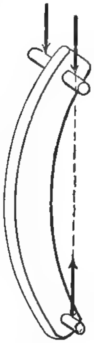

FIG. 312.

Sometimes a column is fixed at one end while the other end is not only round but *incapable of lateral deviation from the tangent line* of the other extremity; this state of end conditions is often spoken of as "**Pin and Square**," Fig. 311, (*c*).

If the rounding* of the ends is produced by a hinge or "pin joint," Fig. 312, both pins lying in the same plane and having immovable bearings at their extremities, the column is to be considered as round-ended as regards flexure in the plane ┐ to the pins, but as square-ended as regards flexure in the plane containing the axes of the pins.

The "moment of inertia" of the section of a column will be understood to be referred to a gravity axis of the section which is ┐ to the plane of flexure (and this corresponds to the "force-plane" spoken of in previous chapters), or plane of the axis of column when bent.

303. Euler's Formula.—Taking the case of a round-ended column, Fig. 313 (*a*), assume the middle of the length as an origin, with the axis X tangent to the elastic curve at that point. The flexure being slight, we may use the form $EI\,d^2y \div dx^2$ for the moment of the stress-couple in any

---

* With round ends, or pin ends, it should be understood that the force at each end must be so applied as to act through the *centre of gravity of the base* (plane figure) of the prismatic column at that end; and continue to do so as the column bends.

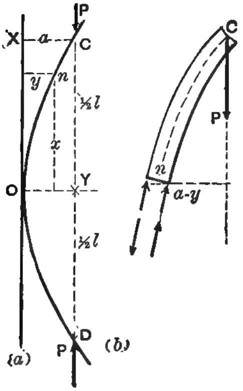

FIG. 313.

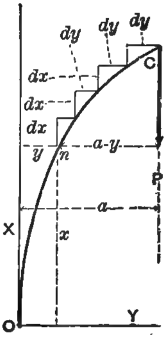

FIG. 314.

section n, remembering that with this notation the axis X must be ∥ to the beam, as in the figure (313). Considering the free body nC, Fig. 313 (b), we note that the shear is zero, that the uniform thrust $=P$, and that $\Sigma(\text{moms.}_n)=0$ gives (a being the deflection at O)

$$EI\frac{d^2y}{dx^2}=P(a-y) \quad . \quad . \quad . \quad (1)$$

Multiplying each side by dy we have

$$\frac{EI}{dx^2}\,dy\,d^2y=Pa\,dy-Py\,dy \quad . \quad . \quad (2)$$

Since this equation is true for the y, dx, dy, and d^2y of any element of arc of the elastic curve, we may suppose it written out for each element from O where $y=0$, and $dy=0$, up to *any* element, (where $dy=dy$ and $y=y$) (see Fig. 314) and then write the sum of the left hand members equal to that of the right hand members, remembering that, since dx is assumed constant, $1\div dx^2$ is a common factor on the left. In other words, integrate between O and any point of the curve, n. That is,

$$\frac{EI}{dx^2}\int_{dy=0}^{dy=dy}[dy]d[dy]=Pa\int_0^y dy-P\int_0^y y\,dy \qquad (3)$$

The product $dy\,d^2y$ has been written $(dy)d(dy)$, (for d^2y is the differential or increment of dy) and is of a form like xdx, or ydy. Performing the integration we have

$$\frac{EI}{dx^2} \cdot \frac{dy^2}{2} = Pay - P\,\frac{y^2}{2} \quad . \quad . \quad . \quad . \quad (4)$$

which is in a form applicable to any point of the curve, and contains the variables x and y and their increments dx and dy. In order to separate the variables, solve for dx, and we have

$$dx = \sqrt{\frac{EI}{P}}\frac{dy}{\sqrt{2ay-y^2}} \text{ or } dx = \sqrt{\frac{EI}{P}} \cdot \frac{d\left(\frac{y}{a}\right)}{\sqrt{2\,\frac{y}{a} - \left(\frac{y}{a}\right)^2}} \quad . \quad (5)$$

$$\therefore \int_0^x dx = \pm\sqrt{\frac{EI}{P}}\int_0^y \frac{d\left(\frac{y}{a}\right)}{\sqrt{2\,\frac{y}{a} - \left(\frac{y}{a}\right)^2}}$$

vers = 1 + cos α

$$\text{i.e., } x = \pm\sqrt{\frac{EI}{P}}\left(\text{vers. sin}^{-1}\frac{y}{a}\right) \quad . \quad . \quad . \quad (6)$$

(6) is the equation of the elastic curve DOC, Fig. 313 (a), and contains the deflection a. If P and a are both given, y can be computed for a given x, and vice versâ, and thus the curve traced out, but we would naturally suppose a to depend on P, for in eq. (6) when $x = \frac{1}{2}l$, y should $= a$. Making these substitutions we obtain

$$\tfrac{1}{2}l = \sqrt{\frac{EI}{P}}\left(\text{vers. sin}^{-1}\,1.00\right);\ \text{i.e., } \tfrac{1}{2}l = \sqrt{\frac{EI}{P}}\ \frac{\pi}{2} \quad (7)$$

Since a has vanished from eq. (7) the value for P obtained from this equation, viz.:

$$P_0 = EI\,\frac{\pi^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

is independent of a, and
is ∴ to be regarded as that force (at each end of the *round-ended* column in Fig. 313) which will *hold* the column at *any* small deflection at which it may previously have been set.

In other words, if the force is less than P_0 no flexure at all will be produced, and hence P_0 is sometimes called the force producing "incipient flexure." [This is roughly verified by exerting a downward pressure with the hand on the upper end of the flexible rod (a T-square-blade for instance) placed vertically on the floor of a room; the pressure must reach a definite value before a decided buckling takes place, and then a very slight increase of pressure occasions a large increase of deflection.]

It is also evident that a force slightly greater than P_0 would very largely increase the deflection, thus gaining for itself so great a lever arm about the middle section as to cause rupture. For this reason eq. (8) may be looked upon as giving the **Breaking Load** of a column with round ends, and is called *Euler's formula.*

Referring now to Fig. 311, it will be seen that if the three parts into which the flat-ended column is divided by its two points of inflection A and B are considered free, individually, in Fig. 315, the forces acting will be as there shown, viz.: At the points of inflection there is no stress-couple, and no shear, but only a thrust, $=P$, and hence the portion AB is in the condition of a round-ended column. Also, the tangents to the elastic curves at O and C being preserved vertical by the frictionless guide-blocks and guides (which are introduced here simply as a theoretical method of preventing the ends from turning, but do not interfere with vertical freedom) OA is in the same state of flexure as half of AB and under the same forces. Hence the length AB must = one half the total length l of the flat-ended column. In other words, the breaking load of a round-ended column of length $=\frac{1}{2}l$, is the same as that of a flat-ended column of length $=l$. Hence for the l of eq. (8) write $\frac{1}{2}l$ and we have as the breaking load of a column with flat-ends and of length $=l$.

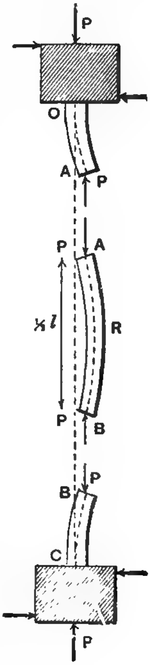

FIG. 315.

$$P_1 = 4\,EI\frac{\pi^2}{l^2} \quad . \quad . \quad . \quad . \quad (9)$$

Similar reasoning, applied to the "pin-and-square" mode of support (in Fig. 311) where the points of inflection are at B, approximately ⅓ l from C, and at the extremity O itself, calls for the substitution of ⅔ l for l in eq. (8), and hence the breaking load of a "*pin-and-square*" column, of length $= l$, is

$$P_2 = \frac{9}{4}\,EI\,\frac{\pi^2}{l^2} \quad . \quad . \quad . \quad (10)$$

Comparing eqs. (8), (9), and (10), and calling the value of P_1 (flat-ends) unity, we derive the following statement :

The breaking loads of a given column are as the numbers

1	9/16	¼	according to the
flat-ends	*pin-and-square*	*round-ends*	mode of support.

These ratios are approximately verified in practice.

Euler's Formula [i.e., eq. (8) and those derived from it, (9) and (10)] when considered as giving the breaking load is peculiar in this respect, that it contains no reference to the stress per unit of area necessary to rupture the material of the column, but merely assumes that the load producing "incipient flexure", i.e., which produces any bending at all, will eventually break the beam because of the greater and greater lever arm thus gained for itself. In the cantilever of Fig. 241 the bending of the beam does not sensibly affect the lever-arm of the load about the wall-section, but with a column, the lever-arm of the load about the mid-section is almost entirely due to the deflection produced.

It is readily seen, from the form of eqs. (8), (9) and (10), that when l is taken quite small the values obtained for P_0, P_1, and P_2 become enormous, and far exceed what would be found from the formula for crushing load of a short block, viz., $P = FC$ (see p. 219), with F denoting the area of section of the prism and C the crushing unit-stress of the material. The degree of slenderness a column must have to justify the use of Euler's relations will appear in the next paragraph.

304. Euler's Formula Tested by Experiment.—Since the "moment of inertia," I, (referred to a certain axis) of the cross-section of the column may be written $I = Fk^2$, where k is the "*radius of gyration*" (see p. 91), and F the area of the plane figure, eq. (8), for "round ends," may be written

$$\frac{P_0}{F} = \frac{\pi^2 E}{(l \div k)^2}. \qquad \qquad (11)$$

Here $P_0 \div F$ is the *average* unit-stress (compressive) on the cross-section and $l \div k$ is a ratio measuring the *slenderness* of the column. (Of course, when the column actually gives way by buckling, the unit-stress on the concave side at the middle of the length is much greater than the average). In the experiments by Christie, described on p. 112 of the *Notes and Examples*, the value of the ratio $l \div k$ ranges from 20 to 480.

As an **example** consider a 3″×3″×½″ angle-bar (or "angle") of wrought iron, with $l = 15$ ft., to be used as a column. Fig. 315*a* shows the cross-section of this shape, with dimensions. C is the center of gravity of this plane figure. Let the force be applied at each end of the column according to Christie's mode of "round ends," i.e., by a ball-bearing device, the force always passing through the point C of the section at each extremity of the column. Since the ends are free to turn in any plane, the axis of the column will deflect in the plane CN ⊥ to the axis 2 . . . 2 (of the plane figure) about which the values of I and of k are *least*. For this shape, we find from the handbook of the Cambria Steel Co., that k about axis 2 . . . 2 is the *least radius of gyration* and $=0.58$ in.; also that

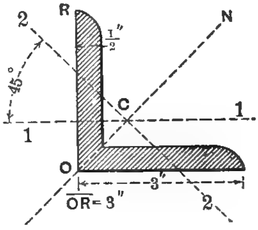

FIG. 315*a*.

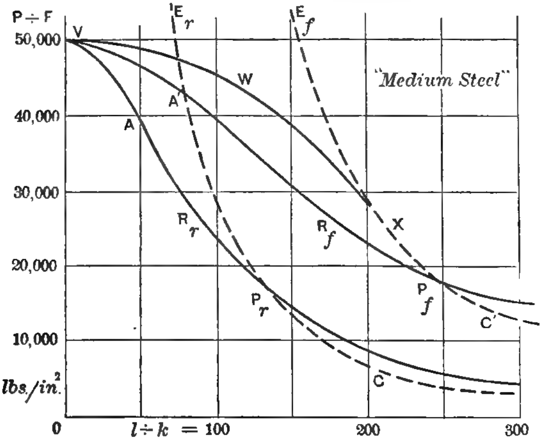

FIG. 315*b*.

the area of the figure is $F=2.75$ sq. in. Hence the "*slenderness-ratio*," $l \div k$, is $180'' \div 0.58'' = 310$; and from eq. (11) we have, with E for wrought iron taken as 25,000,000, lbs./in.^{2} (p. 279),

$$(P_0 \div F) = \pi^2 \times 25{,}000{,}000 \div (310)^2 = 2570 \text{ lbs./in.}^2;$$

while from the Christie experiments we find $(P_0 \div F) = 2650$ lbs./in.^{2} as the *average* unit-stress at rupture; a fairly close agreement with the Euler result. The total rupturing load would then be $P_0 = 2570 \times 2.75 = 7070$ lbs., and the *safe load*, with the "factor of safety" of 8 recommended in the Christie report, would be 884 lbs.

In this way it may be ascertained that for values of $l \div k$ from 200 to 400 for "round ends" and from 300 to 400 for fixed ends there is an approximate agreement between Euler's equations and the Christie experiments. But most of the columns used in engineering practice involve values of $l \div k$ less than 200, so that Euler's formulæ are not adapted to actual columns (though used to some extent in Germany). A formula of such nature as to be available for *all* degrees of slenderness has therefore been established (Rankine's, see next paragraph), based partly on theory and partly on experiment, which has obtained a very wide acceptance among engineers.

In Fig. 315*b* is shown a curve, E_r, resulting from plotting as abscissa and ordinate the values of $P_0 \div F$ and $l \div k$, as related in Euler's formula (11) for columns with round ends, for "medium" structural steel; with $E=30{,}000{,}000$ lbs./in.^{2} E_f is a similar curve plotted from Euler's formula (9) for fixed ends for the same material. Each of these Euler curves is tangent to *both* axes at infinity. The other curves will be referred to later.

305. Rankine's Formula for Columns.—The formula of this name (some times called Gordon's, in some of its forms) has a somewhat more rational basis than Euler's, in that it introduces the maximum normal stress in the outer fibre and is applicable to a column or block of any length, but still contains assumptions not strictly borne out in theory, thus introducing some co-efficients requiring experimental determination. It may be developed as follows:

Since in the flat-ended column in Fig. 315 the middle portion AB, between the inflection points A and B, is acted on at each end by a thrust $= P$, not accompanied by any shear or stress-couple, it will be simpler to treat that

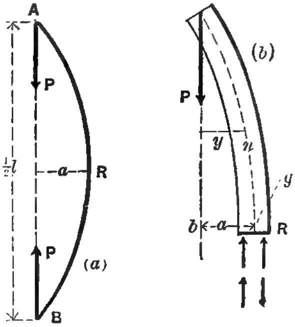

FIG. 316.

portion alone Fig. 316, (*a*), since the thrust and stress-couple induced in the section at *R*, the middle of *AB*, will be equal to those at the flat ends, *O* and *C*, in Fig. 315. Let *a* denote the deflection of *R* from the straight line *AB*. Now consider the portion *AR* as a free body in Fig. 316, (*b*), putting in the elastic forces of the section at *R*, which may be classified into a uniform thrust $= p_1F$, and a stress couple of moment $= \frac{p_2I}{e}$, (see § 294). (The shear is evidently zero, from Σ(hor comps.) $= 0$). Here p_1 denotes the uniform pressure (per unit of area), due to the uniform thrust, and p_2 the pressure or tension (per unit of area), in the elastic forces constituting the stress-couple, on the outermost element of area, at a distance *e* from the gravity axis ($\perp$ to plane of flexure) of the section. *F* is the total area of the section. *I* is the moment of inertia about the said gravity axis, *g*

$$\Sigma \text{ (vert. comps.)} = 0 \text{ gives } P = p_1F \quad . \quad . \quad (1)$$

$$\Sigma \text{ (moms.}_g) \quad = 0 \text{ gives } Pa = \frac{p_2I}{e} \quad . \quad . \quad . \quad . \quad (2)$$

For any section, *n*, between *A* and *R*, we should evidently have the same p_1 as at *R*, but a smaller p_2, since $Py < Pa$ while *e*, *I*, and *F*, do not change, the column being prismatic. Hence the max. (p_1+p_2) is on the concave edge at *R* and for safety should be no more than $C \div n$, where *C* is the Modulus of Crushing (§ 201) and *n* is a "factor of safety." Solving (1) and (2) for p_1 and p_2, and putting their sum $= C \div n$ we have

$$\frac{P}{F} + \frac{Pae}{I} = \frac{C}{n} \quad . \quad . \quad . \quad . \quad (3)$$

We might now solve for *P* and call it the safe load, but it is customary to present the formula in a form for giving the *breaking load*, the factor of safety being applied afterward. Hence, we shall make $n=1$, and solve for P, calling it then the breaking load. Now the deflection a is unknown, but may be expressed approximately, as follows, in terms of e and l.

If we now consider ARB to be a circular arc, of radius $=\rho$, we have from geometry (similar triangles) $a=(l\div 4)^2\div 2\rho$; and if we equate the two expressions for the moment of the stress-couple at R there results $\dfrac{EI}{\rho}=\dfrac{p_2 I}{e}$ (see pp. 249 and 250). A combination of these two relations gives $ae=(p_2\div 32E)l^2$. Now under a safe load the total stress, p_1+p_2, in the outer fibre (concave side) at R will have reached a safe value, R', for the material, and is therefore constant for this material, and if the rude assumption is made that the portion p_2 of this stress is also constant, it follows that the fraction $(p_2\div 32E)$ $=$ a constant; which may be denoted by β, (an abstract number). Let us also write, for convenience, $I=Fk^2$, (k being the radius of gyration of the cross-section about a (gravity) axis through g ⅂ to paper). Hence finally, we have, from eq. (3),

$$\left.\begin{array}{l}\text{Breaking load}\\ \text{for flat ends}\end{array}\right\}=P_1=\frac{FC}{1+\beta(l\div k)^2}. \qquad (4)$$

By the same reasoning as in § 303, for a round-ended column we substitute $2l$ for l; for a column with one end round and the other "flat" or "fixed" (i.e., for a "pin-and-square" column), $\frac{4}{3}l$ for l; and obtain

$$\left.\begin{array}{l}\text{Breaking load for a round-}\\ \text{ended column}\end{array}\right\}=P_0=\frac{FC}{1+4\beta(l\div k)^2}, \qquad (5)$$

$$\left.\begin{array}{l}\text{Breaking load for a ``pin-}\\ \text{and-square'' column}\end{array}\right\}=P_2=\frac{FC}{1+1.78\beta(l\div k)^2}. \qquad (6)$$

Each of these equations (4), (5), and (6), is known as *Rankine's Formula*, for the respective end-conditions mentioned. They find a very extended use among engineers in English-speaking countries; with some variation, however, in the numerical values used for quantities C and β, which are constants for a given material; and also in the fraction of the breaking load which should be taken as the safe, or working, load (the reciprocal of this fraction being called the "factor of safety,") $=n$. A set of fair average values for these constants, as recommended by Rankine and others, is here presented:

	Hard Steel.	Medium Steel.	Soft Steel.	Wrought Iron.	Cast Iron.	Timber.
C (lbs./in.^{2})	70,000	50,000	45,000	36,000	70,000	7,200
β (abstract number)	$\frac{1}{25,000}$	$\frac{1}{36,000}$	$\frac{1}{36,000}$	$\frac{1}{36,000}$	$\frac{1}{6,400}$	$\frac{1}{3,000}$

The factor of safety, n, usually employed with the foregoing formulæ and constants, is $n=4$ for wrought iron and steel in quiescent structures; and 5 under moving loads, as in bridges; while $n=10$ should be used for timber and 8 for cast iron.

In Fig. 315*b* two curves have been plotted, for round ends (R_r) and fixed ends (R_f) in the case of *medium steel;* the above equations (Rankine), with the above values of C and β, having been used. The "slenderness ratio," $l \div k$, is the abscissa; and $P_0 \div F$, or $P_1 \div F$, (the average *breaking* unit-stress), is the ordinate, of any point. These curves may now be compared with the Euler curves, E_r and E_f, (in the same figure) already mentioned as having been plotted for structural steel (of modulus of elasticity $E=30,000,000$ lbs./in.^{2})

306. Examples; under the Rankine Formulæ.—EXAMPLE 1. Let it be required to compute the breaking load of a wrought-iron solid cylinder, used as a column, of length $l=8$ ft. and diameter, $=d$, $=2.4$ inches; with *round ends,* i.e., the pressure acting at each end at the center of the circular base, the ends being free to turn in any direction.

The "end conditions" call for the employment of the "least k," but here k is the same for any gravity axis of the circular section. That is we have

$$k^2 = I \div F = \tfrac{1}{4}\pi r^4 \div \pi r^2 = \tfrac{1}{4}r^2 = \tfrac{1}{4}(1.2)^2 = 0.36 \text{ in.}^2; \quad \therefore\ k = 0.6 \text{ in.}$$

and $(l \div k)$ = "slenderness-ratio" $= 96 \div 0.6 = 160$. Hence from eq. (5)

$$P_0 = \frac{\pi r^2 C}{1+4\beta(160)^2}; \text{ with } \beta = \frac{1}{36{,}000} \text{ and } C = 36{,}000 \text{ lbs./in.}^2; \text{ i.e.,}$$

$$P_0 = \frac{\pi(1.2)^2 36{,}000}{1+2.85} = \frac{162{,}800}{3.85} = 42{,}300 \text{ lbs.}$$

It is seen that, on account of the degree of slenderness of the column, the breaking load is about one quarter of what it would be for a short prism of same section.

With a factor of safety of 5 we should take $\frac{1}{5}$ of 42,300, i.e., 8460 lbs., as safe load.

EXAMPLE 2.—It is required to compute the diameter, d, of a solid cast-iron cylinder, 16 ft. in length, to serve as a column with *flat ends*, whose safe load is to be 6 tons, the factor of safety being 6. This calls for the use of eq. (4) in which we put $P_1 = 6 \times 12{,}000 = 72{,}000$ lbs., the required breaking load; with $C = 70{,}000$ lbs./in.^{2} and $\beta = 1 \div 6400$. The least radius of gryation should be used, but in this case the k^2 is constant for all axes of the section, viz., $k^2 = \frac{1}{4}\pi r^4 \div \pi r^2 = d^2 \div 16$.

Hence from eq. (4) we have (for inch and pound)

$$P_1 = \frac{\frac{1}{4}\pi d^2 C}{1+\beta(l \div k)^2} = \frac{54{,}980 d^2}{1+\frac{1}{6400}[192 \div (\frac{1}{4}d)]^2} = 72{,}000 \text{ lbs.}$$

This on reduction leads to the bi-quadratic equation

$$d^4 - 1.309 d^2 = 120.7;$$

which being solved for d^2 gives $d^2 = 0.654 \pm 11.01$. The upper sign being taken we have, finally, $d = 3.41$ in. as the required diameter.

The "slenderness ratio," therefore, proves to be $192 \div 0.85 = 225$, which though seemingly high is not extreme for a flat-ended column; corresponding, as it does, to 112 for a round-ended column.

EXAMPLE 3.—A prism of medium steel, of uniform rectangular section (solid) with dimensions $b = 3$ in. and $h = 1$ in., is to be subjected to a thrust (connecting-rod of a steam-engine). Its ends are provided with pins (see Fig. 312) capable of turning in firm bearings, the axis of each pin being ⅂ to the "b" dimension of the rectangular section. The length between axes of pins, is $l=6$ ft. It is required to find the breaking load by the Rankine formulæ.

Since the end conditions would be "round-ends" if the axis of the column were to bend in a plane ┐ to the axes of the pins (as in Fig. 312), but "flat-ends" [Fig. 311(b)] in case it bent in the plane containing the axes of the pins; and since the k of the section is different for the two cases, it will be necessary to make each supposition in turn and take the smaller of the two results for breaking load (i.e., as the one to which the factor of safety should be applied).

For round-ended buckling the value of k^2 is $I \div F = [hb^3 \div 12] \div bh = 0.75$ in.^{2}; and, with the values of C and β for medium steel, we have from eq. (5),

$$P_0 = \frac{50{,}000 \times 3.0}{1 + \frac{4}{36{,}000} \cdot \frac{(72)^2}{0.75}} = \frac{150{,}000}{1+0.768} = 84{,}800 \text{ lbs.};$$

while for flat-ended buckling, in the other plane, the k^2 to be used would be $k^2 = [bh^3 \div 12] \div bh = 0.0833$ in.^{2}, and hence from eq. (4)

$$P_1 = \frac{50{,}000 \times 3.0}{1 + \frac{1}{36{,}000} \cdot \frac{(72)^2}{0.0833}} = \frac{150{,}000}{1+1.73} = 54{,}933 \text{ lbs.}$$

It is seen that P_1 is smaller than P_0, so that with a factor of safety of 6 we have for the safe, or working, load, $\frac{1}{6}$ of 54,933, $=9{,}155$ lbs.

307. Radii of Gyration.—The following table, taken from p. 523 of Rankine's Civil Engineering, gives values of k^2, the square of the *least* radius of gyration of the given cross-section about a gravity-axis. By giving the *least* value of k^2 it is implied that the plane of flexure is not determined by the end-conditions of the column (i. e., it is implied that the column has either flat ends or round ends). If either end (or both) is a *pin-joint* the column may need to be treated as having a flat-end as regards flexure in a plane containing the axis of the column and the axis of the pin, if the bearings of the pin are firm; while as regards flexure in a plane perpendicular to the pin it is to be considered round-ended at that extremity.

In the case of a "thin cell" the value of k^2 is strictly true for metal infinitely thin and of *uniform thickness*; still, if that thickness does not exceed $\frac{1}{8}$ of the exterior diameter, the form given is sufficiently near for practical purposes; similar statements apply to the branching forms

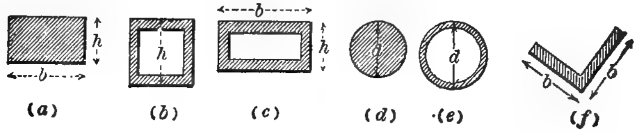

Fig. 317.

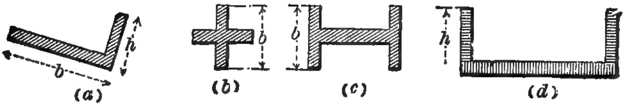

Fig. 318.

Solid Rectangle. $h=$ least side.	Fig. 317 (*a*).	$k^2=\frac{1}{12}h^2$
Thin Square Cell. Side$=h$.	Fig. 317 (*b*).	$k^2=\frac{1}{6}h^2$
Thin Rectangular Cell. $h=$ least side.	Fig. 317 (*c*).	$k^2=\frac{h^2}{12}\cdot\frac{h+3b}{h+b}$
Solid Circular Section. Diameter $=d$.	Fig. 317 (*d*).	$k^2=\frac{1}{16}d^2$
Thin Circular Cell. Exterior diam. $=d$.	Fig. 317 (*e*).	$k^2 \quad \frac{1}{8}d^2$
Angle-Iron of Equal ribs	Fig. 317 (*f*)	$k^2=\frac{1}{24}\cdot b^2$
Angle-Iron of unequal ribs.	Fig. 318 (*a*).	$k^2=\frac{b^2h^2}{12(b^2+h^2)}$
Cross of equal arms.	Fig. 318 (*b*).	$k^2=\frac{1}{24}b^2$
I-Beam as a pillar. Let area of web $=B$. " " " *both* flanges $=A$.	Fig. 318 (*c*).	$k^2=\frac{b^2}{12}\cdot\frac{A}{A+B}$
Channel Iron.	Fig. 318 (*d*).	$k^2=h^2\left[\frac{A}{12(A+B)}+\frac{AB}{4(A+B)^2}\right]$

Let area of web $=B$; of flanges $=A$ (both). h extends from edge of flange to middle of web.

308. Built Columns.—The "compression members" of bridge trusses, and columns in steel framework buildings are generally composed of several pieces of structural steel riveted together, each column being thus formed of a combination of plates, channels, angles, Z-bars, etc. In Figs. 319 and 320 are shown examples of these compound shapes. The Phœnix column is seen to consist of four quadrantal segments riveted together. In Fig. 319 is a combination of two channels and one plate, these three pieces being continuous along the whole length of the column. On the side opposite to the plate are seen lattice bars, arranged in zig-zag, which serve to stiffen the column on that side. The center of gravity of the cross-section of this column is nearer to the edge carrying the plate than to the lattice edge; and if the ends of the column are provided with pins ⟂ to the webs of the channels the axis of each of these pins should be so placed as to contain the center of gravity of the cross-section of the column at that point.

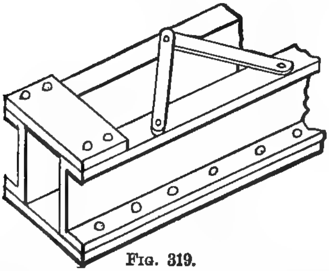
FIG. 319.

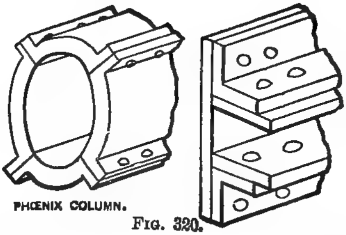

FIG. 320.

The handbooks of the various steel companies present formulæ and tables enabling the breaking loads to be found for their various designs of built columns, and for single I-beams used as columns. For example, the tables given in the handbook of the Cambria Steel Co. for built columns of "medium steel" are stated to be computed from the following formulæ (which are evidently of the Rankine type).

The breaking load for a column of length l and with cross-section of area F and least radius of gyration k is (in pounds):

Square Bearing. Pin and Square Bearing. Pin Bearings.

$$P_1=\frac{50{,}000F}{1+\frac{1}{36{,}000}\cdot\left(\frac{l}{k}\right)^2};\quad P_2=\frac{50{,}000F}{1+\frac{1}{24{,}000}\cdot\left(\frac{l}{k}\right)^2};\quad P_0=\frac{50{,}000F}{1+\frac{1}{18{,}000}\left(\frac{l}{k}\right)^2}.$$

In these formulæ l and k should be in the same unit (both feet, or both inches; since $(l \div k)$ is a ratio) and the proper k to be used for the case of "pin and square bearing" (i.e., one end provided with a pin and the other with a square bearing) should be ascertained as in example 3, p. 371. To obtain the total safe load for the column: "For quiescent loads, as in buildings, divide by 4. For moving loads, as in bridges, divide by 5."

Considerable variety will be found among the formulæ of the Rankine type proposed by different engineers as best satisfying the results of experiment. For accounts of experiments beyond those already quoted in the author's "Notes and Examples in Mechanics," the reader is referred to special works. Kent's Pocket Book for Mechanical Engineers contains much valuable matter on the subject of columns. The handbooks of the Carnegie Steel Co., the Pencoyd Iron Works, and the Phœnix Iron Co., give extensive data relating to steel columns. Osborne's Tables of moments of inertia and radii of gyration of compound sections is a valuable book in this connection.

309. Moment of Inertia of Built Column. Example.—It is proposed to form a column by joining two I-beams by lattice-work, Fig. 321, (*a*). (While the lattice-work is relied upon to cause the beams to act together as one piece, it is not regarded in estimating the area F, or the moment of inertia, of the cross section). It is also required to find the proper distance apart $= x$, Fig. 321, at which these beams must be placed, from centre to centre of webs, that the liability to flexure shall be equal in all axial planes, i.e. that the I of the compound section shall be the same about all gravity axes. This condition will be fulfilled if I_Y can be made $= I_X$* (§ 89), O being the centre of gravity of the compound section, and X perpendicular to the parallel webs of the two equal I-beams.

Let F' = the sectional area of one of the I-beams, I'_Y (see Fig. 321(a) its moment of inertia about its web-axis, that about an axis $\perp$ to web. (These quantities can be found in the hand-book of the iron company, for each size of rolled beam).

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* That is, with flat ends or ball ends; but with pin ends, Fig. 312, if the pin is $\parallel$ to X, put $4I_Y = I_X$; if $\parallel$ to Y, put $4I_X = I_Y$.

Then the

$$\text{total } I_X = 2I'_X\,; \text{ and total } I_Y = 2\left[I'_V + F'\left(\frac{x}{2}\right)^2\right]$$

(see §88 eq. 4.) If these are to be equal, we write them so and solve for x, obtaining

$$x = \sqrt{\frac{4[I'_X - I'_V]}{F'}} \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

310. **Numerically**; suppose each girder to be a 10½ inch light I-beam, 105 lbs. per yard, of the N. J. Steel and Iron Co., in whose hand-book we find that for this beam $I'_X =$ 185.6 biquad. inches, and $I'_V = 9.43$ biquad. inches, while $F' = 10.44$ sq. inches. With these values in eq. (1) we have

$$x = \sqrt{\frac{4\ (185.6 - 9.43)}{10.44}} = \sqrt{67.5} = 8.21 \text{ inches.}$$

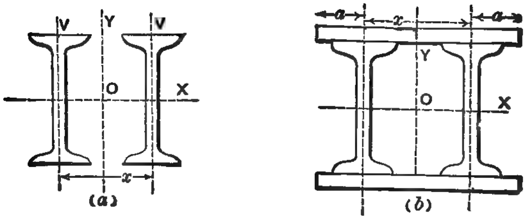

FIG. 321.

The square of the radius of gyration will be

$$k^2 = 2I'_X \div 2F' = 371.2 \div 20.88 = 17.7 \text{ sq. in.} \quad . \quad (2)$$

and is the *same for any gravity axis* (see § 89).

As an additional example, suppose the two I-beams united by plates instead of lattice. Let the thickness of the plate $= t$, Fig. 321, (*b*). Neglect the rivet-holes. The distance a is known from the hand-book. The student may derive a formula* for x, imposing the condition that (total I_X) $= I_Y$.

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* A *cubic* equation.

310a. Design of Columns.—General considerations governing economy and efficiency in the design of built columns are that the various pieces, besides being continuous for the whole length, should be placed as far from the axis of the column as possible, in order to increase the value of k the (least) radius of gyration, thus leading to a larger value of the safe load for a given amount of material, or to a minimum amount of material for a given required safe load; and that the parts should be well fastened together by rivets, preventing all relative motion. The economy secured by placing the material as far from the center as possible also holds, of course, for single pieces used as columns. For example, if the safe load of a hollow cylindrical cast-iron flat-ended column, 20 ft. long, is to be 40 tons, i.e., 80,000 lbs., and the thickness of metal is not to be less than $\frac{7}{8}$ in., we find, after a few trials with Rankine's formula eq. (4), p. 369, taking a factor of safety of 8 (so that the breaking load would be 640,000 lbs.) that an outside diameter of $d=8$ in. is the largest permissible. Thus, taking the least k^2, $(=d^2 \div 8)$, from p. 373, for a thin cylindrical cell, with $l=240$ in., with the sectional area, F, as the quantity to be solved for, we have

$$\frac{70{,}000F}{1+\frac{1}{6400}\cdot\frac{(240)^2}{[8^2\div 8]}}=640{,}000 \text{ lbs.}; \quad \therefore\ F=19.43 \text{ sq. in.}$$

Let d_2 denote the internal diameter of the section; then $\frac{\pi}{4}(8^2-d_2{}^2)=19.43$; whence $d_2=6.26$ in.; i.e., the thickness of metal $=\frac{1}{2}(d-d_2)=0.87$ in., or practically $\frac{7}{8}$ in.

310b. The Merriman-Ritter-Formula for Columns was derived independently by Professors Merriman and Ritter (see *Engineering News*, July 19, 1894) and has a mathematical basis as follows. In Fig. 315*b* curves have been plotted for the Euler and Rankine formulæ for medium steel, both for flat and round ends; and it is seen that each of the Rankine curves is tangent to the horizontal line through V and is roughly parallel to, and not very distant from, the corresponding Euler curve on the extreme right. Professor Merriman derives the equation (of the same form as Rankine's) for a curve which has a horizontal tangent at V, and is *exactly* tangent to the Euler curve at some point on the extreme right (at infinity, in fact) and thus secures a more rational value for the constant called β in Rankine's formula.

With P' denoting the *safe load* for the column and C' the *safe* compressive unit-stress for the material, this, 15,000 formula may be written . . .

$$P' = \frac{FC'}{1+\frac{nC''}{4\pi^2 E}\left(\frac{l}{k}\right)^2}, \quad . \quad . \quad . \quad . \quad (M)$$

30,000

where C'' denotes the unit compressive stress at *elastic limit*, E the modulus of elasticity, F the sectional area, and n an abstract number whose value (as before, in the Rankine formulæ) is 1, 16/9 (or 1.78), and 4, for flat ends, pin-and-square, and round ends, respectively.

If for P' we write P, the breaking load, and correspondingly C for C', and plot values of $P \div F$ and $l \div k$, the curve would not differ greatly from the Rankine curve in Fig. 315*b* for medium steel; and similarly for wrought iron; but for timber and cast iron the variation is considerable, and hence Prof. Merriman does not recommend the use of his formula for the latter two materials. (Crehore's formula differs from the above only in replacing C'' by C'.)

310c. The "Straight-Line Formula."—It will be noticed that in Fig. 315*b* the straight line connecting points A and C (medium steel, round ends) or A' and C' (medium steel, flat ends) would not vary widely from the Rankine curve, so that on account of its simplicity, when restricted to proper limiting values of the ratio $l \div k$, a straight line, or linear relation, between the quantity $P \div F$ and ratio $l \div k$ was proposed by Mr. T. H. Johnson (see Transac. Am. Soc. C. E., 1886, p. 530) for the breaking loads of columns of various materials. Among them are the following:

Wrought iron: Hinged ends, $P_0 = \left[42{,}000 - 157\left(\frac{l}{k}\right)\right]F$;

" " Flat ends, $P_1 = \left[42{,}000 - 128\left(\frac{l}{k}\right)\right]F$;

Mild steel: Hinged ends, $P_0 = \left[52{,}000 - 220\left(\frac{l}{k}\right)\right]F$;

" " Flat ends, $P_1 = \left[52{,}000 - 179\left(\frac{l}{k}\right)\right]F$.

In these formulæ P_0, or P_1, is *breaking load* in lbs., $F=$ sectional area (in *sq. in.*), $l=$the length, and k is the least radius of gyration of the cross-section for flat ends (as for hinged ends, see example 3, § 306); l and k in same unit.

310d. The J. B. Johnson Parabolic Formula for Columns.—If in Fig. 315*b* a parabola be plotted with its axis vertical (and downward) and vertex at the point V of the two Rankine curves, and also made tangent to the Euler curve for the end conditions concerned, the points on such a curve for values of $l \div k$ between zero and the point of tangency to the Euler curve are found to agree fairly well with experiment; and the corresponding formula, or the equation to the curve, is of much simpler form than that of the Rankine types, being almost as simple as the straight line formula. Such a formula was proposed by the late Prof. J. B. Johnson, those for mild steel and wrought iron being given below (breaking load in lbs.).

Mild steel:

$$\text{Pin ends,}\quad P_0=\left[42{,}000-0.97\left(\frac{l}{k}\right)^2\right]F;\ \left(\frac{l}{k}\ \text{not} >150\right)$$

$$\text{Flat ends,}\ P_1=\left[42{,}000-0.62\left(\frac{l}{k}\right)^2\right]F;\ \left(\frac{l}{k}\ \text{not} >190\right)$$

Wrought iron:

$$\text{Pin ends,}\quad P_0=\left[34{,}000-0.67\left(\frac{l}{k}\right)^2\right]F;\ \left(\frac{l}{k}\ \text{not} >170\right)$$

$$\text{Flat ends,}\ P_1=\left[34{,}000-0.43\left(\frac{l}{k}\right)^2\right]F;\ \left(\frac{l}{k}\ \text{not} >210\right)$$

The notation is the same as in the preceding article. The limiting values mentioned for $l \div k$ refer to the points of tangency with the Euler curve. In Fig. 315*b* the curve VWX is a parabola fulfilling the above mathematical condition for medium steel, with flat ends.

311. Solid Wooden Columns and Posts. Formula of U. S. Dept. of Agriculture, Division of Forestry.—This formula was derived by Johnson from the results of experiments conducted by the Division of Forestry and applies to solid wooden columns provided with "*square ends*," the constraint due to which, however, is not to be considered as fully equivalent to that of "fixed ends." The breaking load being denoted by P_1, the sectional area by F, the ratio of length l to the *"least dimension,"* d, of the cross section, by m (i.e., $l \div d = m$), and the unit crushing stress for the material by C, the formula is

$$P_1 = \frac{(700+15m)FC}{700+15m+m^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (F)$$

The values of C to be used for different kinds of timber are given as follows:

White oak and Georgia yellow pine	5000 lbs./in.^{2}
Douglas fir and short-leaf yellow pine	4500 "
Red pine, spruce, hemlock, cypress, chestnut, California redwood, and California spruce.	4000 "
White pine and cedar	3500 "

The fraction of P_1 to be taken as the safe load depends on the wood and the degree of moisture present, four classes being designated in this respect; from Class A (18 per cent of moisture; timber exposed to weather), to Class D (10 per cent; timber at all times protected from the weather). For yellow pine the safe load should be from $0.20P_1$ for Class A to $0.31P_1$ for Class D. For all other timbers, from $0.20P_1$ for Class A to $0.25P_1$ for Class D.

312. Column under Eccentric Loading.—In Fig. 322 let the load P be applied at i, at a distance or "eccentricity" $=c$ from the center of gravity A of the upper base of the column, the reaction at the other end (at k) having an equal eccentricity from B; the ends of the column being free to turn. (In an extreme case Ai and Bk might be brackets fastened to the ends of the column.)

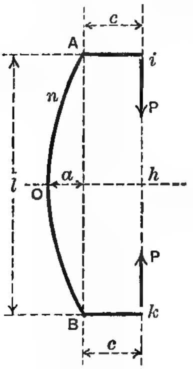


FIG. 322.

AOB is the elastic curve, or bent condition of the axis of the column, originally straight. With O as origin, any point n in the elastic curve has a vertical co-ordinate x and a horizontal co-ordinate y. The unknown lateral deflection of the point O from AB is a. With n as *any* point in the elastic curve, and nAi as free body, we have for the moment of the stress couple in section at n $EI[d^2y \div dx^2] = P(c+a-y)$; which is seen to differ from eq. (1) of p. 362 only in having the constant $c+a$ in place of the constant a. We may therefore use eq. (6) of p. 363 for the present case, after replacing a by $c+a$; and hence, denoting $\sqrt{P \div EI}$ by b, remembering that vers. sin. $=1-\cos$, we may write, as the equation to the elastic curve,

$$y = (c+a)[1-\cos(bx)] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For $x = \frac{1}{2}l$, y should $=$ the deflection a; on substituting which values in (1) there results finally

$$a=c\left[\sec\left(\frac{bl}{2}\right)-1\right]\quad .\ .\ (2);\qquad\text{and}\qquad c+a=c\cdot\sec\left(\frac{bl}{2}\right).\quad .\ .\ .\ (3)$$

Hence the moment of the stress couple at O is $M_0=P(c+a)=Pc\cdot\sec\left(\frac{bl}{2}\right)$ and the unit stress in outer fibre on concave side at O is

$$p=\frac{P}{F}+\frac{M_0e}{I}=\frac{P}{F}+\frac{Pec\,\sec(\frac{1}{2}bl)}{I}.\quad .\ .\ .\ .\ .\ .\ .\ (4)$$

(In this case of eccentric loading, then, the deflection a is *not* indeterminate as was the case in deriving Euler's formula on p. 363. Note that $\frac{1}{2}bl$ is an angle in *radians.*)

Example.—Let the value of P be 10,000 lbs., the length of the column be $l=20$ ft. $=240$ in., and the cross-section be a square cell [see Fig. 317 (*b*)] 4 inches being the side of the outer square; area $F=7$ in.^{2} and $I=14.58$ in.^{4} Let the eccentricity be $c=2$ in., each force P being applied in the middle of a side of the 4 in. square. Let $E=30{,}000{,}000$ lbs./in.^{2}; material, medium steel. With this position of the force plane, $e=2$ in.

Here we have $\frac{1}{2}bl=\frac{1}{2}\left(\sqrt{\frac{10{,}000}{30{,}000{,}000\times 14.58}}\right)\times 240=0.5736$ radians, corresponding to 32° 52′, whose sec. $=1.190$; and therefore $a=2\times(1.190-1)$ $=0.380$ in., and $M_0=10{,}000\times 2\times 1.190=23{,}800$ in.-lbs. Finally

$$p=\frac{10{,}000}{7}+\frac{23{,}800\times 2}{14.58}=1430+3265=4695\text{ lbs./in.}^2$$

With $P=20{,}000$ lbs., we should obtain $\frac{1}{2}bl=0.811$ radians (46° 30′), $a=0.906$ in., $M_0=57{,}120$ in.-lbs., and $p=2860+7835=10{,}695$ lbs./in.^{2}

This latter unit stress is seen to be only moderate in value for the material, leading to the conclusion that 20,000 lbs. for P is a safe load; but on account of the possible original lack of straightness in the column, and of lack of homogeneity, both of which causes might increase a and M_0, it would be better to limit the load to 15,000 lbs.; considering, also, the fact that Rankine's Formula for round ends (with a safety factor of 4) applied to this column for the case of *no* eccentricity would give about 22,000 lbs. as safe load.

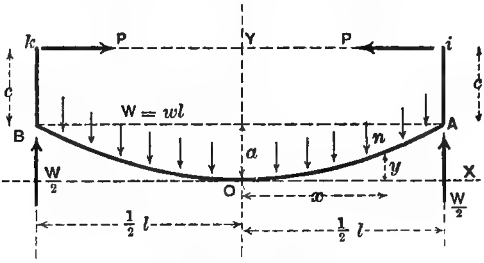


FIG. 322*a*.

313. Beam or Column with Eccentric End Pressures and also under Uniform Transverse Loading.—For example, in Fig. 322*a* let AB be the bent axis of a beam, or column (originally straight), the longitudinal forces P and P being applied at an eccentricity c from A and B, while there is at the same time a vertical loading W, $=wl$, uniformly distributed along the whole length at rate of w lbs. per running inch. The reactions of the two *end supports* will therefore be each $\frac{1}{2}W$. The ends of the column are free to turn. It is required to find the deflection a, the moment M_0 of the couple at middle section O, and the unit stress p on the concave side at O. Take the free body iAn, n being *any point* of the elastic curve AOB, with coordinates x and y referred to the horizontal and vertical axes through O, as an origin, as shown. Then the moment of stress couple at n is

$$EI(d^2y \div dx^2)=P(c+a-y)+(\tfrac{1}{8})w(l^2-4x^2) \quad . \quad . \quad . \quad . \quad (5)$$

Since $(d^2y \div dx^2)$ is a variable, let us denote $(-EI \div P)(d^2y \div dx^2)$ by u, as an auxiliary variable; and eq. (5) will now read

$$y-u=c+a+[(\tfrac{1}{8})w(l^2-4x^2)] \div P \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Differentiating (6) twice, with respect to x, we have

$$\frac{d^2y}{dx^2}-\frac{d^2u}{dx^2}=-\frac{w}{P}; \quad \text{that is,} \quad \frac{d^2u}{dx^2}=-\frac{P}{EI}u+\frac{w}{P}. \quad . \quad . \quad . \quad . \quad (7)$$

Multiplying (7) by $2du$, and denoting $P \div EI$ by b^2 and $2w \div P$ by h, we have by integration, (dx^2 is a constant, x being the independent variable), $(du)^2 \div (dx)^2=-b^2u^2+hu+C$, where C is a constant of integration; and hence $dx=du \div (\sqrt{C+hu-b^2u^2})$, which integrates into

$$x=\frac{1}{b}\cdot\sin^{-1}\left(\frac{2b^2u-h}{\sqrt{h^2+4Cb^2}}\right)+C', \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where C' is a constant. Transformation of (8) gives

$$(\sqrt{h^2+4Cb^2})\sin[b(x-C')]+h=2b^2u \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eliminating u by aid of eqs. (6) and (9) we have

$$2b^2y=\sqrt{h^2+4Cb^2}\cdot\sin[b(x-C')]+h+2b^2(c+a)+(\tfrac{1}{8})b^2h(l^2-4x^2) \quad (10)$$

from which

$$2b^2(dy/dx)=b\sqrt{h^2+4Cb^2}\cdot\cos[b(x-C')]-b^2hx \quad . \quad . \quad . \quad (11)$$

To determine the three constants C, C', and a, we now make use of the facts that in (10) when $x=0$, y also $=0$, and for $x=\frac{1}{2}l$, $y=a$; and that in (11) for $x=0$, dy/dx, must $=0$. The three equations thus obtained, containing constants only, enable us to determine C, C', and a, and insert their values in (10); thus giving us as *the equation to the elastic curve AOB*,

$$y=\left(c+\frac{h}{2b^2}\right)\left[\frac{1-\cos(bx)}{\cos(\frac{1}{2}bl)}\right]-\tfrac{1}{4}hx^2; \quad . \quad . \quad . \quad . \quad . \quad (12)$$

as also the value of the deflection

$$a=[c+(h \div 2b^2)][\sec(\tfrac{1}{2}bl)-1]-(\tfrac{1}{16})hl^2 \quad . \quad . \quad . \quad . \quad (13)$$

To find the moment of stress couple, M_0, at O, we have now only to substitute $x=0$ and $y=0$ in eq. (5), and for a its value from (13); and thus obtain

$$M_0=P\left[\left(c+\frac{h}{2b^2}\right)\cdot\sec(\tfrac{1}{2}bl)-\frac{h}{2b^2}\right], \quad . \quad . \quad . \quad . \quad . \quad (14)$$

With F as the sectional area of the cross-section of the (prismatic) column (or beam, as it might also be called in this connection), and e as the distance of the outer fibre from the gravity axis of the section, we now have for p, the stress in outer fibre on concave side at O,

$$p=\frac{P}{F}+\frac{M_0e}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Since b and h denote $\sqrt{P \div EI}$ and $2w \div P$, respectively, it is seen that when w is zero, h is zero and eq. (13) reduces to eq. (2) of the previous article. Again, if the two forces P are central, i.e., applied at A and B, we put $c=0$; in which case an approximate result may be reached by writing for the deflection a the value it would have if the end forces P were not present, i.e., $\frac{5}{384}\cdot\frac{Wl^3}{EI}$, as due to the uniform load W alone (see p. 260). On this basis the value of M_0 is $Pa+(\frac{1}{8})Wl$.

(In case the vertical load on the beam or column in Fig. 322*a* is a single load Q concentrated in the middle at O, a treatment similar to the foregoing may be applied, but is somewhat more complicated. For details of such a case the reader is referred to *Mécanique Appliquée*, by Bresse, Tome I. p. 384.)

314. Buckling of Web-Plates in Built Girders.—In §257 mention was made of the fact that very high web plates in built beams, such as *I* beams and box-girders, might need to be stiffened by riveting "angles" on the sides of the web. (The girders here spoken of are horizontal ones, such as might be used for carrying a railroad over a short span of 20 to 50 feet.

An approximate method of determining whether such stiffening is needed to prevent lateral buckling of the web may be based upon Rankine's formula for a long column and will now be given.

In Fig. 323 we have, free, a portion of a bent I-beam, between two vertical sections at a distance apart $= h_1 =$ the height of the web. In such a beam under forces ⊥ to its axis it has been proved (§256) that we may consider the web to sustain all the shear, J, at any section, and the flanges to take all the tension and compression, which form the "*stress-couple*" of the section. These couples and the two shears are shown in Fig. 323, for the two exposed sections. There is supposed to be no load on this portion of the beam, hence the shears at the two ends are

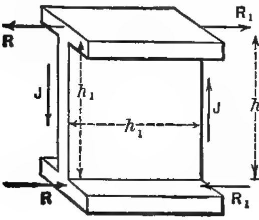


FIG. 323.

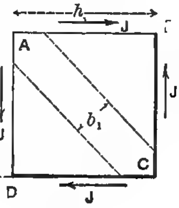


FIG. 324.

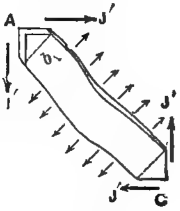


FIG. 325.

equal. Now the shear acting between each flange and the horizontal edge of the web is equal in intensity per square inch to that in the vertical edge of the web; hence if the web alone, of Fig. 323, is shown as a free body in Fig. 324, we must insert two horizontal forces $= J$, in opposite directions, on its upper and lower edges. Each of these $= J$ since we have taken a horizontal length $h_1 =$ height of web. In this figure, 324, we notice that the effect of the acting forces is to lengthen the diagonal BD and shorten the diagonal AC, both of those diagonals making an angle of 45° with the horizontal.

Let us now consider this buckling tendency along AC, by treating as free the strip AC, of small width $= b_1$. This is shown in Fig. 325. The only forces acting in the direction of its length AC are the components along AC of the four forces J' at the extremities. We may therefore treat the strip as a long column of a length $l = h_1 \sqrt{2}$, of a sectional area $F = bb_1$, (where b is the thickness of the web plate), with a value of $k^2 = {}^1/_{12}\, b^2$ (see § 307), and with fixed (or flat) ends. Now the sum of the longitudinal components of the two J'.'s at A is $Q = 2\ J'\ \frac{1}{2}\ \sqrt{2}$ $= J'\ \sqrt{2}$; but J' itself $= \frac{J}{bh_1} \cdot b\ \frac{1}{2}\ b_1 \sqrt{2}$. since the small rectangle on which J' acts has an area $= b\ \frac{1}{2}\ b_1 \sqrt{2}$, and the shearing stress on it has an intensity of $(J \div bh_1)$ per unit of area. Hence the longitudinal force at each end of this long column is

$$Q = \frac{b_1}{h_1} J \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

According to eq. (4) and the table in § 305, the *safe load* (factor of safety $=4$) for a *medium steel* column of this form, with flat ends, would be (pound and inch)

$$P_1 = \frac{\frac{1}{4} bb_1 50{,}000}{1 + \frac{1}{36{,}000} \cdot \frac{2h_1^2}{{}^1/_{12} b^2}} = \frac{12{,}500 bb_1}{1 + \frac{1}{1{,}500} \cdot \frac{h_1^2}{b^2}} \quad . \quad . \quad (2)$$

If, then, in any particular locality of the girder (of *medium steel*) we find that Q is $> P_1$, i.e.

$$\text{if } \frac{J}{h_1} \text{ is } > \frac{12{,}500b}{1+\frac{1}{1{,}500}\cdot\frac{h_1^2}{b^2}} \text{ (pound and inch).} \quad . \quad (3)$$

then vertical stiffeners will be required laterally.

When these are required, they are generally placed at intervals equal to h_1, (the depth of web), along that part of the girder where Q is $>P_1$.

Example Fig. 326.—Will stiffening pieces be required in a plate girder of 20 feet span, bearing a uniform load of 40 tons, and having a web 24 in. deep and $\frac{3}{8}$ in. thick?

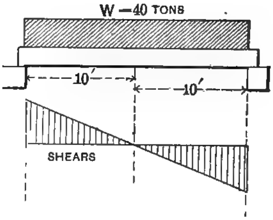


Fig. 326.

From § 242 we know that the greatest shear, J max., is close to either pier, and hence we investigate that part of the girder first.

J max. $=\frac{1}{2}W=20$ tons $=40{,}000$ lbs.

∴ (inch and lb.), see (3),

$$\frac{J}{h_1}=\frac{40{,}000}{24}=1666.6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

while, see (3), (inch and pound),

$$\frac{12{,}500\times\frac{3}{8}}{1+\frac{1}{1{,}500}\cdot\frac{24^2}{(\frac{3}{8})^2}}=1270 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is less than 1666.66.

Hence stiffening pieces will be needed near the extremities of the girder. Also, since the shear for this case of loading diminishes uniformly toward zero at the middle they will be needed from each end up to a distance of $\frac{1270}{1666}$ of 10 ft. from the middle.

CHAPTER VII.

LINEAR ARCHES (OF BLOCKWORK).

315. A Blockwork Arch is a structure, spanning an opening or gap, depending, for stability, upon the resistance to *compresssion* of its blocks, or *voussoirs*, the material of which, such as stone or brick, is not suitable for sustaining a tensile strain. Above the voussoirs is usually placed a load of some character, (e.q. a roadway,) whose pressure upon the voussoirs will *be considered as vertical*, only. This condition is not fully realized in practice, unless the load is of cut stone, with vertical and horizontal joints resting upon voussoirs of corresponding shape (see Fig. 327), but sufficiently so to warrant its assumption in theory. Symmetry of form about a vertical axis will also be assumed in the following treatment.

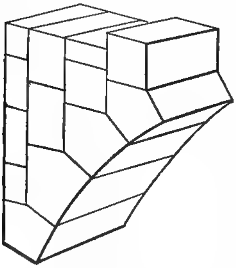


Fig. 327.

316. Linear Arches.—For purposes of theoretical discussion the voussoirs of Fig. 327 may be considered to become infinitely small and infinite in number, thus forming a "linear arch," while retaining the same shapes, their depth ┐ to the face being assumed constant that it may not appear in the formulae. The joints between them are ┐ to the curve of the arch, i.e., adjacent voussoirs can exert pressure on each other only in the direction of the tangent-line to that curve.

317. Inverted Catenary, or Linear Arch Sustaining its Own Weight Alone.—Suppose the infinitely small voussoirs to have weight, uniformly distributed along the curve, weighing q lbs. per running linear unit. The equilibrium of such a structure, Fig. 328, is of course unstable but theoretically possible. Required the form of the curve when equilibrium exists. The conditions of equilibrium are, obviously: 1st. The thrust or mutual pressure T between any two adjacent voussoirs at any point, A, of the curve must be tangent to the curve; and 2ndly, considering a portion BA as a free body, the resultant of H_0 the pressure at B the crown, and T at A, must balance R the resultant of the ‖ vertical forces (i.e., weights of the elementary voussoirs) acting between B and A.

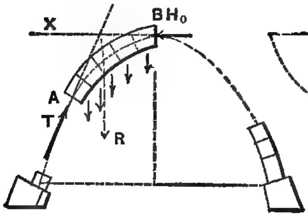


FIG. 328.

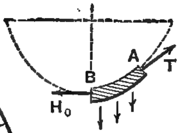


FIG. 329.

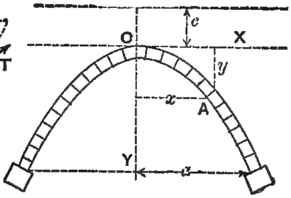


Fig. 330.

But the conditions of equilibrium of a flexible, inextensible and uniformly loaded cord or chain are the very same (weights uniform along the curve) the forces being reversed in direction. Fig. 329. Instead of compression we have tension, while the ‖ vertical forces act toward instead of away from, the axis X. Hence the curve of equilibrium of Fig. 328 is an inverted catenary (see § 48) whose equation is

$$y+c=\tfrac{1}{2}c\left[e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

See Fig. 330. $e = 2.71828$ the Naperian Base. The "parameter" c may be determined by putting $x = a$, the half span, and $y = OY$, the *rise*, then solving for c by successive approximations. The "*horizontal thrust*," or H_0, is $= qc$, while if s = length of arch OA, along the curve, the thrust T at any point A is

$$T = \sqrt{H_0^2 + q^2 s^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

From the foregoing it may be inferred that a series of voussoirs of *finite dimensions*, arranged so as to contain the catenary curve, with joints ┐ to that curve and of equal weights for equal lengths of arc will be in equilibrium, and moreover in *stable* equilibrium on account of friction, and the finite width of the joints; see Fig. 331.

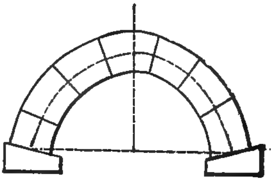


FIG. 331.

318. Linear Arches under Given Loading.—The linear arches to be considered further will be treated as without weight themselves but as bearing vertically pressing loads (each voussoir its own).

Problem.—Given the form of the linear arch itself, it is required to find the law of vertical depth of loading under which the given linear arch will be in equilibrium. Fig. 332, given the curve ABC, i.e., the linear arch itself, required the form of the curve MON, or upper limit of loading, such that the linear arch ABC shall be in equilibrium under the loads lying between the two curves. The loading is supposed homogeneous and of constant depth ┐ to paper; so that the ordinates z between the two curves are proportional to the *load per horizontal linear unit.* Assume a height of load z_0 at the crown, at pleasure; then required the z of any point m as a function of z_0 and the curve ABC.

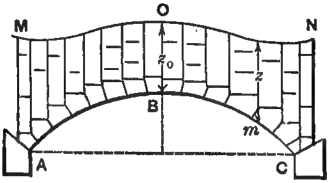


FIG. 332.

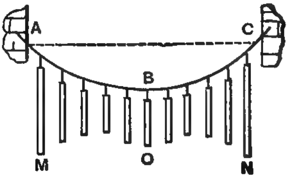


FIG. 333.

Practical Solution.—Since a linear arch under vertical pressures is nothing more than the inversion of the curve assumed by a cord loaded in the same way, this problem might be solved mechanically by experimenting with a light cord, Fig. 333, to which are hung other *heavy* cords, or bars of uniform weight per unit length, and at equal horizontal distances apart *when in equilibrium.* By varying the lengths of the bars, and their points of attachment, we may finally find the curve sought, *MON.* (See also § 343.)

Analytical Solution.—Consider the structure in Fig. 334. A number of rods of finite length, in the same plane, are in equilibrium, bearing the weights P, P_1, etc., at the connecting joints, each piece exerting a thrust T against the adjacent joint. The joint A, (the "pin" of the hinge), imagined separated from the contiguous rods and *hence free,* is held in equilibrium by the vertical force P (a load) and the two thrusts T and T', making angles $= \theta$ and θ' with the vertical; Fig. 335 shows the joint A free. From Σ(horizontal comps.)$=0$, we have

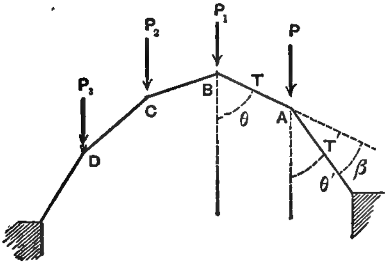


FIG. 334.

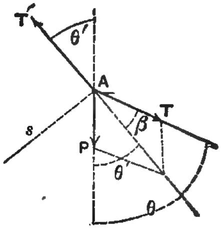


FIG. 335.

$$T \sin \theta = T' \sin \theta'.$$

That is, the horizontal component of the thrust in any rod is the same for all; call it H_0. ∴

$$T = \frac{H_0}{\sin \theta} \qquad (1)$$

Now draw a line As ⊥ to T' and write Σ (compons. ∥ to As) $=0$; whence $P\sin\theta' = T\sin\beta$, and [see (1)]

$$\therefore P = \frac{H_0 \sin\beta}{\sin\theta \sin\theta'} \qquad (2)$$

Let the rods of Fig. 334 become infinitely small and infinite in number and the load continuous. The length of each rod becomes $=ds$ an element of the linear arch. β is the angle between two consecutive ds's, θ is the angle between the tangent line and the vertical, while P becomes the load resting on a single dx, or horizontal distance between the middles of the two ds's. That is, Fig. 336, if $\gamma =$ weight of a cubic unit of the loading, $P=\gamma z dx$. (The lamina of arch and load considered is unity, ⊥ to paper, in thickness.) $H_0 =$ a constant $=$ thrust at crown O; $\theta=\theta'$, and $\sin\beta = ds \div \rho$, (since the angle between two consecutive tangents is $=$ that between two consecutive radii of curvature). Hence eq. (2) becomes

$$\gamma z dx = \frac{H_0 ds}{\rho\sin^2\theta}\text{ ; but } dx = ds\sin\theta,$$

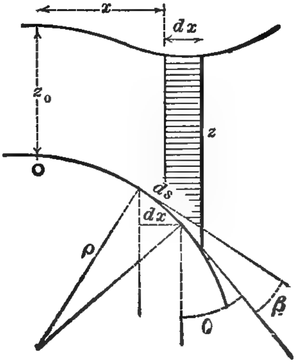


Fig. 336.

$$\therefore \gamma z = \frac{H_0}{\rho\sin^3\theta} \quad . \quad . \quad . \quad . \quad (3)$$

Call the radius of curvature at the crown ρ_0, and since there $z=z_0$ and $\theta=90°$, (3) gives $\gamma z_0\rho_0 = H_0$; hence (3) may be written

$$z = \frac{z_0\rho_0}{\rho\sin^3\theta} \quad . \quad . \quad . \quad . \quad (4)$$

This is the law of vertical depth of loading required. For a point of the linear arch where the tangent line is vertical, $\sin\theta=0$ and z would $=\infty$; i.e., the load would be *in-*

finitely high. Hence, in practice, a full semi-circle, for instance, could not be used as a linear arch.

319. Circular Arc as Linear Arch.—As an example of the preceding problem let us apply eq. (4) to a circular arc, Fig. 337, as a linear arch. Since for a circle ρ is constant $=r$, eq. (4) reduces to

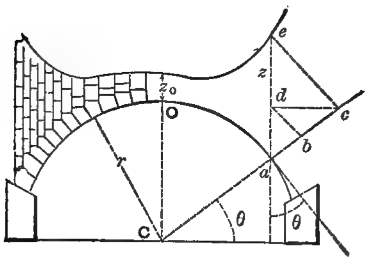


FIG. 337.

$$z=\frac{z_0}{\sin^3\theta} \quad . \quad (5)$$

Hence the depth of loading must vary inversely as the cube of the sine of the angle θ made by the tangent line (of the linear arch) with the vertical.

To find the depth z by **construction.**—Having z_0 given, C being the centre of the arch, prolong Ca and make $ab = z_0$; at b draw a ┐ to Cb, intersecting the vertical through a at some point d; draw the horizontal dc to meet Ca at some point c. Again, draw ce ┐ to Cc, meeting ad in e; then $ae = z$ required; a being any point of the linear arch. For, from the similar right triangles involved, we have

$$z_0=\overline{ab}=\overline{ad}\sin\theta=\overline{ac}\sin\theta.\sin\theta=\overline{ae}\sin\theta\sin\theta\sin\theta$$

$$\therefore \overline{ae}=\frac{z_0}{\sin^3\theta};\ \text{i.e., } \overline{ae}=z. \qquad Q.E.D. \quad [\text{see } (5.)]$$

320. Parabola as Linear Arch.—To apply eq. 4 § 318 to a parabola (axis vertical) as linear arch, we must find values of ρ and ρ_0 the radii of curvature at any point and the crown respectively. That is, in the general formula,

$$\rho=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} \div \frac{d^2y}{dx^2}.$$

we must substitute the forms for the first and second differential co-efficients, derived from the equation of the

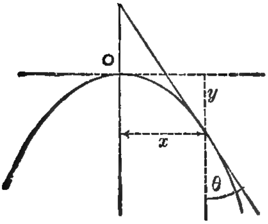


FIG. 338.

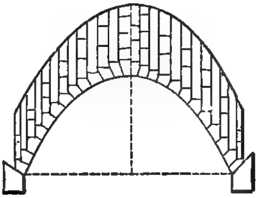


FIG. 339.

curve (parabola) in Fig. 338, i.e. from $x^2 = 2py$; whence we obtain

$$\frac{dy}{dx}, \text{or } \cot\theta, = \frac{x}{p} \text{ and } \frac{d^2y}{dx^2} = \frac{1}{p}$$

$$\text{Hence } \rho = \frac{(\sqrt{1+\cot^2\theta})^3}{1 \div p} = p \operatorname{cosec.}^3\theta, \text{ i.e. } \rho = \frac{p}{\sin^3\theta} \quad . \quad . \quad . \quad (6)$$

At the vertex $\theta = 90° \therefore \rho_0 = p$. Hence by substituting for ρ and ρ_0 in eq. (4) of § 318 we obtain

$z = z_0 =$ **constant** [Fig. 339] . . . . . . (7)

for a parabolic linear arch. Therefore the depth of homogeneous loading must be the same at all points as at the crown; i.e., the load is uniformly distributed with respect to the horizontal. This result might have been anticipated from the fact that a cord assumes the parabolic form when its load (as approximately true for suspension bridges) is uniformly distributed horizontally. See § 46 in Statics and Kinetics.

321. Linear Arch for a Given Upper Contour of Loading, the arch itself being the unknown lower contour. Given the upper curve or limit of load and the depth z_0 at crown, required the form of linear arch which will be in equilibrium under the homogenous load between itself and that upper curve. In Fig. 340 let MON be the given upper contour of load, z_0 is given or assumed, z' and z'' are the respective ordinates of the two curves BAC and MON. Required the equation of BAC.

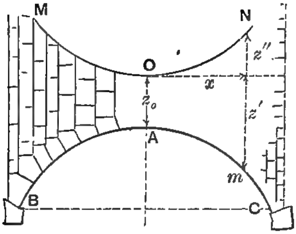


FIG. 340.

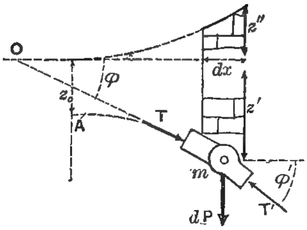


FIG. 341.

As before, the loading is homogenous, so that the weights of any portions of it are proportional to the corresponding areas between the curves. (Unity thickness ⅂ to paper.) Now, Fig. 341, regard two consecutive *ds's* of the linear arch as two links or consecutive blocks bearing at their junction m the load $dP = \gamma\,(z' + z'')\,dx$ in which γ denotes the heaviness, or weight of a cubic unit of the loading. If T and T' are the thrusts exerted on these two blocks by their neighbors (here supposed removed) we have the three forces dP, T and T', forming a system in equilibrium. Hence from $\Sigma X = 0$,

$$T \cos\varphi = T' \cos\varphi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\Sigma Y = 0 \text{ gives } T' \sin\varphi' - T \sin\varphi = dP \quad . \quad . \quad . \quad (2)$$

From (1) it appears that $T \cos\varphi$ is constant at all points of the linear arch (just as we found in § 318) and hence $=$ the thrust at the crown, $= H$, whence we may write

$$T = H \div \cos\varphi \text{ and } T' = H \div \cos\varphi' \quad . \quad . \quad . \quad (3)$$

Substituting from (3) in (2) we obtain

$$H\,(\tan\varphi' - \tan\varphi) = dP \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But $\tan\varphi = \dfrac{dz'}{dx}$ and $\tan\varphi' = \dfrac{dz' + d^2z'}{dx}$, ($dx$ constant) while $dP = \gamma\,(z' + z'')\,dx$. Hence, putting for convenience $H = \gamma a^2$, (where a = side of an imaginary *square* of the loading, whose thickness = unity and whose weight = H) we have.

$$\frac{d^2z'}{dx^2}=\frac{1}{a^2}(z'+z'') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

as a relation holding good for any point of the linear arch which is to be in equilibrium under the load included between itself and the given curve whose ordinates are z'', Fig. 340.

322. Example of Preceding. Upper Contour a Straight Line.—Fig. 342. Let the upper contour be a right line and horizontal; then the z'' of eq. 5 becomes zero at all points of ON. Hence drop the accent of z' in eq. (5) and we have

$$\frac{d^2z}{dx^2}=\frac{z}{a^2}$$

Multiplying which by dz we obtain

$$\frac{dz\,d^2z}{dx^2}=\frac{1}{a^2}zdz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This being true of the z, dz, d^2z and dx of each element of the curve $O'B$ whose equation is desired, conceive it written out for each element between O' and *any point m*, and put the sum of the left-hand members of these equations = to that of the right-hand members, remembering that a^2 and dx^2 are the same for each element. This gives

$$\frac{1}{dx^2}\int_{dz=0}^{dz=dz} dz\,d^2z=\frac{1}{a^2}\int_{z=z_0}^{z=z} zdz\,;\ \text{i.e.,}\ \frac{1}{dx^2}.\frac{dz^2}{2}=\frac{1}{a^2}\left[\frac{z^2}{2}-\frac{z_0^2}{2}\right]$$

$$\therefore\ dx=\frac{adz}{\sqrt{z^2-z_0^2}}=a\,.\,\frac{d\left(\frac{z}{z_0}\right)}{\sqrt{\left(\frac{z}{z_0}\right)^2-1}} \quad . \quad . \quad . \quad . \quad (7.)$$

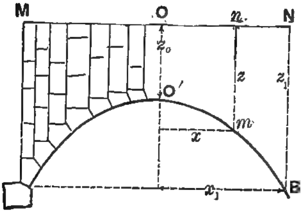


FIG. 342.

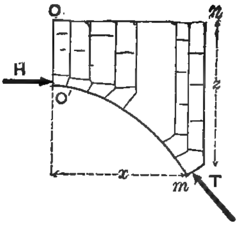


FIG 343

Integrating (7.) between O' and any point m

$$\Big[_0^x x = a\Big[_{z_0}^{z} \log._e \left(\frac{z}{z_0} + \sqrt{\left(\frac{z}{z_0}\right)^2 - 1}\right) \quad . \quad . \quad (8)'$$

$$\text{i.e., } x = a \log._e \left[\frac{z}{z_0} + \sqrt{\left(\frac{z}{z_0}\right)^2 - 1}\,\right]; \quad . \quad . \quad . \quad . \quad (8.)$$

$$\text{or } z = \frac{z_0}{2}\left[e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right] = z_0 \cosh\left(\frac{x}{a}\right); \text{ (see p. 48).} \quad . \quad . \quad (9.)$$

This curve is called the **transformed catenary** since we may obtain it from a common catenary by altering all the ordinates of the latter in a constant ratio, just as an ellipse may be obtained from a circle. If in eq. (9) a were $= z_0$ the curve would be a common catenary.

Supposing z_0 and the co-ordinates x_1 and z_1 of the point B (abutment) given, we may compute a from eq. 8 by putting $x = x_1$ and $z = z_1$, and solving for a. Then the crown-thrust $H = \gamma a^2$ becomes known, and a can be used in eqs. (8) or (9) to plot points in the curve or linear arch. From eq. (9) we have

$$\left.\begin{matrix}\text{area} \\ OO'mn\end{matrix}\right\} = \int_0^x z dx = \frac{z_0}{2}\int_0^x \left[e^{\frac{x}{a}} dx + e^{-\frac{x}{a}} dx\right] = \frac{az_0}{2}\left[e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right] \quad . \quad (10)$$

Fig. 343.

Call this area, A. As for the thrusts at the different joints of the linear arch, see Fig. 343, we have crown-thrust $= H = \gamma a^2$ . . . ; . . . . (11)
and at any joint m the thrust

$$T = \sqrt{H^2 + (\gamma A)^2} = \gamma\sqrt{a^4 + A^2} \quad . \quad . \quad . \quad . \quad (12)$$

323. Remarks.—The foregoing results may be utilized with arches of finite dimensions by making the arch-ring contain the imaginary linear arch, and the joints ┐ to the curve of the same. Questions of friction and the resistance of the material of the voussoirs are reserved for a succeeding chapter, (§ 344) in which will be advanced a more practical theory dealing with approximate linear arches or "equilibrium polygons" as they will then be called. Still, a study of exact linear arches is valuable on many accounts. By inverting the linear arches so far presented we have the forms assumed by flexible and inextensible cords loaded in the same way.

CHAPTER VIII.

ELEMENTS OF GRAPHICAL STATICS.

324. Definition.—In many respects graphical processes have advantages over the purely analytical, which recommend their use in many problems where celerity is desired without refined accuracy. One of these advantages is that gross errors are more easily detected, and another that the relations of the forces, distances, etc., are made so apparent to the eye, in the drawing, that the general effect of a given change in the data can readily be predicted at a glance.

Graphical Statics is the system of geometrical constructions by which problems in Statics may be solved by the use of drafting instruments, forces as well as distances being represented in amount and direction by lines on the paper, of proper length and position, according to arbitrary scales; so many feet of distance to the linear inch of paper, for example, for distances; and so many pounds or tons to the linear inch of paper for forces.

Of course results should be interpreted by the same scale as that used for the data. The parallelogram of forces is the basis of all constructions for combining and resolving forces.

325. Force Polygons and Concurrent Forces in a Plane.—If a material point is in equilibrium under three forces P_1 P_2 P_3 (in the same plane of course) Fig. 344, any one of them, as P_1, must be equal and opposite to R the **resultant of** the other two (diagonal of their parallelogram). If now we lay off to some convenient scale a line in Fig. 345 = P_1 and $\parallel$ to P_1 in Fig. 344; and then from the *pointed* end of P_1 a line equal and $\parallel$ to P_2 and laid off *pointing the same way*, we note that the line remaining to close the triangle in Fig. 345 must be = and $\parallel$ to P_3, since that triangle is nothing more than the left-hand half-parallelogram of Fig. 344. Also, in 345, to *close* the triangle properly the directions of the arrows must be continuous *Point to Butt*, round the periphery. Fig. 345 is called a **force polygon**; of three sides only in this case. By means of it, given any two of the three forces which hold the point in equilibrium, the third can be found, being equal and $\parallel$ to the side necessary to "close" the force polygon.

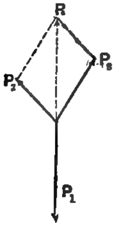


FIG. 344.

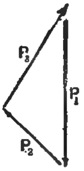


FIG. 345.

Similarly, if a number of forces in a plane hold a material point in equilibrium, Fig. 346, their force polygon,

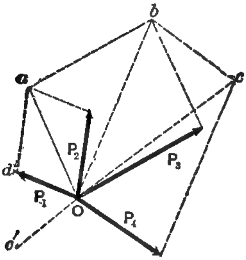


FIG. 346.

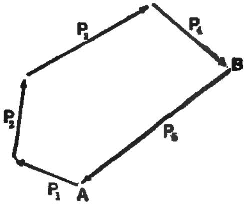


FIG. 347.

Fig. 347, must close, whatever be the order in which its sides are drawn. For, if we combine P_1 and P_2 into a resultant *Oa*, Fig. 346, then this resultant with P_3 to form a resultant *Ob*, and so on; we find the resultant of P_1, P_2, P_3, and P_4 to be *Oc*, and if a fifth force is to produce equilibrium it must be equal and opposite to *Oc*, and would close the polygon *OdabcO*, in which the sides are equal and parallel respectively to the forces mentioned. To utilize this fact we can dispense with all parts of the parallelograms in Fig. 346 except the sides mentioned, and then proceed as follows in Fig. 347:

If P_5 is the unknown force which is to balance the other four (i.e, is their *anti-resultant*), we draw the sides of the force polygon from A round to B, making each line parallel and equal to the proper force and pointing the same way; then the line BA represents the required P_5 in amount and direction, since the arrow BA must follow the continuity of the others (point to butt).

If the arrow BA were pointed at the extremity B, then it gives, obviously, the amount and direction of the *resultant* of the four forces $P_1 \ldots P_4$. The foregoing shows that if a system of **Concurrent Forces in a Plane** is in equilibrium, its *force polygon must close.*

326. Non-Concurrent Forces in a Plane.—Given a system of non-concurrent forces in a plane, acting on a rigid body, required graphic means of finding their resultant and anti-resultant; also of expressing conditions of equilibrium. The resultant must be found in *amount* and *direction*; and also in *position* (i.e., its *line of action* must be determined). E. g., Fig. 348 shows a curved rigid beam fixed in a vise at T, and also under the action of forces P_1 P_2 P_3 and P_4 (*besides* the action of the vise); required the resultant of P_1, P_2, P_3, and P_4. By the ordinary parallelogram of forces we combine P_1 and P_2 at a, the intersection of their lines of action, into a resultant R_a; then R_a with P_3 at b, to form R_b; and finally R_b with P_4 at c to form R_c which is ∴ the resultant required, i.e., of $P_1 \ldots . P_4$; and $c \ldots F$ is its line of action.

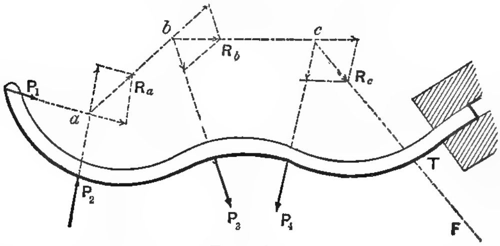


FIG. 348.

The separate force triangles (half-parallelograms) by which the successive partial resultants R_a, etc., were found, are again drawn in Fig. 349. Now since R_c, acting in the line c . . F, Fig. 348, is the resultant of P_1 . . P_4, it is plain that a force R_c' equal to R_c and acting along c . . F, but in the opposite direction, would balance the system P_1 . . . P_4, (is their anti-resultant). That is, the forces $P_1\ P_2\ P_3\ P_4$ and R_c' would form a system in equilibrium. The force R_c' then, represents the action of the vise T upon the beam. Hence replace the vise by the force R_c' acting in the line . . . F . . . c; to do which requires us to imagine a rigid prolongation of that end of the beam, to intersect F . . . c. This is shown in Fig. 350 where the whole beam is *free*, in equilibrium, under the forces shown, and in precisely the same state of stress, part for part, as in Fig. 348. Also, by combining in one force diagram, in Fig. 351, all the force triangles of Fig. 349 (by making their common sides coincide, and putting R_c' instead of R_c, and dotting all forces other than those of Fig. 350), we have a figure to be interpreted in connection with Fig. 350.

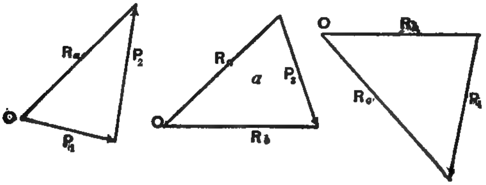
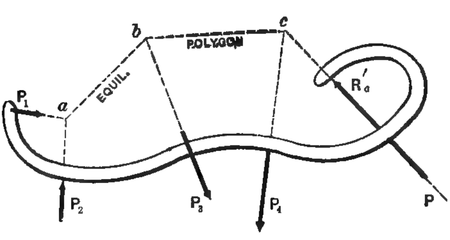
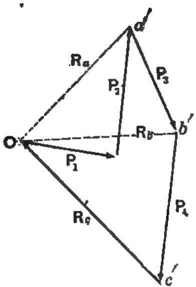


FIG. 349.



SPACE DIAGRAM
FIG. 350.



FORCE DIAGRAM
FIG. 351.

Here we note, first, that in the figure called a force-diagram, $P_1P_2P_3P_4$ and R_c' form a closed polygon and that their arrows follow a continuous order, point to butt, around the perimeter ; which proves that one condition of equilibrium of a system of non-concurrent forces in a plane is that its force polygon must close. Secondly, note that *ab* is ‖ to *Oa′*, and *bc* to *Ob′* ; hence if the force-diagram has been drawn (including the **rays**, dotted) in order to determine the amount and direction of R_c', or any other one force, we may then find its line of action in the space-diagram, as follows: (N. B.—By space diagram is meant the figure showing to a true scale the form of the rigid body and the lines of action of the forces concerned). Through *a*, the intersection of P_1 and P_2, draw a line ‖ to *Oa′* to cut P_3 in some point *b* ; then through *b* a line ‖ to *Ob′* to cut P_4 at some point *c*; *cF* drawn ‖ to *Oc′* is the required line of action of R_c', the anti-resultant of P_1, P_2, P_3, and P_4.

abc is called an **equilibrium polygon**; this one having but two *segments*, *ab* and *bc* (sometimes the lines of action of P_1 and R_c' may conveniently be considered as segments.) *The segments of the equilibrium polygon are parallel to the respective rays of the force diagram.*

Hence for the equilibrium of a system of **non-concurrent forces** in a plane *not only must its force polygon close,* but also the first and last segments of the corresponding equilibrium polygon must coincide with the resultants of the first two forces, and of the last two forces, respectively, of the system. *E.g.*, *ab* coincides with the line of action of the resultant of P_1 and P_2 ; *bc* with that of P_4 and R_c'. Evidently the equil. polygon will be different with each different order of forces in the force polygon or different choice of a **pole**, *O*. But if the order of forces be taken as above, *as they occur along the beam,* or structure, and the pole taken at the "*butt*" of the first force in the force polygon, there will be only one ; (and this one will be called the **special equilibrium polygon** in the chapter on arch-ribs, and the "true linear arch" in dealing with the stone arch.) After the **rays** (dotted in Fig. 351) have been added, by joining the pole to each vertex with which it is not already connected, the final figure may be called the *force diagram.*

It may sometimes be convenient to give the name of rays to the two forces of the force polygon which meet at the pole, in which case the first and last segments of the corresponding equil. polygon will coincide with the lines of action of those forces in the **space-diagram** (as we may call the representation of the body or structure on which the forces act). This "space diagram" shows the real field of action of the forces, while the force diagram, which may be placed in any convenient position on the paper, shows the *magnitudes* and directions of the forces acting in the former diagram, its lines being interpreted on a scale of so many *lbs.* or *tons* to the inch of paper; in the space-diagram we deal with a scale of so many *feet* to the inch of paper.

We have found, then, that if any vertex or corner of the closed force polygon be taken as a pole, and rays drawn from it to all the other corners of the polygon, and a corresponding equil. polygon drawn in the space diagram, the first and last segments of the latter polygon must co-incide with the first and last forces according to the order adopted (or with the resultants of the first two and last two, if more convenient to classify them thus). It remains to utilize this principle.

327. To Find the Resultant of Several Forces in a Plane.—This might be done as in § 326, but since frequently a given set of forces are parallel, or nearly so, a special method will now be given, of great convenience in such cases. Fig. 352. Let P_1 P_2 and P_3 be the given forces whose resultant is required. Let us first find their *anti-resultant,* or force which will balance

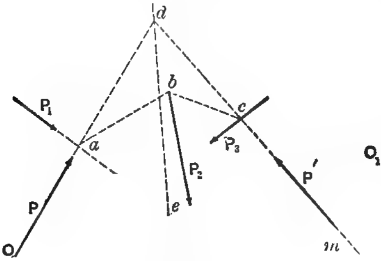


FIG. 352.

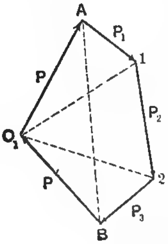


FIG. 353.

them. This anti-resultant may be conceived as decomposed into two components P and P' *one of which*, say P, is arbitrary in amount and position. Assuming P, then, at convenience, in the space diagram, it is required to find P'. The five forces must form a balanced system; hence if beginning at O_1, Fig. 353, we lay off a line $O_1A = P$ by scale, then $A1 =$ and ∥ to P_1, and so on (point to butt), the line BO_1 necessary to close the force polygon is $= P'$ required. Now form the corresponding equil. polygon in the space diagram in the usual way, viz.: through a the intersection of P and P_1 draw ab ∥ to the ray $O_1 \ldots 1$ (which connects the pole O_1 with the point of the last force mentioned). From b, where ab intersects the line of P_2, draw bc, ∥ to the ray $O_1 \ldots 2$, till it intersects the line of P_3. A line mc drawn through c and ∥ to the P' of the force diagram is the line of action of P'.

Now the resultant of P and P' is the anti-resultant of P_1, P_2 and P_3; $\therefore$ d, the intersection of the lines of P and P', is a point in the line of action of the anti-resultant required, while its direction and magnitude are given by the line BA in the force diagram; for BA forms a closed polygon both with P_1 P_2 P_3, and with PP'. Hence a line through d ∥ to BA, viz., de, is the line of action of the anti-resultant (and hence of the resultant) of P_1, P_2, P_3.

Since, in this construction, P is arbitrary, we may first choose O_1, arbitrarily, in a *convenient position*, i.e., in such a position that by inspection the segments of the resulting equil. polygon shall give fair intersections and not pass off the paper. If the given forces are parallel the device of introducing the oblique P and P' is quite necessary.

328.—The result of this construction may be stated as follows, (regarding Oa and cm as segments of the equil. polygon as well as ab and bc): *If any two segments of an equil. polygon be prolonged*, their intersection is a point in the line of action of the *resultant of those forces* acting at the vertices intervening between the given segments. **Here**, the resultant of P_1 P_2 P_3 acts through *d*.

329. Vertical Reaction of Piers, etc.—Fig. 354. Given the vertical forces or loads P_1 P_2 and P_3 acting on a rigid body (beam, or truss) which is supported by two piers having smooth horizontal surfaces (so that the reactions must be vertical), required the reactions V_0 and V_n of the piers. For an instant suppose V_0 and V_n known; they are in equil. with P_1 P_2 and P_3. The introduction of the equal and opposite forces P and P' in the same line will not disturb the equilibrium. Taking the seven forces in the order P V_0 P_1 P_2 P_3 V_n and P', a force polygon formed with them will close (see (*b*) in Fig. where the forces which really lie on the same line are slightly separated). With O, the butt of P, as a pole, draw the rays of the force diagram OA, OB, etc. The corresponding equil. polygon begins at *a*, the intersection of P and V_0 in (*a*) (the space diagram), and ends at *n* the intersection of P' and V_n. Join *an*. Now since P and P' act in the same line, *an* must be that line and must be ∥ to P and P' of the force diagram. Since the amount and direction of P and P' are arbitrary, the position of the pole O is arbitrary, while P_1, P_2, and P_3 are the only forces known in advance in the force diagram.

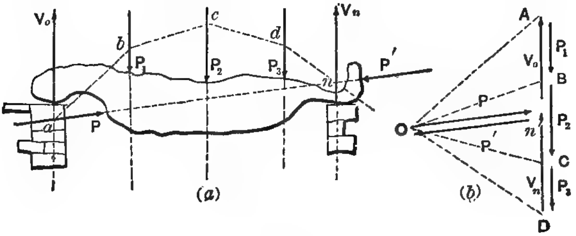


FIG. 354.

Hence V_0 and V_n may be determined as follows: Lay off the given loads P_1, P_2, etc., in the order of their occurrence in the space diagram, to form a “load-line” *AD* (see (*b*.) Fig. 354) as a beginning for a force-diagram ; take any convenient pole O, draw the rays OA, OB, OC and OD. Then beginning at any convenient point a in the vertical line containing the unknown V_0, draw $ab \parallel$ to OA, $bc \parallel$ to OB, and so on, until the last segment (dn in this case) cuts the vertical containing the unknown V_n in some point n. Join an (this is sometimes called a *closing line*) and draw a $\parallel$ to it through O, in the force-diagram. This last line will cut the "load-line" in some point n', and divide it in two parts $n'A$ and Dn', which are respectively V_0 and V_n required.

Corollary.—Evidently, for a given system of loads, in given vertical lines of action, and for two given piers, or abutments, having *smooth horizontal surfaces*, the location of the point n' on the load line is *independent of the choice of a pole.*

Of course, in treating the stresses and deflection of the rigid body concerned, P and P' are left out of account, as being imaginary and serving only a temporary purpose.

330. Application of Foregoing Principles to a Roof Truss.—Fig. 355. W_1 and W_2 are wind pressures, P_1 and P_2 are loads, while the remaining external forces, viz., the re-

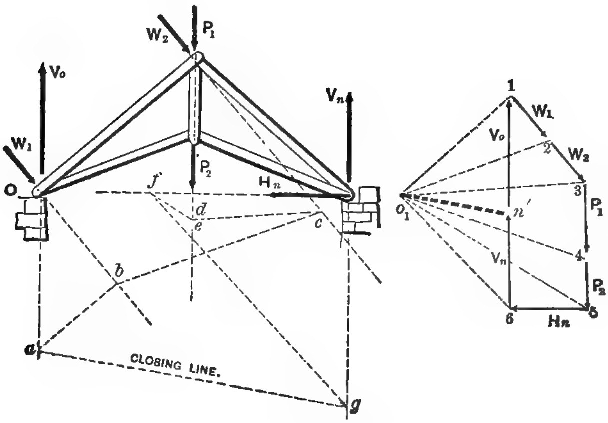


FIG. 355.

actions, or supporting forces, V_0, V_n and H_n, may be found by preceding §§. (We here suppose that the right abutment furnishes all the horizontal resistance; none at the left).

Lay off the forces (known) W_1, W_2, P_1, and P_2 in the usual way, to form a portion of the closed force polygon. To close the polygon it is evident we need only draw a horizontal through 5 and limit it by a vertical through 1. This determines H_n but it remains to determine n' the point of division between V_0 and V_n. Select a convenient pole O_1, and draw rays from it to 1, 2, etc. Assume a convenient point a in the line of V_0 in the space diagram, and through it draw a line ∥ to $O_1 1$ to meet the line of W_1 in some point b; then a line ∥ to $O_1 2$ to meet the line of W_2 in some point c; then through c ∥ to $O_1 3$ to meet the line of P_1 in some point d; then through d ∥ to $O_1 4$ to meet the line of P_2 in some point e, (e is identical with d, since P_1 and P_2 are in the same line); then ef ∥ to $O_1 5$ to meet H_n in some point f; then fg ∥ to $O_1 6$ to meet V_n in some point g.

abcdefg is an equilibrium polygon corresponding to the pole O_1.

Now join ag, the "closing-line," and draw a ∥ to it through O_1 to determine n', the required point of division between V_0 and V_n on the vertical 1 6. Hence V_0 and V_n are now determined as well as H_n.

[The use of the arbitrary pole O_1 implies the temporary employment of a pair of opposite and equal forces in the line ag, the amount of either being $= O_1 n'$].

Having now all the external forces acting on the truss, and assuming that it contains no "redundant parts," i.e., parts unnecessary for rigidity of the frame-work, we proceed to find the pulls and thrusts in the individual pieces, on the following plan. The truss being *pin-connected*, no piece extending beyond a joint, and all loads being considered to act at joints, the action, pull or thrust, of each piece on the joint at either extremity will be in the direction of the piece, i.e., in a *known direction*, and the pin of each joint is in equilibrium under a system of concurrent forces consisting of the loads (if any) at the joint and the pulls or thrusts exerted upon it by the pieces meeting there. Hence we may apply the principles of § 325 to each joint in turn. See Fig. 356. In constructing and interpreting the various force polygons, Mr. R. H Bow's convenient notation will be used; this is as follows: In the space diagram a capital letter [ABC, etc.] is placed in each triangular cell of the truss, and also in each angular space in the outside outline of the truss between the external forces and the adjacent truss-pieces. In this way we can speak of the force W_1 as the force BC, of W_2 as the force CE, the stress in the piece $\alpha\beta$ as the force CD, and so on. That is, the stress in any one piece can be named from the letters in the spaces bordering its two sides. Corresponding to these capital letters in the *spaces* of the space-diagram, small letters will be used at the *vertices* of the closed force-polygons (one polygon for each joint) in such a way that the stress in the piece CD, for example, shall be the force cd of the force polygon belonging to any joint in which that piece terminates; the stress in the piece FG by the force fg in the proper force polygon, and so on.

In Fig. 356 the whole truss is shown free, in equilibrium under the external forces. To find the pulls or thrusts (i.e., tensions or compressions) in the pieces, consider that if all but two of the forces of a closed force polygon are known in magnitude and direction, while the directions, only, of those two are known, the *whole force polygon may be drawn,* thus determining *the amounts* of those two forces by the lengths of the corresponding sides.

We must ∴ begin with a joint where no more than two pieces meet, as at α; [call the joints α, β, γ, δ, and the corcorresponding force polygons α', β' etc. Fig. 356.] Hence at α' (anywhere on the paper) make $ab \parallel$ and $=$ (by scale) to the known force AB (i.e., V_0) *pointing it at the upper end,* and from this end draw $bc =$ and $\parallel$ to the known force BC (i.e., W_1) pointing this at the lower end.

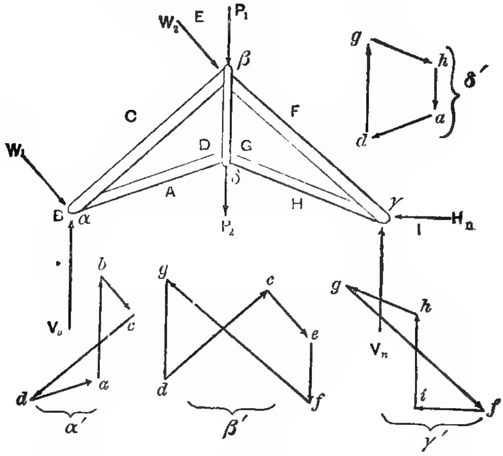


Fig. 356.

To close the polygon draw through *c* a ‖ to the piece *CD*, and through *a* a ‖ to *AD*; their intersection determines *d*, and the polygon is closed. Since the arrows must be point to butt round the periphery, the force with which the piece *CD* acts on the pin of the joint *a* is a force of an amount = *cd* and in a direction from *c* toward *d*; hence the piece *CD* is *in compression*; whereas the action of the piece *DA* upon the pin at *a* is from *d* toward *a* (direction of arrow) and hence *DA* is *in tension*. Notice that in constructing the force polygon *a′* a right-handed (or clock-wise) rotation has been observed in considering in turn the spaces *A B C* and *D*, round the joint *a*. A similar order will be found convenient in each of the other joints.

Knowing now the stress in the piece *CD*, (as well as in *DA*) all but two of the forces acting on the pin at the joint *β* are known, and accordingly we begin a force polygon, *β′*, for that joint by drawing *dc*,= and ‖ to the *dc* of polygon *a′*, *but pointed in the opposite direction*, since the action of *CD* on the joint *β* is equal and opposite to its action on the joint *a* (this disregards the weight of the piece). Through *c* draw *ce* = and ‖ to the force *CE* (i.e., W_2) and pointing the same way; then ef, = and ∥ to the load EF (i.e. P_1) and pointing downward. Through f draw a ∥ to the piece FG and through d, a ∥ to the piece GD, and the polygon is closed, thus determining the stresses in the pieces FG and GD. Noting the pointing of the arrows, we readily see that FG is in compression while GD is in tension.

Next pass to the joint δ, and construct the polygon δ', thus determining the stress gh in GH and that ad in AD; this last force ad should check with its equal and opposite ad already determined in polygon a'. Another check consists in the proper closing of the polygon γ', all of whose sides are now known.

[A compound stress-diagram may be formed by superposing the polygons already found in such a way as to make equal sides co-incide; but the character of each stress is not so readily perceived then as when they are kept separate].

In a similar manner we may find the stresses in any pin-connected frame-work (in one plane and having no redundant pieces) under given loads, provided all the supporting forces or reactions can be found. In the case of a braced-arch (truss) as shown in Fig. 357, hinged to the abutments at both ends and not free to slide laterally upon them, the reactions at O and B depend, in amount and direction, not only upon the equations of Statics, but on *the form* and *elasticity* of the arch-truss. Such cases will be treated later under arch-ribs, or curved beams.

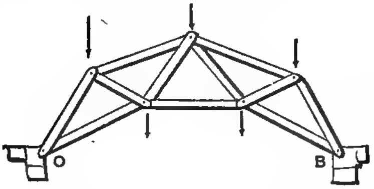
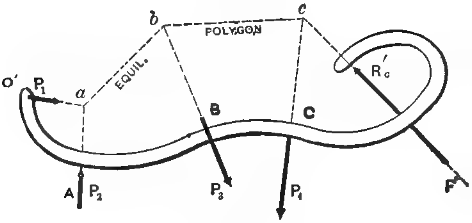


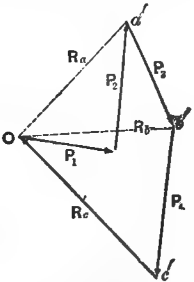
FIG. 357.

332. The Special Equil. Polygon. Its Relation to the Stresses in the Rigid Body.—Reproducing Figs. 350 and 351 in Figs. 358 and 359, (where a rigid curved beam is in equilibrium under the forces P_1, P_2, P_3, P_4 and R'_c) we call $a \ldots b \ldots c$

the *special equil. polygon* because it corresponds to a force diagram in which the same order of forces has been observed as that in which they occur along the beam (from left to right here). From the relations between the force diagram and equil. polygon, this special equil. polygon in the space diagram has the following properties in connection with the corresponding rays (dotted lines) in the force diagram.



SPACE DIAGRAM
FIG. 358.



FORCE DIAGRAM
FIG. 359.

The stresses in any cross-section of the portion $O'A$ of the beam, are due to P_1 alone; those of any cross-section on AB to P_1 and P_2, i.e., to *their resultant* R_a, whose magnitude is given by the line Oa' in the force diagram, while its line of action is ab the first segment of the equil. polygon. Similarly, the stresses in BC are due to P_1, P_2 and P_3, i.e., to their resultant R_b acting along the segment bc, its magnitude being $= Ob'$ in the force diagram. E.g., if the section at m be exposed, considering $O'ABm$ as a free body, we have (see Fig. 360) the elastic stresses (or internal forces) at m balancing the exterior or "applied forces" P_1, P_2 and P_3. Obviously, then, the stresses at m are just

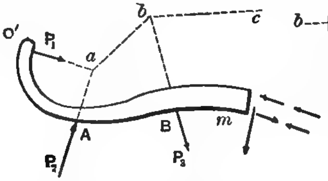


FIG. 360.

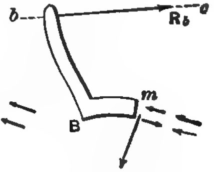


FIG. 361.

the same as if R_b the resultant of P_1, P_2 and P_3, acted upon an imaginary rigid prolongation of the beam intersecting *bc* (see Fig. 361). R_b might be called the "*anti-stress-resultant*" for the portion *BC* of the beam. We may ∴ state the following: *If a rigid body is in equilibrium under a system of* **Non-Concurrent Forces** *in a plane, and the special equilibrium polygon has been drawn, then each ray of the force diagram is the anti-stress-resultant of that portion of the beam which corresponds to the segment of the equilibrium polygon to which the ray is parallel; and its line of action is the segment just mentioned.*

Evidently if the body is not one rigid piece, but composed of a ring of uncemented blocks (or voussoirs), it may be considered rigid only so long as no slipping takes place or disarrangement of the blocks; and this requires that the "anti-stress-resultant" for a given joint between two blocks shall not lie outside the bearing surface of the joint, nor make too small an angle with it, lest tipping or slipping occur. For an example of this see Fig. 362, showing a line of three blocks in equilibrium under five forces. The pressure borne at the joint *MN*, is $= R_a$ in the force-diagram and acts in the line *ab*. The construction supposes all the forces given except one, in amount and position, and that this one could easily be found in *amount*, as being the side remaining to close the force polygon, while its *position* would depend on the equil. polygon. But in practice the *two* forces P_1 and R'_c are generally unknown, hence the point *O*, or pole of the force diagram, can not be fixed, nor the special equil. polygon located, until other considerations, outside of those so far presented, are brought into play. In the progress of such a problem, as will be seen, it will be necessary to use arbitrary *trial* positions for the pole *O*, and corresponding *trial* equilibrium polygons.

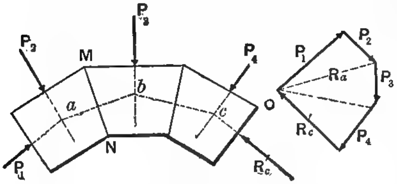


FIG. 362.

CHAPTER IX.

GRAPHICAL STATICS OF VERTICAL FORCES.

333. Remarks.—(With the exception of § 378 *a*) in problems to be treated subsequently (either the stiff arch-rib, or the block-work of an arch-ring, of masonry) when the body is considered *free* all the forces holding it in equil. will be *vertical* (loads, due to gravity) except the reactions at the two extremities, as in Fig. 363; but for convenience each reaction will be replaced by its horizontal and vertical components (see Fig. 364). The two H's are of course equal, since they are the only horizontal forces in the system. *Henceforth, all equil. polygons under discussion will be understood to imply this kind of system of forces.* P_1, P_2, etc., will represent the "loads"; V_0 and V_n the vertical components of the abutment reactions; H the value of either horizontal component of the same. (We here suppose the pressures T_0 and T_n resolved along the horizontal and vertical.)

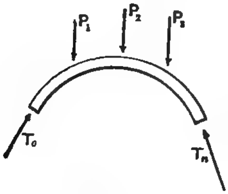


FIG. 363.

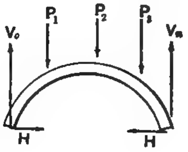


FIG. 364.

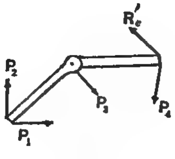


FIG. 364a.

334. Concrete Conception of an Equilibrium Polygon.—Any equilibrium polygon has this property, due to its mode of construction, viz.: If the *ab* and *bc* of Fig. 358 were imponderable straight rods, jointed at *b* without friction, they would be in equilibrium under the system of forces there given. (See Fig. 364*a*). The rod *ab* suffers a compression equal to the R_a of the force diagram, Fig. 359, and *bc* a compression $= R_b$. In some cases these rods might be in tension, and would then form a set of links playing the part of a suspension-bridge cable. (See § 44).

335. Example of Equilibrium Polygon Drawn to Vertical Loads—Fig. 365. [The structure bearing the given loads is not shown, but simply the imaginary rods, or segments of an equilibrium polygon, which would support the given loads in equilibrium if the abutment points *A* and *B*, to which the terminal rods are hinged, were firm. In the present case this equilibrium is unstable since the rods form a *standing structure*; but if they were *hanging*, the equilibrium would be *stable*. Still, in the present case, a *very light bracing*, or a little friction at all joints would make the equilibrium stable.

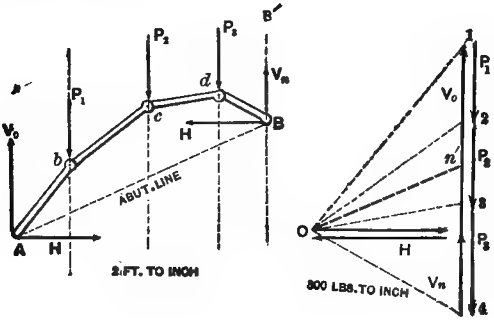


FIG. 365.

Given three loads P_1, P_2, and P_3, and two "**abutment verticals**" *A'* and *B'*, in which we desire the equil. polygon to terminate, lay off as a "*load-line*," to scale, P_1, P_2, and P_3 end to end in their order. Then selecting any pole,

O, draw the rays $O1$, $O2$, etc., of a force diagram (the V's and P's, though really on the same vertical, are separated slightly for distinctness; also the H's, which both pass through O and divide the load-line into V_0 and V_n). We determine a corresponding equilibrium polygon by drawing through A (any point in A') a line $\parallel$ to $O\,.\,.\,1$, to intersect P_1 in some point b; through b a $\parallel$ to $O\,.\,.\,2$, and so on, until B' the other abutment-vertical is struck in some point B. AB is the "*abutment-line*" or "*closing-line.*"

By choosing another point for O, another equilibrium polygon would result. As to which of the infinite number (which could thus be drawn, for the given loads and the A' and B' verticals) is the *special equilibrium polygon* for the arch-rib or stone-arch, or other structure, on which the loads rest, is to be considered hereafter. In any of the above equilibrium polygons the imaginary series of jointed rods would be in equilibrium.

336. Useful Property of an Equilibrium Polygon for Vertical Loads.—(Particular case of § 328). See Fig. 366. In any equil. polygon, supporting vertical loads, consider as free any number of consecutive segments, or rods, with the loads at their joints, e. g., the 5th and 6th and portions of the 4th and 7th which, we suppose cut and the compressive forces in them put in, T_4 and T_7, in order to consider 4 5 6 7 as a free body. For equil., according to Statics, the lines of action of T_4 and T_7 (the compression in those rods) must intersect in a point, C, in the line of action of the resultant of P_4, P_5, and P_6; i.e., of the loads occurring at the intervening vertices. That is, the point C must lie in the vertical containing the centre of gravity of those loads. Since the position of this vertical must be independent of the particular equilibrium polygon used, any other (dotted lines in Fig. 366) for the same loads will give the same re-

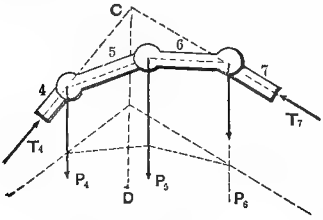


FIG. 366.

sults. Hence the vertical *CD*, containing the centre of gravity of any number of consecutive loads, is easily found by drawing the equilibrium polygon corresponding to any convenient force diagram having the proper load-line.

This principle can be advantageously applied to finding a gravity-line of any plane figure, by dividing the latter into parallel strips, whose areas may be treated as loads applied in their respective centres of gravity. If the strips are quite numerous, the centre of gravity of each may be considered to be at the centre of the line joining the middles of the two long sides, while their areas may be taken as proportional to the lengths of the lines drawn through these centres of gravity parallel to the long sides and limited by the end-curves of the strips. Hence the "load-line" of the force diagram may consist of these lines, or of their halves, or quarters, etc., if more convenient (§ 376).

USEFUL RELATIONS BETWEEN FORCE DIAGRAMS AND EQUILIBRIUM POLYGONS.

(for vertical loads.)

337. Résumé of Construction.—Fig. 367. Given the loads P_1, etc., their verticals, and the two abutment verticals *A'* and *B'*, in which the abutments are to lie; we lay off a load-line 1 . . . 4, take any convenient pole, *O*, for a force-diagram and complete the latter. For a corresponding equilibrium polygon, assume any point *A* in the vertical *A'*, for an abutment, and draw the successive segments *A*1, 2, etc., respectively parallel to the inclined lines of the force diagram (rays), thus determining finally the abutment *B*, in *B'*, which (*B*) will not in general lie in the horizontal through *A*.

Now join *AB*, calling *AB* the abutment-line, and draw a parallel to it through *O*, thus fixing the point *n'* on the

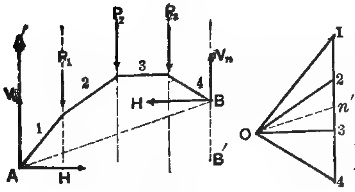


FIG. 367.

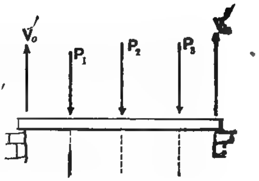


FIG. 368.

load-line. This point n', as above determined, is ***independent** of the location of the pole, O,* (proved in § 329) and divides the load-line into two portions ($V'_0 = 1 \ldots n'$, and $V'_n = n' \ldots 4$) which are the vertical pressures which two supports in the verticals A' and B' *would sustain if the given loads rested on a horizontal rigid bar,* as in Fig. 368.

See § 329. Hence to find the point n' we may use *any convenient pole O.*

[N. B.—The forces V_0 and V_n of Fig. 367 are not identical with V'_0 and V'_n, but may be obtained by dropping a ┐ from O to the load-line, thus dividing the load-line into two portions which are V_0 (upper portion) and V_n. However, if A and B be connected by a *tie-rod,* in Fig. 367, the abutments in that figure will bear vertical pressures only and they will be the same as in Fig. 368, while the tension in the tie-rod will be $= On'$.]

338. Theorem.—*The vertical dimensions of any two equilibrium polygons, drawn to the same loads, load-verticals, and abutment-verticals, are inversely proportional to their H's (or "pole distances").* We here regard an equil. polygon and its abutment-line as a closed figure. Thus, in Fig. 369, we have two force-diagrams (with a common load-line, for convenience) and their corresponding equil. polygons, for the same loads and verticals. From § 337 we know that On' is ∥ to AB and O_0n' is ∥ to A_0B_0. Let CD be any vertical cutting the first segments of the two equil. polygons.

Denote the intercepts thus determined by z' and z'_0, respectively. From the parallelisms just mentioned, and others more familiar, we have the triangle $O\ 1n'$ similar to the triangle Az' (shaded), and the triangle $O_0 1n'$ similar to the triangle $A_0z'_0$. Hence the proportions between bases and altitudes $\left\{ \frac{1n'}{H} = \frac{z'}{h} \text{ and } \frac{1n'}{H_0} = \frac{z''_0}{h} \right\}$

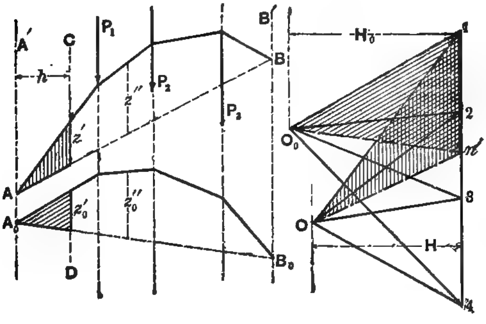


FIG. 369.

$\therefore z' : z'_0 :: H_0 : H$. The same kind of proof may easily be applied to the vertical intercepts in any other segments, e. g., z'' and z''_0. Q. E. D.

339. Corollaries to the foregoing. It is evident that:

(1.) If the pole of the force-diagram be moved along a vertical line, the equilibrium polygon changing its form in a corresponding manner, the vertical dimensions of the equilibrium polygon remain unchanged; and

(2.) If the pole move along a straight line which contains the point n', the direction of the abutment-line remains constantly parallel to the former line, while the vertical dimensions of the equilibrium polygon change in inverse proportion to the pole distance, or H, of the force-diagram. [H is the ┐ distance of the pole from the load-line, and is called the pole-distance].

§ **340. Linear Arch as Equilibrium Polygon.**—(See § 316.) If the given loads are infinitely small with infinitely small horizontal spaces between them, any equilibrium polygon becomes a **linear arch.** Graphically we can not deal with these infinitely small loads and spaces, but from § 336 it is evident that if we replace them, in successive groups,

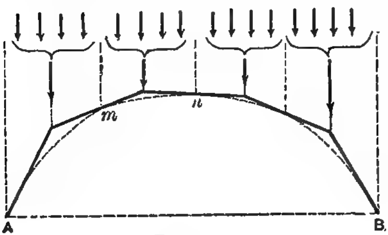


Fig. 370.

by finite forces, each of which = the sum of those composing one group and is applied through the centre of gravity of that group, we can draw an equilibrium polygon whose segments will be tangent to the curve of the corresponding linear arch, and indicate its position with sufficient exactness for practical purposes. (See Fig. 370). The successive points of tangency A, m, n, etc., lie vertically under the points of division between the groups. This relation forms the basis of the graphical treatment of voussoir, or blockwork, arches.

341. To Pass an Equilibrium Polygon Through Three Arbitrary Points.—(In the present case the forces are vertical. For a construction dealing with any plane system of forces see construction in § 378*a*.) Given a system of loads, it is required to draw an equilibrium polygon for them through any three points, two of which may be considered as abutments, outside of the load-verticals, the third point being between the verticals of the first two. See Fig. 371. The loads P_1, etc., are given, with their verticals, while A, p, and B are the three points. Lay off the load-line, and with any convenient pole, O_1, construct a force-diagram, then a corresponding **preliminary** equilibrium polygon beginning at A. Its right abutment B_1, in the vertical through B, is thus found. $O_1\ n'$ can now be drawn ∥ to AB_1, to determine n'. Draw $n'O$ ∥ to BA. The pole of the required equilibrium polygon must lie on $n'O$ (§ 337).

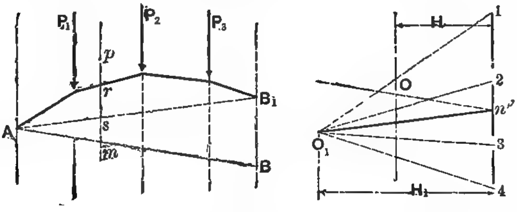


Fig. 371.

Draw a vertical through *p*. The H of the required equilibrium polygon must satisfy the proportion $H : H_1 :: \overline{rs} : \overline{pm}$. (See § 338). Hence construct or compute H from the proportion and draw a vertical at distance H from the load-line (on the left of the load-line here); its intersection with $n'\,O$ gives O the desired pole, for which a force diagram may now be drawn. The corresponding equilibrium polygon beginning at the first point A will also pass through p and B; it is not drawn in the figure.

342. Symmetrical Case of the Foregoing Problem.—If two points A and B are *on a level*, the third, p, on the middle vertical between them; and the loads (an *even* number) *symmetrically disposed both in position and magnitude*, about p, we may proceed more simply, as follows: (Fig. 372).

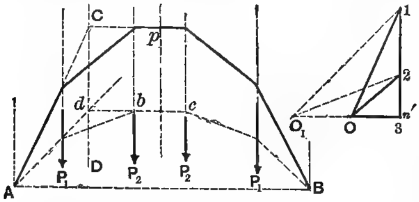


FIG. 372.

From symmetry n' must occur in the middle of the load-line, of which we need lay off only the upper half. Take a convenient pole O_1, in the horizontal through n', and draw a half force diagram and a corresponding half equilibrium polygon (both dotted). The upper segment bc of the latter must be horizontal and being prolonged, cuts the prolongation of the first segment in a point d, which determines the vertical CD containing the centre of gravity of the loads occurring over the half-span on the left. (See § 336). In the required equilibrium polygon the segment containing the point p must be horizontal, and its intersection with the first segment must lie in CD. Hence determine this intersection, C, by drawing the vertical CD and a horizontal through p; then join CA, which is the *first segment* of the required equil. polygon. A parallel to CA through 1 is the *first ray* of the corresponding force diagram, and determines the pole O on the horizontal through n'. Completing the force diagram for this pole (half of it only here), the required equil. **polygon** is easily finished afterwards.

343. To Find a System of Loads Under Which a Given Equilibrium Polygon Would be in Equilibrium.—Fig. 373. Let *AB* be the given equilibrium polygon. Through any point *O* as a pole draw a parallel to each segment of the equilibrium polygon. *Any* vertical, as *V*, cutting these lines will have, intercepted upon it, a load-line 1, 2, 3, whose parts 1..2, 2 . . 3, etc., are proportional to the successive loads which, placed on the corresponding joints of the equilibrium polygon would be supported by it in equilibrium (unstable).

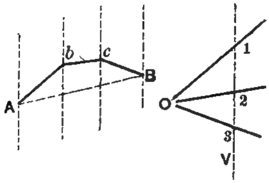


FIG. 373.

One load may be assumed and the others constructed. A hanging, as well as a standing, equilibrium polygon may be dealt with in like manner, but will be in *stable* equilibrium. The problem in § 44 may be solved in this way, the various steps and final results being as follows (Fig. 50 is here repeated):—

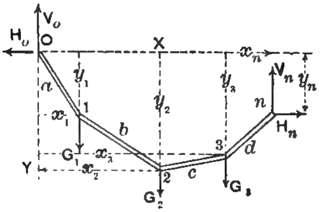


FIG. 50.

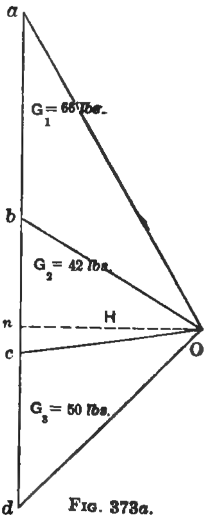


FIG. 373a.

Let weight G_1 be given, $=66$ lbs., and the positions of the cord segments be as in Fig. 50. We first lay off (see Fig. 373*a*) vertically, $ab=66$ lbs., by some convenient scale, and prolong this vertical line indefinitely downward. *aO* is then drawn parallel to 0 . . . 1 of Fig. 50, and *bO* parallel to 1 . . . 2. Their intersection determines a pole, *O*, through which *Oc* and *Od*, parallel respectively to 2 . . . 3 and 3 . . . *n*, are drawn, to intersect *ad* in *c* and *d*.

We also draw the horizontal *On*, in Fig. 373*a*. By scaling, we now find the results:—$G_2=bc=42$ lbs.; $G_3=cd=50$ lbs.; $H=58.5$ lbs., ($=H_0$ and H_n of Fig. 50); while $V_0=an=100$ lbs. and $V_n=58$ lbs.

CHAPTER X.

RIGHT ARCHES OF MASONRY.

Note.—The treatment given in this chapter is by many engineers considered sufficiently exact for ordinary masonry arches, the more refined methods of the "elastic theory" being reserved for arches of fairly continuous material, such as those of metal and of concrete (re-inforced or otherwise); and is accordingly retained in this revised edition.

344.—In an ordinary "right" stone-arch (i.e., one in which the faces are ⏋ to the axis of the cylindrical soffit, or under surface), the successive blocks forming the arch-ring are called *voussoirs*, the joints between them being planes which, prolonged, meet generally in one or more horizontal lines; e.g., those of a three-centred arch in three ∥ horizontal lines; those of a circular arch in one, the axis of the cylinder, etc. Elliptic arches are sometimes used. The inner concave surface is called the *soffit*, to which the radiating joints between the voussoirs are made perpendicular. The curved line in which the soffit is intersected by a plane ⏋ to the axis of the arch is the **Intrados.** The curve in the same plane as the intrados, and bounding the outer extremities of the joints between the voussoirs, is called the **Extrados.**

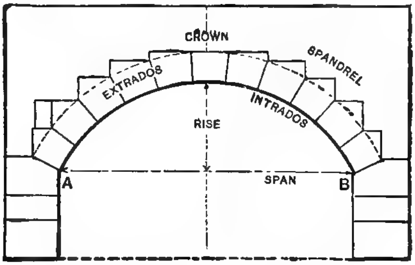


Fig. 374.

Fig. 374 gives other terms in use in connection with a stone arch, and explains those already given. *AB* is the "springing-line."

345. Mortar and Friction.—As common mortar hardens very slowly, no reliance should be placed on its tenacity as an element of stability in arches of any considerable size; though hydraulic mortar and *thin* joints of ordinary mortar can sometimes be depended on. *Friction,* however, between the surfaces of contiguous voussoirs, plays an essential part in the stability of an arch, and will therefore be considered.

The stability of voussoir-arches must ∴ be made to depend on the resistance of the voussoirs to compresssion and to sliding upon each other; as also of the blocks composing the piers, the foundations of the latter being firm.

346. Point of Application of the Resultant Pressure between two consecutive voussoirs; (or pier blocks). Applying Navier's principle (as in flexure of beams) that the pressure per unit area on a joint varies uniformly from the extremity under greatest compression to the point of least compression (or of no compression); and remembering that negative pressures (i.e., tension) can not exist, as they might in a curved beam, we may represent the pressure per unit area at successive points of a joint (from the intrados toward the extrados, or vice versâ) by the ordinates of a straight line, forming the surface of a trapezoid or triangle, in which figure the foot of the ordinate of the centre of gravity is the *point of application of the resultant pressure.* Thus, where the least compression is supposed

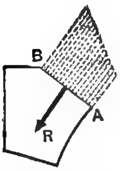


Fig. 375.

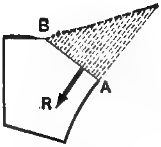


Fig. 376.

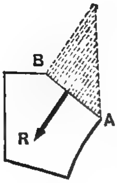


Fig. 377.

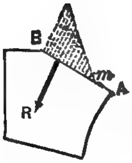


Fig. 378.

to occur at the intrados A, Fig. 375, the pressures vary as the ordinates of a trapezoid, increasing to a maximum value at B, in the extrados. In Fig. 376, where the pressure is zero at B, and varies as the ordinates of a *triangle*, the resultant pressure acts through a point *one-third* the joint-length from A. Similarly in Fig. 377, it acts one-third the joint-length from B. Hence, when the pressure is not zero at either edge the resultant pressure acts within the **middle third** of the joint. Whereas, if the resultant pressure falls *without* the middle third, it shows that a portion Am of the joint, see Fig. 378, receives no pressure, i.e., the joint tends to open along Am.

Therefore that no joint tend to open, the resultant pressure must *fall within the middle third.*

It must be understood that the joint surfaces here dealt with are rectangles, seen edgewise in the figures.

347. Friction.—By experiment it has been found the angle of friction (see § 156) for two contiguous voussoirs of stone or brick is about 30°; i.e., the coefficient of friction is f = tan. 30°. Hence if the direction of the pressure exerted upon a voussoir by its neighbor makes an angle α less than 30° with the *normal to the joint surface,* there is no danger of rupture of the arch by the sliding of one on the other. (See Fig. 379).

348. Resistance to Crushing.—When the resultant pressure falls at its extreme allowable limit, viz.: the edge of the middle third, the pressure per unit of area at n, Fig. 380, is double the mean pressure per unit of area. Hence, in designing an arch of masonry, we must be assured that at every joint (taking 10 as a factor of safety)

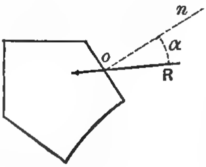


FIG. 379.

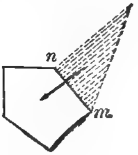


FIG. 380.

$$\left\{\begin{array}{l}\text{Double the mean press-}\\ \text{ure per unit of area}\end{array}\right\}\text{ must be less than } {}^{1}/_{10}\, C$$

C being the ultimate resistance to crushing, of the material employed (§ 201) (Modulus of Crushing).

Since a lamina *one foot* thick will always be considered in what follows, careful attention must be paid to the units employed in applying the above tests.

EXAMPLE.—If a joint is 3 ft. by 1 foot, and the resultant pressure is 22.5 tons the mean pressure per sq. foot is

$$p=22.5\div3=7.5 \text{ tons per sq. foot}$$

$\therefore$ its double=15 tons per sq. foot=208.3 lbs. sq. inch, which is much less than $^1/_{10}$ of C for most building stones; see § 203, and below.

At joints where the resultant pressure falls at the middle, the max. pressure per square inch would be equal to the mean pressure per square inch; but for safety it is best to assume that, at times, (from moving loads, or vibrations) it may move to the edge of the middle third, causing the max. pressure to be double the mean (per square inch).

Gen. Gillmore's experiments in 1876 gave the following results, among many others:

NAME OF BUILDING STONE.	C IN LBS. PER SQ. INCH.
Berea sand-stone, 2-inch cube,	8955
" " 4 " "	11720
Limestone, Sebastopol, 2-inch cube (*chalk*),	1075
Limestone from Caen, France,	3650
Limestone from Kingston, N. Y.,	13900
Marble, Vermont, 2-inch cube,	8000 to 13000
Granite, New Hampshire, 2-inch cube,	15700 to 24000

349. The Three Conditions of Safe Equilibrium for an arch of uncemented voussoirs.

Recapitulating the results of the foregoing paragraphs, we may state, as follows, the three conditions which must be satisfied at every joint of arch-ring and pier, for each of any possible combination of loads upon the structure:

(1). The resultant pressure must pass within the middle-third.

(2). The resultant pressure must not make an angle > 30° with the normal to the joint.

(3). The mean pressure per unit of area on the surface of the joint must not exceed $^1/_{20}$ of the Modulus of crushing of the material.

350. The True Linear-Arch, or Special Equilibrium Polygon; and the resultant pressure at any joint. Let the weight of each voussoir and its load be represented by a vertical force passing through the centre of gravity of the two, as in Fig. 381. Taking any two points A and B, A being in the first joint and B in the last; also a third point, p, in the crown joint (supposing such to be there, although generally a key-stone occupies the crown), through these three points can be drawn [§ 341] an equilibrium polygon for the loads given; suppose this equil. polygon nowhere passes outside of the arch-ring (the arch-ring is the portion between the intrados, mn, and the (dotted) extrados $m'n'$) intersecting the joints at b, c, etc. Evidently if such be the case, and small metal rods (not round) were inserted at A, b, c, etc., so as to separate the arch-stones slightly, the arch would stand, though in unstable equilibrium, the piers being firm; and by a different choice of A, p, and B, it might be possible to draw other equilibrium polygons with segments cutting the joints within the arch-ring, and if the metal rods were shifted to these new intersections the arch would again stand (in unstable equilibrium).

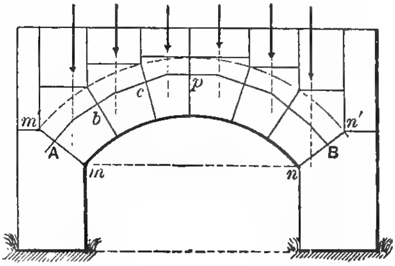


FIG. 381.

In other words, if an arch stands, it may be possible to draw a great number of linear arches within the limits of the arch-ring, since three points determine an equilibrium polygon (or linear arch) for given loads. The question arises then: *which linear arch is the locus of the actual resultant pressures at the successive joints?*

[Considering the arch-ring as an elastic curved beam inserted in firm piers (i.e., the blocks at the springing-line are incapable of turning) and having secured a close fit at all joints before the centering is lowered, the most satisfactory answer to this question is given in Prof. Greene's "Arches," p. 131; viz., to consider the arch-ring as an arch rib of fixed ends and no hinges; see § 380 of next chapter; but the lengthy computations there employed (and the method demands a simple algebraic curve for the arch) may be most advantageously replaced by Prof. Eddy's graphic method ("New Constructions in Graphical Statics," published in Van Nostrand's Magazine for 1877), which applies to arch curves of any form.

This method will be given in a subsequent chapter, on Arch Ribs, or Curved Beams; but for arches of masonry a much simpler procedure is sufficiently exact for practical purposes and will now be presented].

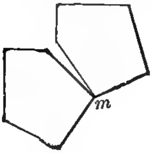


Fig. 382.

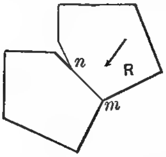


Fig. 383.

If two elastic blocks of an arch-ring touch at one edge, Fig. 382, their adjacent sides making a small angle with each other, and are then gradually pressed more and more forcibly together at the edge *m*, as the arch-ring settles, the centering being gradually lowered, the surface of contact becomes larger and larger, from the compression which ensues (see Fig. 383); while the resultant pressure between the blocks, first applied at the extreme edge *m*, has now probably advanced nearer the middle of the joint in the mutual adjustment of the arch-stones. With this in view we may reasonably deduce the following theory of the location of the **true linear arch** (sometimes called the "line of pressures" and "curve of pressure") in an arch under given loading and with *firm piers.* (Whether the piers are really unyielding, under the oblique thrusts at the springing-line, is a matter for subsequent investigation.

351. Location of the True Linear Arch.—Granted that the voussoirs have been closely fitted to each other over the centering (sheets of lead are sometimes used in the joints to make a better distribution of pressure); and that the piers are firm ; and that the arch can stand at all without the centering ; then we assume that in the mutual accommodation between the voussoirs, as the centering is lowered, the resultant of the pressures distributed over any joint, if at first near the extreme edge of the joint, advances nearer to the middle as the arch settles to its final position of equilibrium under its load; and hence the following

352. Practical Conclusions.

I. If for a given arch and loading, with firm piers, an equilibrium polygon can be drawn (by proper selection of the points A, p, and B, Fig. 381) entirely within the *middle third* of the arch ring, not only will the arch stand, but the resultant pressure at every joint will be within the middle third (Condition 1, § 349); and among all possible equilibrium polygons which can be drawn within the middle third, that is the "true" one which most nearly coincides with the middle line of the arch-ring.

II. If (with firm piers, as before) no equilibrium polygon can be drawn within the middle third, and only one within the arch-ring at all, the arch may stand, but chipping and spawling are likely to occur at the edges of the joints. The design should ∴ be altered.

III. If no equilibrium polygon can be drawn within the arch-ring, the design of either the arch or the loading must be changed; since, although the arch may stand from the resistance of the spandrel walls, such a stability must be looked upon as precarious and not countenanced in any large important structure. (Very frequently, in small arches of brick and stone, as they occur in buildings, the cement is so tenacious that the whole structure is virtually a single continuous mass).

When the "true" linear arch has once been determined, the amount of the resultant pressure on any joint is given by the length of the proper ray in the force diagram.

ARRANGEMENT OF DATA FOR GRAPHIC TREATMENT.

353. Character of Load.—In most large stone arch bridges the load (permanent load) does not consist exclusively of masonry up to the road-way but partially of earth filling above the masonry, except at the faces of the arch where the spandrel walls serve as retaining walls to hold the earth. (Fig. 384). If the intrados is a half circle or half-ellipse, a compactly-built masonry backing is carried up beyond the springing-line to AB about 60° to 45° from the crown, Fig. 385; so that the portion of arch ring below AB may be considered as part of the abutment, and thus AB is the virtual springing-line, for graphic treatment.

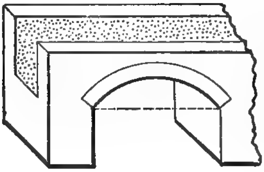


FIG. 384.

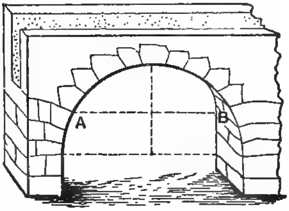


FIG. 385.

Sometimes, to save filling, small arches are built over the haunches of the main arch, with earth placed over them, as shown in Fig. 386. In any of the preceding cases it is customary to consider that, on account of the bonding of the stones in the arch *shell*, the loading at a given distance from the crown is *uniformly distributed over the width of the roadway*.

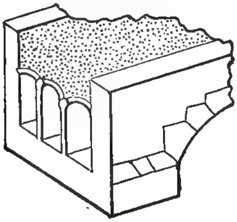


FIG. 386.

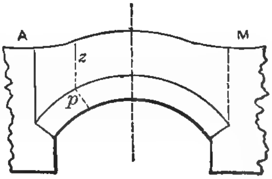


FIG. 387.

354. Reduced Load-Contour.—In the graphical discussion of a proposed arch we consider a lamina one foot thick, this lamina being vertical and ⏋ to the axis of the arch; i.e., the lamina is ∥ to the spandrel walls. For graphical treatment, equal areas of the elevation (see Fig. 387) of this lamina must represent equal weights. Taking the material of the arch-ring as a standard, we must find for each point p of the extrados an imaginary height z of the arch-ring material, which would give the same pressure (per running horizontal foot) at that point as that due to the actual load above that point. A number of such ordinates, each measured vertically upward from the extrados determine points in the "**Reduced Load-Contour,**" i.e., the imaginary line, AM, the area between which and the extrados of the arch-ring represents a homogeneous load of the same density as the arch-ring, and equivalent to the actual load (above extrados), *vertical by vertical.*

355. Example of Reduced Load-Contour.—Fig. 388. Given an arch-ring of granite (heaviness = 170 lbs. per cubic foot) with a dead load of rubble (heav. = 140) and earth (heav. = 100), distributed as in figure. At the point p, of the extrados, the depth 5 feet of rubble is equivalent to a depth of $[\frac{140}{170}\times 5]=4.1$ ft. of granite, while the 6 feet of earth is equivalent to $[\frac{100}{170}\times 6]=3.5$ feet of granite. Hence the **Reduced Load-Contour** has an ordinate, above p, of 7.6 feet. That is, for each of several points of the arch-ring extrados reduce the rubble ordinate in the ratio of 170 : 140, and the earth ordinate in the ratio 170 : 100 and add the results, setting off the sum vertically from the points in the extrados*. In this way Fig. 389 is obtained and the area

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This is most conveniently done by graphics, thus: On a right-line set off 17 equal parts (of any convenient magnitude.) Call this distance OA. Through O draw another right line at any convenient angle (30° to 60°) with OA, and on it from O set off OB equal to 14 (for the rubble; or 10 for the earth) of the same equal parts. Join AB. From O toward A set off all the rubble ordinates to be reduced, (each being set off from O) and through the other extremity of each draw a line parallel to AB. The reduced ordinates will be the respective lengths, from O, along OB, to the intersections of these parallels with OB.

* With the dividers.

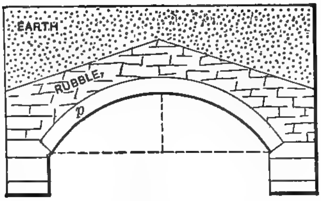

Fig. 388.

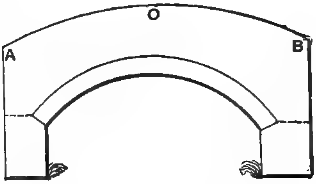

Fig. 389.

there given is to be treated as representing homogeneous granite one foot thick. This, of course, now includes the arch-ring also. *AB* is the "reduced load-contour."

356. Live Loads.—In discussing a railroad arch bridge the "live load" (a train of locomotives, e.g., to take an extreme case) can not be disregarded, and for each of its positions we have a separate **Reduced Load-Contour.**

Example.—Suppose the arch of Fig. 388 to be 12 feet wide (not including spandrel walls) and that a train of locomotives weighing 3,000 lbs. per running foot of the track covers one half of the span. Uniformly * distributed laterally over the width, 12 ft., this rate of loading is equivalent to a masonry load of one foot high and a heaviness of 250 lbs. per cubic ft., i.e., is equivalent to a height of 1.4 ft. of granite masonry [since $\frac{250}{170}\times 1.0=1.4$] over the half span considered. Hence from Fig. 390 we obtain Fig. 391 in an obvious manner. Fig. 391 is now ready for graphic treatment.

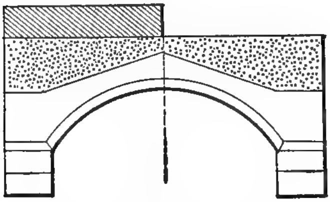

Fig. 390.

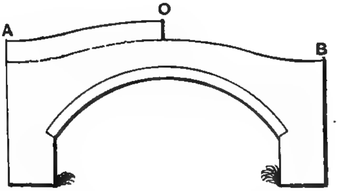

Fig. 391.

357. Piers and Abutments.—In a series of equal arches the pier between two consecutive arches bears simply the weight of the two adjacent semi-arches, plus the load immediately above the pier, and ∴ does not need to be as large as the abutment of the first and last arches, since these latter must be prepared to resist the oblique thrusts of their arches without help from the thrust of another on the other side.

---

* If the earth-filling is shallow, the laminæ directly under the track probably receive a greater pressure than the others.

In a very long series of arches it is sometimes customary to make a few of the intermediate piers large enough to act as abutments. These are called "abutment piers," and in case one arch should fall, no others would be lost except those occurring between the same two abutment piers as the first. See Fig. 392. *A* is an abutment-pier.

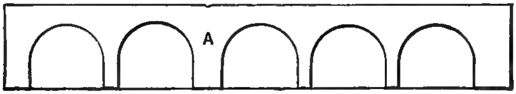

FIG. 392.

GRAPHICAL TREATMENT OF ARCH.

358.—Having found the "reduced load-contour," as in preceding paragraphs, for a given arch and load, we are ready to proceed with the graphic treatment, i.e., the first given, or assumed, form and thickness of arch-ring is to be investigated with regard to stability. It may be necessary to treat, separately, a lamina under the spandrel wall, and one under the interior loading. The constructions are equally well adapted to arches of all shapes, to Gothic as well as circular and elliptical.

359.—Case I. Symmetrical Arch and Symmetrical Loading.—(The "steady" (permanent) or "dead" load on an arch is usually symmetrical). Fig. 393. From symmetry we need

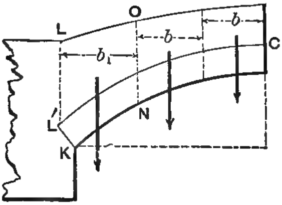

FIG. 393.

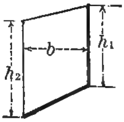

FIG. 394.

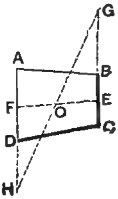

FIG. 395.

deal with only one half (say the left) of the arch and load. Divide this semi-arch and load into six or ten divisions by vertical lines; these divisions are *considered* as *trapezoids* and should have the same horizontal width $= b$ (a convenient whole number of feet) except the last one, *LKN*, next the abutment, and this is a pentagon of a different width b_1, (the remnant of the horizontal distance *LC*). The weight of masonry in each division is equal to (the area of division)×(unity thickness of lamina)×(weight of a cubic unit of arch-ring). For example for a division having an area of 20 sq. feet, and composed of masonry weighing 160 lbs. per cubic foot, we have $20 \times 1 \times 160 = 3{,}200$ lbs., applied through the centre of gravity of the division. The area of a trapezoid, Fig. 394, is $\frac{1}{2}b(h_1+h_2)$, and its centre of gravity may be found, Fig. 395, by the construction of Prob. 6, in § 26; or by § 27*a*. The weight of the pentagon *LN*, Fig. 393, and its line of application (through centre of gravity) may be found by combining results for the two trapezoids into which it is divided by a vertical through *K*. See § 21.

Since the weights of the respective trapezoids (*excepting LN*) are proportional to their middle vertical intercepts [such as $\frac{1}{2}(h_1+h_2)$ Fig. 394] these intercepts (transferred with the dividers) may be used directly to form the load-line, Fig. 396, or proportional parts of them if more convenient. The force scale, which this implies, is easily computed, and a proper length calculated to represent the weight of the odd division *LN*; i.e., 1 . . . 2 on the load-line.

Now consider *A*, the middle point of the abutment joint, Fig. 396, as the starting point of an equilibrium polygon (or abutment of a linear arch) for a given loading, and require that this equilibrium polygon shall pass through *p*, the middle of the crown joint, and through the middle of the abutment joint on the right (not shown in figure).

Proceed as in § 342, thus determining the polygon *Ap* for the half-arch. Draw joints in the arch-ring through those points where the extrados is intersected by the ver-

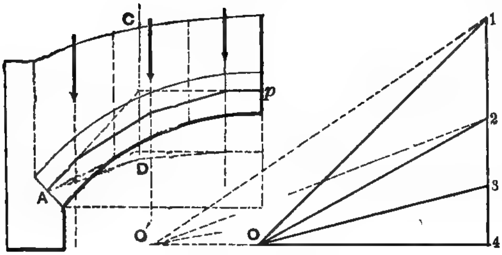

FIG. 396.

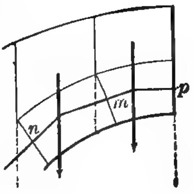

FIG. 397.

tical separating the divisions (not the gravity verticals). The points in which these joints are cut by the segments of the equilibrium polygon, Fig. 397, are (very nearly, if the joint is not more than 60° from *p*, the crown) the points of application in these joints, respectively, of the resultant pressures on them, (if this is the "true linear arch" for this arch and load) while the amount and direction of each such pressure is given by the proper ray in the force-diagram.

If at any joint so drawn the linear arch (or equilibrium polygon) passes outside the middle third of the arch-ring, the point *A*, or *p*, (or both) should be judiciously moved (within the middle third) to find if possible a linear arch which keeps within limits at all joints. If this is found impossible, the thickness of the arch-ring may be increased at the abutment (giving a smaller increase toward the crown) and the desired result obtained; or a change in the distribution or amount of the loading, if allowable, may gain this object. If but one linear arch can be drawn within the middle third, it may be considered the "true" one; if several, the one most nearly co-inciding with the middles of the joints (see §§ 351 and 352) is so considered.

360.—Case II. Unsymmetrical Loading on a Symmetrical Arch (e.g., arch with live load covering one half-span as in Figs. 390 and 391). Here we must evidently use a full force diagram, and the full elevation of the arch-ring and load

See Fig. 398. Select three points *A*, *p*, and *B*, as follows, to determine a *trial equilibrium polygon*:

Select *A* at the *lower limit* of the middle third of the abutment-joint at the end of the span which is the more heavily-loaded; in the other abutment-joint take *B* at the upper limit of the middle third; and take *p* in the middle of the crown-joint. Then by § 341 draw an equilibrium polygon (i.e., a linear arch) through these three points for the given set of loads, and if it does not remain within the middle third, try other positions for *A*, *p*, and *B*, within the middle third. As to the "true linear arch" alterations of the design, etc., the same remarks apply as already given in Case I. Very frequently it is not necessary to draw more than one linear arch, for a given loading, for even if one could be drawn nearer the middle of the arch-ring than the first, that fact is almost always apparent on mere inspection, and the one already drawn (if within middle third) will furnish values sufficiently accurate for the pressures on the respective joints, and their direction angles.

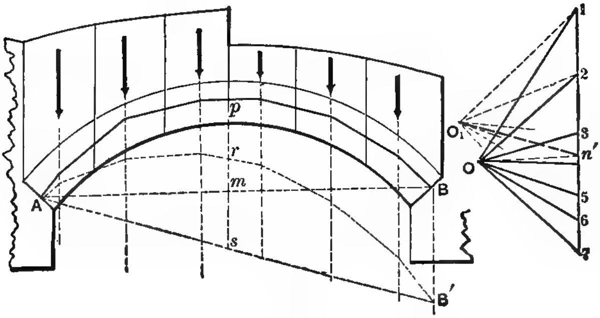

FIG. 398.

360a.—The design for the arch-ring and loading is not to be considered satisfactory until it is ascertained that for the dead load and any possible combination of live-load (in addition) the pressure at any joint is

(1.) Within the middle third of that joint;

(2.) At an angle of < 30° with the normal to joint-surface.

(3.) Of a *mean* pressure per square inch not > than $^1/_{20}$ of the ultimate crushing resistance. (See § 348.)

§ 361. **Abutments.**—The abutment should be compactly and solidly built, and is then treated as a single rigid mass. The pressure of the lowest voussoir upon it (considering a lamina one foot thick) is given by the proper ray of the force diagram (O .. 1, e. g., in Fig. 396) in amount and direction. The stability of the abutment will depend on the amount and direction of the resultant obtained by combining that pressure P_a with the weight G of the abutment and its load, see Fig. 399. Assume a probable width RS for the abutment and compute the weight G of the corresponding abutment $OBRS$ and $MNBO$, and find the centre of gravity of the whole mass C. Apply G in the vertical through C, and combine it with P_a at their intersection D. The resultant P should not cut the base RS in a point beyond the middle third (or, if this rule gives too massive a pier, take such a width that the pressure per square inch at S shall not exceed a safe value as computed from § 362.) After one or two trials a satisfactory width can be obtained. We should also be assured that the angle PDG is less than 30°. The horizontal joints above RS should also be tested as if each were, in turn, the lowest base, and if necessary may be inclined (like mn) to prevent slipping. On no joint should the maximum pressure per square inch be > than $^1/_{10}$ the crushing strength of the cement. Abutments of firm natural rock are of course to be preferred where they can be had. If water penetrates under an abutment its buoyant effort lessens the weight of the latter to a considerable extent.

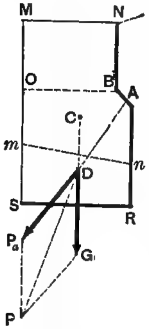

Fig. 399.

362. Maximum Pressure Per Unit of Area When the Resultant Pressure Falls at Any Given Distance from the Middle; according to Navier's theory of the distribution of the pressure; see § 346. Case I. Let the resultant pressure P, Fig. 400, (*a*), fall within the middle third, a distance $= nd$ ($< \frac{1}{6}\ d$) from the middle of joint (d = depth of joint.) Then we have the following relations:

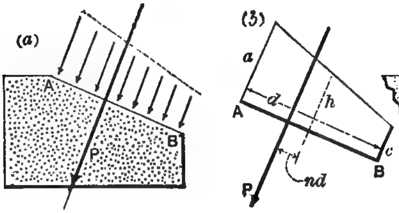

FIG. 400.

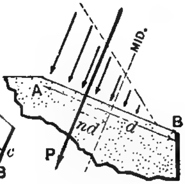

FIG. 401.

p (the mean press. per. sq. in.), p_m (max. press. per sq. in.), and p_n (least press. per sq. in.) are proportional to the lines h (mid. width), a (max. base), and c (min. base) respectively, of a trapezoid, Fig. 400, (*b*), through whose centre of gravity P acts. But (§ 26)

$$nd = \frac{d}{6} \cdot \frac{a-c}{a+c} \text{ i.e., } n = \frac{1}{6}\ \frac{a-h}{h} \text{ or } a = h\ (6n+1)$$

$\therefore\ p_m = p\ (6n+1)$. Hence the following table:

If $nd =$	$\frac{1}{6}\ d$	$^1/_9\ d$	$^1/_{18}\ d$	then the max.
press. $p_m =$	2	$^5/_3$	$^4/_3$	times the mean pressure.

Case II. Let P fall outside the mid. third, a distance $=$ nd ($> \frac{1}{6}\ d$) from the middle of joint. Here, since the joint is not considered capable of withstanding tension, we have a triangle, instead of a trapezoid. Fig. 401. First compute the mean press. per sq. in.

$$p = \frac{P \text{ (lbs.)}}{(1-2n)\ 18\ d \text{ inches}}$$ or from this table: (lamina one foot thick).

For $nd =$	$\frac{4}{18}\,d$	$\frac{5}{18}\,d$	$\frac{6}{18}\,d$	$\frac{7}{18}\,d$	$\frac{8}{18}\,d$	$\frac{9}{18}\,d$
$p =$	$\frac{1}{10} \cdot \frac{P}{d}$	$\frac{1}{8} \cdot \frac{P}{d}$	$\frac{1}{6} \cdot \frac{P}{d}$	$\frac{1}{4} \cdot \frac{P}{d}$	$\frac{1}{2} \cdot \frac{P}{d}$	infinity.

(*d* in inches and *P* in lbs.; with arch lamina 1 ft. in thickness.)

Then the maximum pressure (at A, Fig. 401) $p_m, = 2p$, becomes known, in lbs. per sq. in.

362a. Arch-ring under Non-vertical Forces.—An example of this occurs when a vertical arch-ring is to support the pressure of a liquid on its extrados. Since water-pressures are always at right angles to the surface pressed on, these pressures on the extradosal surface of the arch-ring form a system of non-parallel forces which are normal to the curve of the extrados at their respective points of application and lie in parallel vertical planes, parallel to the faces of the lamina. We here assume that the extradosal surface is a cylinder (in the most general sense) whose rectilinear elements are $\urcorner$ to the faces of the lamina. If, then, we divide the length of the extrados, from crown to each abutment, into from six to ten parts, the respective pressures on the corresponding surfaces are obtained by multiplying the area of each by the depth of its centre of gravity from the upper free surface of the liquid, and this product by the weight of a unit of volume of the liquid; and each such pressure may be considered as acting through the centre of the area. Finally, if we find the resultant of each of these pressures and the weight of the corresponding portion of the arch-ring, these resultants form a series of non-vertical forces in a plane, for which an equilibrium polygon can then be passed through three assumed points by § 378a, these three points being taken in the crown-joint and the two abutment-joints. As to the "true linear arch" see § 359.

As an extreme theoretic limit it is worth noting that if the extrados and intrados of the arch-ring are concentric circles; if the weights of the voussoirs are neglected; and if the rise of the arch is very small compared with the depth of the crown below the water surface, then the *circularcentre-line of the arch-ring is the "true linear arch."*

CHAPTER XI.

ARCH-RIBS.

Note.—The methods used in this chapter for the treatment of the "elastic arch" are practically the equivalent of those based on the theory of "Least Work."

364. Definitions and Assumptions.—An **arch-rib** (or *elastic-arch*, as distinguished from a block-work arch) is a rigid curved beam, either solid, or built up of pieces like a truss (and then called a braced arch) the stresses in which, under a given loading and with prescribed mode of support it is here proposed to determine. The rib is supposed symmetrical about a vertical plane containing its axis or middle line, and the Moment of Inertia of any cross section is understood to be referred to a gravity axis of the section, which (the axis) is perpendicular to the said vertical plane. It is assumed that in its strained condition under a load, the shape of the rib differs so little from its form when unstrained that the change in the abscissa or ordinate of any point in the rib axis (a curve) may be neglected when added (algebraically) to the coordinate itself; also that the dimensions of a cross-section are small compared with the radius of curvature at any part of the curved axis, and with the span.

365. Mode of Support.—Either extremity of the rib may be *hinged* to its pier (which gives freedom to the end-tangent-line to turn in the vertical plane of the rib when a load is applied); or may be *fixed*, i.e., so built-in, or bolted rigidly to the pier, that the end-tangent-line is incapable of changing its direction when a load is applied. A hinge may be inserted anywhere along the rib, and of course destroys the rigidity, or resistance to bending at that point. (A hinge having its pin horizontal ┐ to the axis of the rib is meant). Evidently no more than three such hinges could be introduced along an arch-rib between two piers; unless it is to be a *hanging* structure, acting as a suspension-cable.

366. Arch Rib as a Free Body.—In considering the whole rib free it is convenient, for graphical treatment, that no section be conceived made at its extremities, if fixed; hence in dealing with that mode of support the end of the rib will be considered as having a rigid prolongation reaching to a point vertically above or below the pier junction, an unknown distance from it, and there acted on by a force of such unknown amount and direction as to preserve the actual extremity of the rib and its tangent line in the same position and direction as they really are. As an illustration of this Fig. 402 shows *free* an arch rib. *ONB*, with its extremities *O* and *B fixed* in the piers, with no hinges, and bearing two loads P_1 and P_2. The other forces of the system holding it in equilibrium are the horizontal and vertical components, of the pier reactions (H, V, H_n, and V_n), and in this case of fixed ends each of these two reactions is a single force not intersecting the end of the rib, but cutting the vertical through the end in some point F (on the left; and in G on the right) at some vertical distance c, (or d), from the end. Hence the utility of these imaginary prolongations OQF, and BRG, the pier being supposed removed. Compare Figs. 348 and 350.

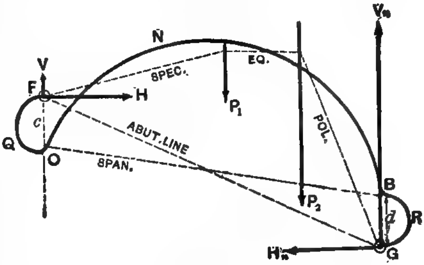

FIG. 402.

The imaginary points, or hinges, F and G, will be called *abutments* being such for the special equilibrium polygon (dotted line), while O and B are the real ends **of the curved** beam, or rib.

In this system of forces there are five **unknowns, viz.:** V, V_n, $H = H_n$, and the distances c and d. **Their determination** by analysis, even if the rib is a **circular arc, is extremely** intricate and tedious; but by graphical statics (Prof. Eddy's method; see § 350 for reference), it is comparatively simple and direct and applies to *any shape* of rib, and is sufficiently accurate for practical purposes. This method consists of constructions leading to the location of the "special equilibrium polygon" and its force diagram. In case the rib is hinged to the piers, the reactions of the latter act through these hinges, **Fig. 403,** i e., the abutments of the special equilibrium polygon coincide with the ends of the rib O and B, and for a given rib and load the unknown quantities are only three V, V_n, and H; (strictly there are four; but $\Sigma X = 0$ gives $H_n = H$). The solution by analytics is possible only for ribs of simple algebraic curves and is long and cumbrous; whereas Prof. Eddy's graphic method is comparatively brief and simple and is applicable to any shape of rib whatever.

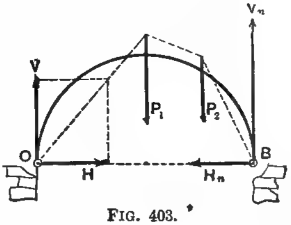

FIG. 403.

367. Utility of the Special Equilibrium Polygon and its force diagram. The use of locating these will now be illustrated [See § 332]. As proved in §§ 332 and 334 the compression in each "rod" or segment of the "special equilibrium polygon" is the anti-stress resultant of the cross sections in the corresponding portion of the beam, rib, or other structure, the value of this compression (in lbs. or tons) being measured by the length of the parallel ray in the force diagram. Suppose that in some way (to be explained subsequently) the special equilibrium polygon and its force diagram have been drawn for the arch-rib in Fig. 404 having **fixed ends,** O and B, and no hinges; required the elastic stresses in any cross-section of the rib as at m. Let the

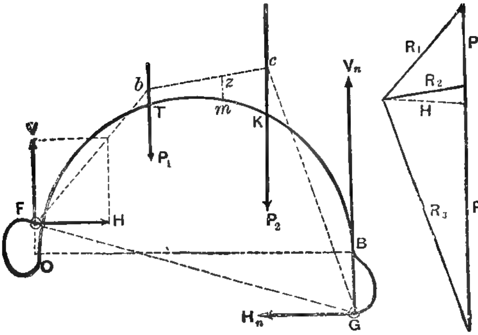

FIG. 404.

scale of the force-diagram on the right be 200 lbs. to the inch, say, and that of the space-diagram (on the left) 30 ft. to the inch.

The cross section m lies in a portion TK, of the rib, corresponding to the rod or segment bc of the equilibrium polygon; hence its anti-stress-resultant is a force R_2 acting in the line bc, and of an amount given in the force-diagram. Now R_2 is the resultant of V, H, and P_1, which with the elastic forces at m form a system in equilibrium, shown in Fig. 405; the portion $FOTm$ being considered free. Hence taking the tangent line and the normal at m as axes we should have Σ (tang. comps.) $= 0$; Σ (norm. comps.) $= 0$; and Σ (moms. about gravity axis of the section at m) $= 0$, and could thus find the unknowns p_1, p_2, and J, which appear in the expressions p_1F the thrust, $\frac{p_2 I}{e}$ the moment of

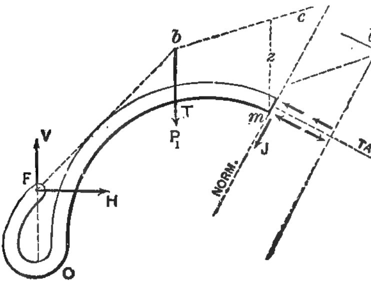

FIG. 405.

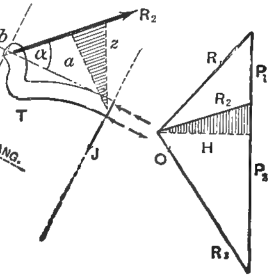

FIG. 406.

the stress-couple, and J the shear. These elastic stresses are classified as in § 295, which see. p_1 and p_2 are *lbs. per square inch*, J is *lbs.*, e is the distance from the horizontal gravity axis of the section to the outermost element of area, (where the compression or tension is p_2 lbs. per sq. in., as due to the stress-couple alone) while I is the "moment of inertia" of the section about that gravity axis. [See §§ 247 and 295; also § 85]. Graphics, however, gives us a more direct method, as follows: Since R_2, in the line bc, is the equivalent of V, H, and P_1, the stresses at m will be just the same as if R_2 acted directly upon a lateral prolongation of the rib at T (to intersect bc Fig. 405) as shown in Fig. 406, this prolongation Tb taking the place of TOF in Fig. 405. The force diagram is also reproduced here. Let a denote the length of the ┐ from m's gravity axis upon bc, and z the vertical intercept between m and bc. For this imaginary free body, we have,

$$\text{from } \Sigma\,(\text{tang. compons.})=0,\ R_2 \cos a = p_1 F$$

$$\text{and from } \Sigma\,(\text{norm. compons.})=0,\ R_2 \sin a = J$$

$$\left.\begin{array}{l}\text{while from } \Sigma\,(\text{moms. about}\\ \text{the gravity axis of } m)=0,\end{array}\right\}\ \text{we have } R_2 a = \frac{p_2 I}{e}.$$

But from the two similar triangles (shaded; one of them is in force diagram) $a:z::H:R_2 \ \therefore\ R_2a = Hz$, whence we may rewrite these relations as follows (with a general statement), viz.:

If the Special Equilibrium Polygon and Its Force Diagram Have Been Drawn for a given arch-rib, of given mode of support, and under a given loading, then in any cross-section of the rib, we have (F = area of section):

(1.) The Thrust, i.e., $p_1F=$ $\left\{\right.$ The projection of the *proper ray* (of the force diagram) upon the *tangent line* of the rib **drawn at the given section.**

(2.) **The Shear**, i.e., J, = (upon which depends the shearing stress in the web). (See §§ 253 and 256).	**The projection of the proper** ray (of the force diagram) **up**on the *normal* to the rib curve at the given section.
(3.) **The Moment of the stress couple**, i.e., $\frac{p_2 I}{e}$, =	The product (Hz) of the H (or pole-distance) of the force-diagram by the vertical distance of the gravity axis of the section from the spec. equilibrium polygon.

By the "proper ray" is meant that ray which is parallel to the segment (of the equil. polygon) immediately under or above which the given section is situated. Thus in Fig. 404, the proper ray for any section on TK is R_2; on KB, R_3; on TO, R_1. The *projection* of a ray upon any given tangent or normal, is easily found by drawing through each end of the ray a line ⏋ to the tangent (or normal); the length between these ⏋'s on the tangent (or normal) is the force required (by the scale of the force diagram). We may thus construct a shear diagram, and a thrust diagram for a given case, while the successive vertical intercepts between the rib and special equilibrium polygon form a moment diagram. For example if the z of a point m is ½ inch in a space diagram drawn to a scale of 20 feet to the inch, while H measures 2.1 inches in a force diagram constructed on a scale of ten tons to the inch, we have, for the moment of the stress-couple at m, $M = Hz = [2.1 \times 10]$ tons $\times [\frac{1}{2} \times 20]$ ft. $= 210$ ft. tons.

368.—It is thus seen how a location of the special equilibrium polygon, and the lines of the corresponding force diagram, lead directly to a knowledge of the stresses in al the cross-sections of the curved beam under consideration bearing a given load; or, vice versâ, leads to a statement of conditions to be satisfied by the dimensions of the rib for proper security.

It is here supposed that the rib has sufficient later bracing (with others which lie parallel with it) to prevent buckling sideways in any part like a long column. Before proceeding to the complete graphical analysis of the different cases of arch-ribs, it will be necessary to devote the next few paragraphs to developing a few analytical relations in the theory of flexure of a curved beam, and to giving some processes in "graphical arithmetic."

369. Change in the Angle Between Two Consecutive Rib Tangents when the rib is loaded, as compared with its value before loading. Consider any small portion (of an arch rib) included between two consecutive cross-sections; Fig. 407. *KHGW* is its unstrained form. Let *EA*, $= ds$, be the original length of this portion of the rib axis. The length of all the fibres ($\parallel$ to rib-axis) was originally $=ds$ (nearly) and the two consecutive tangent-lines, at *E* and *A*, made an angle $= d\theta$ originally, with each other. While under strain, however, all the fibres are shortened *equally* an amount $d\lambda_1$, by the uniformly distributed tangential thrust, but are *unequally* shortened (or lengthened, according as they are on one side or the other of the gravity axis *E*, or *A*, of the section) by the system of forces making what we call the "stress couple," among which the stress at the distance e from the gravity axis *A* of the section is called p_2 per square inch; so that the tangent line at *A′* now takes the direction *A′D* $\perp$ to *H′A′G′* instead of *A′C* (we suppose the section at *E* to remain fixed, for conveni-

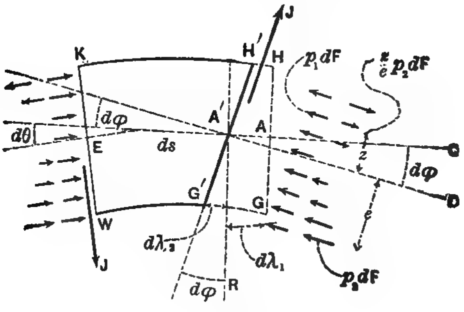

FIG. 407.

ence, since the change of angle between the two tangents depends on the stresses acting, and not on the new position in space, of this part of the rib), and hence the angle between the tangent-lines at E and A (originally $= d\theta$) is now increased by an amount $CA'D = d\varphi$ (or $G'A'R = d\varphi$); $G'H'$ is the new position of GH. We obtain the value of $d\varphi$ as follows: That part ($d\lambda_2$) of the shortening of the fibre at G, at distance e from A due to the force p_2dF, is § 201 eq. (1), $d\lambda_2 = \frac{p_2ds}{E}$. But, geometrically, $d\lambda_2$ also $=ed\varphi$,

$$\therefore \quad Eed\varphi = p_2ds \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

But, letting M denote the moment of the stress-couple at section A (M depends on the loading, mode of support, etc., in any particular case) we know from § 295 eq. (6) that $M=\frac{p_2I}{e}$, and hence by substitution in (1) we have

$$d\varphi = \frac{Mds}{EI} \quad . \quad . \quad . \quad . \quad (2)$$

[**If the arch-rib** in question has less than three hinges, the equal shortening of the fibres due to the thrust (of the block in last figure) p_1F, will have an indirect effect on the angle $d\varphi$. This will be considered later.]

370. Total Change $\left[\text{i.e.} \int d\varphi\right]$ **in the Angle Between the End Tangents of a Rib,** before and after loading. Take the example in Fig. 408 of a rib fixed at one end and hinged at

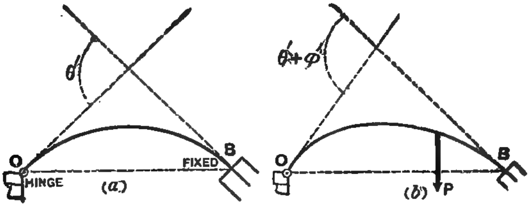

FIG. 408.

the other. When the rib is unstrained (as it is supposed to be, on the left, its own weight being neglected; it is not supposed *sprung* into place, but is entirely without strain) then the angle between the end-tangents has some value $\theta' = \int_0^B d\theta =$ the sum of the successive small angles $d\theta$ for each element ds of the rib curve (or axis). After loading [on the right, Fig. 408], this angle has increased having now a value

$$\theta' + \int_0^B d\varphi, \text{ i.e., a value} = \theta' + \int_0^B \frac{Mds}{EI} \quad . \quad . \quad . \quad \text{(I.)}$$

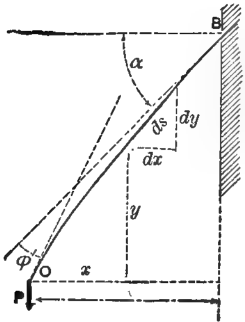

FIG. 409.

There must be no hinge *between* O and B.

§ 371. Example of Equation (I.) in Analysis.—A *straight*, homogeneous, prismatic beam, Fig. 409, its own weight neglected, is fixed obliquely in a wall. After placing a load P on the free end, required the angle between the end-tangents. This was zero before loading ∴ its value after loading is

$$= 0 + \varphi' = 0 + \frac{1}{EI}\int_0^B Mds$$

By considering free a portion between O and any ds of the beam, we find that $M = Px =$ mom. of the stress couple. The flexure is so slight that the angle between any ds and its dx is still practically $= \alpha$ (§ 364), and ∴ $ds = dx \sec \alpha$. Hence, by substitution in eq. (I.) we have

$$\varphi' = \frac{1}{EI}\int_0^B Mds = \frac{P \sec \alpha}{EI}\int_0^{l \cos \alpha} xdx = \frac{P \sec \alpha}{EI}\left[\frac{x^2}{2}\right]_0^{l \cos \alpha};$$

$$\therefore \varphi' = \frac{P(\cos \alpha)l^2}{2EI} \text{ [Compare with § 237].}$$

It is now apparent that if *both* ends of an arch rib are *fixed*, when unstrained, and the rib be then loaded (within elastic limit, and deformation slight) we must have

$$\int_0^B (Mds \div EI) = \text{zero, since } \varphi' = 0.$$

372. Projections of the Displacement of any Point of a Loaded Rib Relatively to Another Point and the Tangent Line at the Latter.—(There must be no hinge between O and B). Let O be the point whose displacement is considered and B the other point. Fig. 410. If B's tangent-line is fixed while the extremity O is not supported in any way (Fig. 410) then a load P put on, O is displaced to a new position O_n.

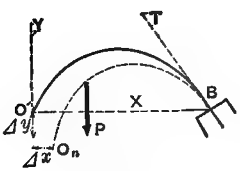

FIG. 410.

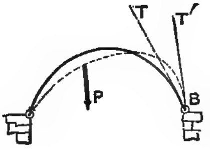

FIG. 411.

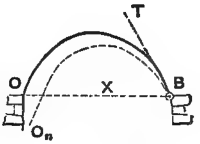

FIG. 412.

With O as an origin and OB as the axis of X, the *projection* of the displacement OO_n upon X, will be called $\varDelta x$, that upon Y, $\varDelta y$.

In the case in Fig. 410, O's displacement with respect to B and its tangent-line BT, is *also* its *absolute displacement in space*, since neither B nor BT has moved as the rib changes form under the load. In Fig. 411, however, the extremities O and B are both hinged to piers, or supports, the dotted line showing its form when deformed under a load. The hinges are supposed immovable, the rib being free to turn about them without friction. The dotted line is the changed form under a load, and the *absolute* displacement of O is zero; but not so its displacement relatively to B and B's tangent BT, for BT has moved to a new position BT'. To find this relative displacement conceive the new curve of the rib superposed on the old in a way that B and BT may coincide with their original positions, Fig. 412. It is now seen that O's displacement relatively to B and BT is not zero but $=OO_{\prime\prime}$, and has a small Δx but a comparatively large Δy. In fact for this case of hinged ends, *piers immovable,* rib continuous between them, and deformation slight, we shall write $\Delta x=$ zero as compared with Δy, the *axis X passing through OB*).

373. Values of the X and Y Projections of O's Displacement Relatively to B and B's Tangent; the origin being taken at O. Fig. 413. Let the coordinates of the different points E, D, C, etc., of the rib, referred to O and an arbitrary X axis, be x and y, their *radial distances* from O being u (i.e., u for C, u' for D, etc.; in general, u). $OEDC$ is the *unstrained form* of the rib, (e.g., the form it would assume if it lay flat on its side on a level platform, under no straining forces), while $O_{\prime\prime}E''D'CB$ is its form under some loading, i.e., under strain. (The superposition above mentioned (§ 372) is supposed already made if necessary, so that BT is tangent at B to both forms). Now conceive the rib OB to pass into its strained condition by the successive bending of each ds in turn. The straining or bending of the first ds, BC, through the small angle $d\varphi$ (dependent on the moment of the stress couple at C in the strained condition) causes the whole finite piece OC to turn about C as a centre through the same small angle $d\varphi$; hence the point O describes a small linear arc $OO'=\delta v$, whose radius $=u$ the hypothenuse of the x and y of C, and whose value $\therefore$ is $\delta v=u\,d\varphi$.

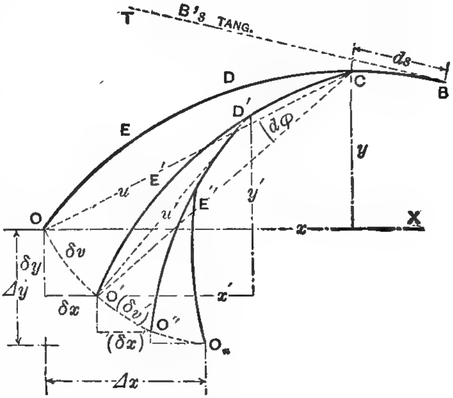

FIG. 413.

Next let the section D, now at D', turn through its proper angle $d\varphi'$ (dependent on its stress-couple) carrying with it the portion $D'O'$, into the position $D'O''$, making O' describe a linear arc $O'O''=(\delta v)'=u'd\varphi'$, in which $u'=$ the hypothenuse on the x' and y' (of D), (the deformation is so slight that the co-ordinates of the different points referred to O and X are not appreciably affected). Thus, each section having been allowed to turn through the angle proper to it, O finally reaches its position, O_n, of displacement. Each successive δv, or linear arc described by O, has a shorter radius. Let δx, $(\delta x)'$, etc., represent the projections of the successive (δv)'s upon the axis X; and similarly δy, $(\delta y)'$ etc., upon the axis Y. Then the total X projection of the curved line $O \ldots . O_n$ will be

$$\varDelta x = \int \delta x \text{ and similarly } \varDelta y = \int \delta y \quad . \quad . \quad . \quad (1)$$

But $\delta v = u\, d\, \varphi$, and from similar right-triangles, $\delta x : \delta v :: y : u$ and $\delta y : \delta v :: x : u \therefore \delta x = yd\varphi$ and $\delta y = xd\varphi$; whence, (see (1) and (2) of §369)

$$\varDelta x = \int \delta x = \int yd\varphi = \int_0^B \frac{Myds}{EI} \quad . \quad . \quad . \quad (\text{II.})$$

$$\text{and } \varDelta y = \int \delta y = \int xd\varphi = \int_0^B \frac{Mxds}{EI} \quad . \quad . \quad . \quad (\text{III.})$$

If the rib is homogeneous E is constant, and if it is of constant cross-section, all sections being similarly cut by the vertical plane of the rib's axis (i.e., if it is a "curved prism") I, the moment of inertia is also constant.

374. Recapitulation of Analytical Relations, for reference. (*Not applicable if there is a hinge between O and B*)

Total Change in Angle between tangent-lines O and B $\Big\} = \int_0^B \frac{Mds}{EI} \quad . \quad . \quad . \quad . \quad (\text{I.})$

The X-Projection of O's Displacement Relatively to B and B's tangent-line; (*the origin being at O*) and the axes X and Y ⅂ to each other) $\Bigg\} = \int_0^B \frac{Myds}{EI} \quad . \quad . \quad . \quad (\text{II.})$

$$\left.\begin{array}{l}\textbf{The Y-Projection of O's Displacement,}\\ \text{etc., as above.}\end{array}\right\} = \int_0^B \frac{Mxds}{EI} \quad . \quad . \quad \textbf{(III.)}$$

Here x and y are the co-ordinates of points in the rib-curve, ds an element of that curve, M the moment of the stress-couple in the corresponding section as induced by the loading, or constraint, of the rib.

(The results already derived for deflections, slopes, etc., for straight beams, could also be obtained from these formulae, I., II. and III. In these formulae also it must be remembered that no account has been taken of the shortening of the rib-axis by the thrust, nor of the effect of a change of temperature.)

374a. Résumé of the Properties of Equilibrium Polygons and their Force Diagrams, for Systems of Vertical Loads.—See §§ 335 to 343. Given a system of loads or vertical forces, P_1, P_2, etc., Fig. 414, and two abutment verticals, F' and G'; if we lay off, vertically, to form a "load-line," $1..2 = P_1$, $2...3 = P_2$, etc., select any **Pole**, O_1, and join $O_1 ... 1$, $O_1 ... 2$, etc.; also, beginning at any point F_1 in the vertical F', if we draw $F_1 ... a \parallel$ to $O_1 .. 1$ to intersect the line of P_1; then $ab \parallel$ to $O_1 .. 2$, and so on until finally a point G_1, in G', is determined; then the figure $F_1 .abc\ G_1$ is an *equilibrium polygon* for the given loads and load verticals, and $O_1 ... 1234$ is its "force diagram." The former is so called because the short segments F_1a ab, etc., if considered to be rigid and imponderable rods, in a vertical plane, hinged to each other and the terminal ones to abutments F_1 and G_1, would be in equilibrium under the given loads hung at the joints. An infinite number of equilib-

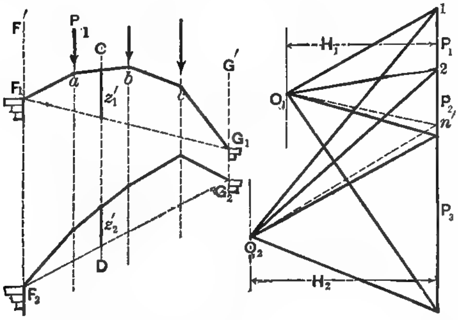

FIG. 414.

rium polygons may be drawn for the given loads and abutment-verticals, by choosing different poles in the force diagram. [One other is shown in the figure; O_2 is its pole. (F_1 G_1 and F_2 G_2 are abutment lines.)] For all of these the following statements are true:

(1.) A line through the pole, ‖ to the abutment line cuts the load-line in the same point n', whichever equilibrium polygon be used (∴ any one will serve to determine n'

(2.) If a vertical CD be drawn, giving an intercept z' in each of the equilibrium polygons, the product Hz' is the same for all the equilibrium polygons. That is, (see Fig. 414) for any two of the polygons we have

$$H_1 : H_2 :: z_2' : z_1' \;;\ \text{or}\ H_2 z_2' = H_1 z_1'.$$

(3.) The compression in each rod is given by that "ray" (in the force diagram) to which it is parallel.

(4.) The "pole distance" H, or ┐ let fall from the pole upon the load-line, divides it into two parts which are the vertical components of the compressions in the abutment-rods *respectively* (the other component being horizontal); H is the horizontal component of each (and, in fact, of each of the compressions in all the other rods). The compressions in the extreme rods may also be called the abutment reactions (oblique) and are given by the extreme rays.

(5.) **Three Points** [not all in the same segment (or rod)] determine an equilibrium polygon for given loads. Having given, then, three points, we may draw the equilibrium polygon by §341.

375. Summation of Products. Before proceeding to treat graphically any case of arch-ribs, a few processes in graphical arithmetic, as it may be called, must be presented, and thus established for future use.

To make a summation of products of two factors in each by means of an equilibrium polygon.

Construction. Suppose it required to make the summation $\Sigma (x z)$ *i. e.*, to sum the series

$$x_1 z_1 + x_2 z_2 + x_3 z_3 + \ . \ . \ . \quad \text{by graphics.}$$

Having first arranged the terms in the order of magnitude of the x's, we proceed as follows: Supposing, for illustration, that two of the z's (z_3 and z_4) are negative (dotted in figure) see Fig. 415. These quantities x and z may be of any nature whatever, anything capable of being represented by a length, laid off to scale.

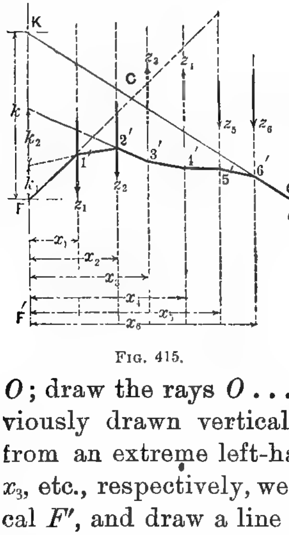

FIG. 415.

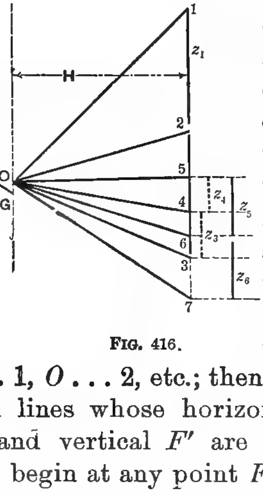

FIG. 416.

First, in Fig 416, lay off the z's in their order, end to end, on a vertical load-line taking care to lay off z_3 and z_4 *upward* in their turn. Take any convenient pole O; draw the rays $O \dots 1$, $O \dots 2$, etc.; then, having previously drawn vertical lines whose horizontal distances from an extreme left-hand vertical F' are made $= x_1$, x_2, x_3, etc., respectively, we begin at any point F, in the vertical F', and draw a line ‖ to $O \dots 1$ to intersect the x_1 vertical in some point; then $1'\ 2'$ ‖ to $O \dots 2$, and so on, following carefully the proper order. Produce the last segment ($6' \dots G$ in this case) to intersect the vertical F' in some point K. Let $KF = k$ (measured on the same scale as the x's), then the summation required is

$$\Sigma_1^6 (xz) = Hk.$$

H is measured on the scale of the z's, which need not be the same as that of the x's; in fact the z's may not be the same kind of quantity as the x's.

[PROOF.—From similar triangles $H : z_1 :: x_1 : k_1$, ∴ $x_1 z_1 = Hk_1$;
and “ “ “ $H : z_2 :: x_2 : k_2$, ∴ $x_2 z_2 = Hk_2$.

and so on. But $H(k_1+k_2+\text{etc.})=H\times\overline{FK}=Hk$].

376. Gravity Vertical.—From the same construction in Fig. 415 we can determine the line of action (or gravity vertical) of the resultant of the parallel vertical forces z_1, z_2, etc. (or loads); by prolonging the first and last segments to their intersection at C. The resultant of the system of forces or loads acts through C and is vertical in this case; its value being $= \Sigma\ (z)$, that is, it = the length 1 . . . 7 in the force diagram, interpreted by the proper scale. It is now supposed that the z's represent forces, the x's being their respective lever arms about F. If the z's represent the areas of small finite portions of a large plane figure, we may find a gravity-line (through C) of that figure by the above construction; each z being-applied through the centre of gravity of its own portion.

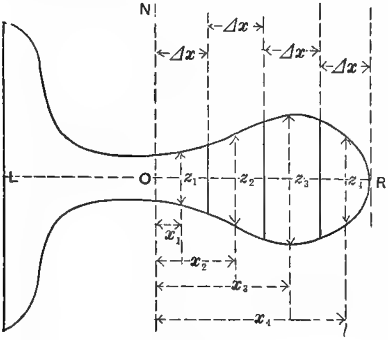

FIG. 416 *a.*

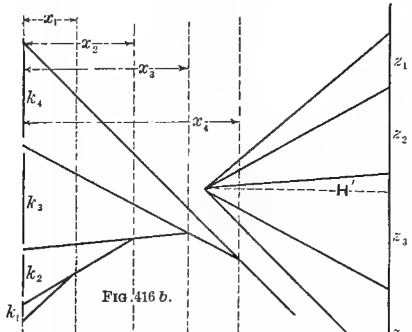

FIG. 416 *b.*

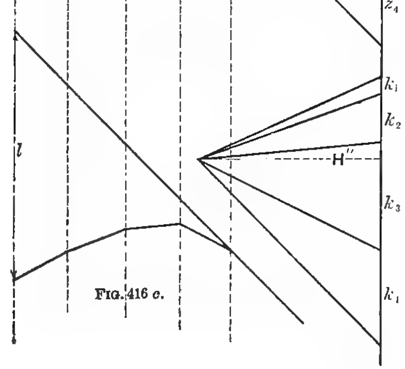

FIG. 416 *c.*

Calling the distance $\bar{x}$ between the verticals through C and F, we have also $\bar{x}\,.\,\Sigma\ (z) = \Sigma\ (xz)$ because $\Sigma\ (z)$ is the resultant of the ∥ z's. This is also evident from the proportion (similar triangles)

$$H : (1\,.\,.\,7) :: \bar{x} : k.$$

376a. Moment of Inertia (of Plane Figure) by Graphics.*— Fig. 416*a*. $I_N = ?$ First, for the portion on right. Divide OR into equal parts each $= \Delta x$. Let z_1, z_2, etc., be the middle ordinates of the strips thus obtained, and x_1, etc. their abscissas (of middle points).

Then we have approximately

$$I_N \text{ for } OR = \Delta x . z_1 x_1^2 + \Delta x . z_2 x_2^2 + \dots\dots$$
$$= \Delta x [(z_1 x_1) x_1 + (z_2 x_2) x_2 + \dots] \quad . . (1)$$

But by §375 we may construct the products z_1x_1, z_2x_2, etc., taking a convenient H', (see Fig. 416, (*b*)), and obtain k_1, k_2, etc., such that $z_1x_1 = H'k_1$, $z_2x_2 = H'k_2$, etc. Hence eq. (1) becomes :

$$I_N \text{ for } OR \text{ approx.} = H' \Delta x [k_1 x_1 + k_2 x_2 + \dots] \quad . . . (2)$$

By a second use of § 375 (see Fig. 416 *c*) we construct l, such that $k_1 x_1 + k_2 x_2 + \dots . = H''l$ [H'' taken at convenience]. ∴ from eq. (2) we have finally, (approx.),

$$I_N \text{ for } OR = H'H''l\Delta x \quad . . . . (3)$$

For example if $OR = 4$ in., with four strips, Δx would $=$ 1 in.; and if $H' = 2$ in., $H'' = 2$ in., and $l = 5.2$ in., then

$$I_N \text{ for } OR = 2 \times 2 \times 5.2 \times 1.0 = 20.8 \text{ biquad. inches.}$$

The I_N for OL, on the left of N, is found in a similar manner and added to I_N for OR to obtain the total I_N. The position of a gravity axis is easily found by cutting the shape out of sheet metal and balancing on a knife edge ; or may be obtained graphically by § 336 ; or 376.

377. Construction for locating a line vm (Fig. 417) at (*a*), in the polygon FG in such a position as to satisfy the two following conditions with reference to the vertical intercepts at 1, 2, 3, 4, and 5, between it and the given points 1, 2, 3, etc., of the perimeter of the polygon.

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* Another graphic method for this purpose will be found in § 76 (p. 80), of the author's *Notes and Examples in Mechanics.*

Condition I.—(Calling these intercepts u_1, u_2, etc., and their horizontal distances from a given vertical F, x_1, x_2, etc.) $\Sigma(u)$ is to $= 0$; i.e., the sum of the positive u's must be numerically $=$ that of the negative (which here are at 1 and 5). An infinite number of positions of vm will satisfy condition I.

Condition II.—$\Sigma(ux)$ is to $= 0$; i.e., the sum of the moments of the positive u's about F must $=$ that of the negative u's. i.e., the moment of the resultant of the positive u's must $=$ that of the resultant of the negative; and $\therefore$ (Condit. I being already satisfied) these two resultants must be *directly opposed* and *equal*. But the ordinates u in (a) are individually equal to the difference of the *full* and *dotted* ordinates in (b) with the same x's $\therefore$ the conditions may be rewritten:

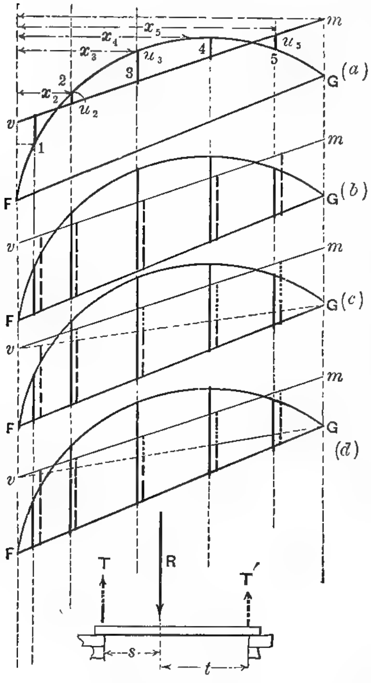

FIG. 417.

I. Σ (full ords. in (b)) $=$ Σ (dotted ords. in (b))

II. Σ [each full ord. in (b) $\times$ its x] $= \Sigma$ [each dotted ord. in (b) $\times$ its x] i.e., the centres of gravity of the full and of the dotted in (b) must lie in the *same vertical.*

Again, by joining vG, we may divide the dotted ordinates of (b) into two sets which are dotted, and broken, respectively, in (c) Then, finally, drawing in (d),

R, the resultant of full ords. of (c)
T, " " " broken " " "
T', " " " dotted " " "

we are prepared to state in still another and final form the conditions which vm must fulfil, viz.:

(I.) $T+T'$ must $= R$; and (II.) The resultant of T and T' must act in the same vertical as R.

In short, the quantities T, T', and R must form a balanced system, considered as forces. All of which amounts practically to this: that if the verticals in which T and T' act are known and R be conceived as a load supported by a horizontal beam (see foot of Fig. 417, last figure) resting on piers in those verticals, then T and T' are the respective *reactions of those piers.* It will now be shown that the verticals of T and T' are easily found, *being independent of the position* of vm; and that both the vertical and the magnitude of R, being likewise independent of vm, are determined with facility in advance. For, if v be shifted up or down, all the broken ordinates in (c) or (d) will change in the same proportion (viz. as vF changes), while the dotted ordinates, though shifted along their verticals, do not change in value; hence the shifting of v affects neither the vertical nor the value of T', nor the vertical of T. The *value* of T, however, is proportional to vF. Similarly, if m be shifted, up or down, T' will vary proportionally to mG, but its vertical, or line of action, remains the same. T is unaffected in any way by the shifting of m. R, depending for its value and position on the full ordinates of (c) Fig. 417, is independent of the location of vm. We may ∴ proceed as follows:

1st. Determine R graphically, in amount and position, by means of § 376.

2ndly. Determine the *verticals* of T and T' by *any trial* position of vm (call it v_2m_2), and the corresponding trial values of T and T' (call them T_2 and T'_2).

3rdly. By the fiction of the horizontal beam, construct (§ 329) or compute the true values of T and T', and then determine the true distances vF and mG by the proportions

$$vF : v_2F :: T : T_2 \text{ and } mG : m_2G :: T' : T'_2.$$

Example of this. Fig. 418. (See Fig. 417 for *s* and *t*.)

From *A* toward *B* in (*e*) Fig. 418, lay off the lengths (or lines proportional to them) of the full ordinates 1, 2, etc., of (*f*). Take any pole O_1, and draw the equilibrium polygon (*f*)′ and prolong its extreme segments to find *C* and thus determine *R*'s vertical. *R* is represented by *AB*. In (*g*) [same as (*f*) but shifted to avoid complexity of lines] draw a trial v_2m_2 and join v_2G_2. Determine the sum T_2 of the broken ordinates (between v_2G_2 and F_2G_2) and its vertical line of application, precisely as in dealing with *R*; also T'_2 that of the dotted ordinates (five) and its vertical. Now the true $T=Rt\div(s+t)$ and the true $T'=Rs\div(s+t)$. Hence compute $\overline{vF}=(T\div T_2)\,\overline{v_2F_2}$ and $\overline{mG}=(T'\div T'_2)\,\overline{m_2G_2}$, and by laying them off vertically upward from *F* and *G* respectively we determine *v* and *m*, i.e., the line *vm* to fulfil the conditions imposed at the beginning of this article, relating to the vertical ordinates intercepted between *vm* and given points on the perimeter of a polygon or curve.

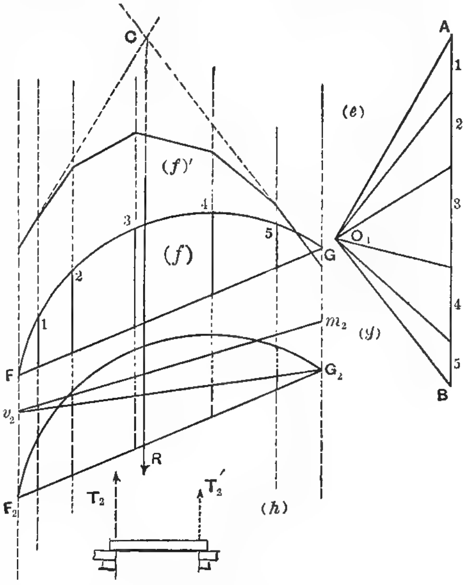

Fig. 418.

Note (a). If the verticals in which the intercepts lie are equidistant and quite numerous, then the lines of action of T_2 and T'_2 will divide the horizontal distance between *F* and *G* into three equal parts. This will be *exactly true* in the application of this construction to § 390.

Note (b). Also, if the verticals are symmetrically placed about a vertical line, (as will usually be the case) v_2m_2 is best drawn parallel to FG, for then T_2 and T'_2 will be equal and equi-distant from said vertical line.

378. Classification of Arch-Ribs, or Elastic Arches, according to continuity and modes of support. In the accompanying figures the *full* curves show the *unstrained* form of the rib (before any load, even its own weight, is permitted to come upon it); the dotted curve shows its shape (much exaggerated) when bearing a load. For a given loading **Three Conditions** must be given to determine the special equilibrium polygon (§§ 366 and 367).

Class A.—Continuous rib, free to slip laterally on the piers, which have smooth horizontal surfaces, Fig. 420.

This is chiefly of theoretic interest, its consideration being therefore omitted. The pier reactions are necessarily vertical, just as if it were a straight horizontal beam.

Class B. Rib of Three Hinges, two at the piers and one intermediate (usually at the crown) Fig. 421. Fig. 36 also is an example of this. That is, the rib is discontinuous and of two segments. Since at each hinge the moment of the stress couple must be zero, the special equilibrium polygon must pass through the hinges. Hence as three points fully determine an equilibrium polygon for given load, the special equilibrium is drawn by § 341.

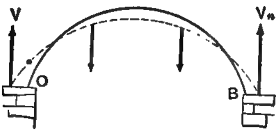

FIG. 420.

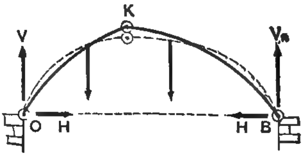

FIG. 421.

[§ 378*a* will contain a construction for arch-ribs of three hinges, when the forces are not all vertical.]

Class C. Rib of Two Hinges, these being at the piers, the rib continuous between. The piers are considered immovable, i.e., the span cannot change as a consequence of loading. It is also considered that the rib is fitted to its hinges at a definite temperature, and is then under no constraint from the piers (as if it lay flat on the ground), not even its own weight being permitted to act when it is finally put into position. When the "false works" or temporary supports are removed, stresses are induced in the rib both by its loading, including its own weight, and by a change of temperature. Stresses due to temperature may be ascertained separately and then combined with those due to the loading. [Classes A and B are not subject to temperature stresses.] Fig. 422 shows a rib of two hinges, at ends. Conceive the *dotted* curve (form and position under strain) to be superposed on the continuous curve (form before strain) in such a way that B and its tangent line (which has been displaced from its original position) may occupy their previous position. This gives us the broken curve O_nB. OO_n is ∴ O's displacement relatively to B and B's tangent. Now the *piers being immovable* O_nB (right line)$=OB$; i.e., the X projection (or Δx) of OO_n upon OB (*taken as an axis of* X) is zero compared with its Δy. Hence as one condition to fix the special equilibrium polygon for a given loading we have (from § 373)

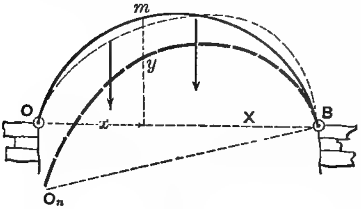

FIG. 422.

$$\int_O^B [Myds \div EI]=0 \qquad . \quad . \quad . \quad (1)$$

The other two are that the } must pass through O . (2)
special equilibrium polygon } " " " B . (3)

Class D. Rib with **Fixed Ends** and *no hinges*, i.e., continuous. Piers immovable. The ends may be *fixed* by being inserted, or built, in the masonry, or by being fastened to large plates which are bolted to the piers. [The St. Louis Bridge and that at Coblenz over the Rhine are of this class.] Fig. 423. In this class there being no hinges we have no point given in advance through which the special equilibrium polygon must pass. However, since O's displacement relatively (and absolutely) to B and B's tangent is zero, both Δx and Δy [see § 373] = zero. Also the tangent-lines both at O and B being *fixed in direction*, the angle between them is the same under loading, or change of temperature, as when the rib was first placed in position under no strain and at a definite temperature.

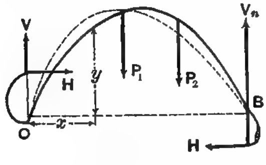

FIG. 423.

Hence the conditions for locating the special equilibrium polygon are

$$\int_0^B \frac{Mds}{EI} = 0 : \int_0 \frac{Myds}{EI} = 0 ; \int_0^B \frac{Mxds}{EI} = 0.$$

In the figure the imaginary rigid prolongations at the ends are shown [see § 366].

Other designs than those mentioned are practicable (such as: one end fixed, the other hinged; both ends fixed and one hinge between, etc.), but are of unusual occurrence.

378a. Rib of Three Hinges, Forces not all Vertical.* If the given rib of three hinges upholds a roof, the wind-pressure on which is to be considered as well as the weights of the materials composing the roof-covering, the forces will not all be vertical. To draw the special equil. polygon in such a case the following construction holds: **Required to draw an equilibrium polygon, for *any plane system* of forces, through three arbitrary points, A, p and B; Fig. 423a. Find the line of action of R_1, the resultant of all the forces occurring between A and p;** also,

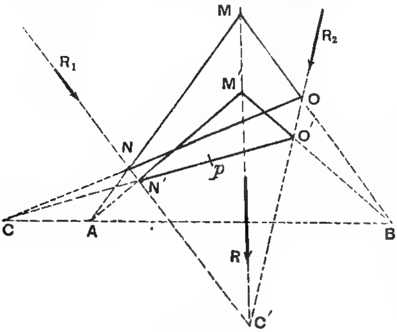

FIG. 423a.

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* See p. 117 of the author's "Notes and Examples in Mechanics" for a detailed example of the following construction.

that of R_2, the resultant of all forces between p and B; also the line of action of R, the resultant of R_1 and R_2, [see § 328.] Join any point M in R with A and also with B, and join the intersections N and O. Then $A\,N$ will be the direction of the first segment, $O\,B$ that of the last, and $N\,O$ itself is the segment corresponding to p (in the desired polygon) of an equilibrium polygon for the given forces. See § 328. If $A\,N'\,p\,O'\,B$ are the corresponding segments (as yet unknown) of the desired equil. polygon, we note that the two triangles MNO and $M'N'O'$, having their vertices on three lines which meet in a point [i.e., R meets R_1 and R_2 in C'], are homological [see Prop. VII. of Introduc. to Modern Geometry, in Chauvenet's Geometry,] and that ∴ the three intersections of their corresponding sides must lie on the same straight line. Of those intersections we already have A and B, while the third must be at C, found at the intersection of AB and NO. Hence by connecting C and p, we determine N' and O'. Joining $N'A$ and $O'B$, the first ray of the required force diagram will be ‖ to $N'A$, while the last ray will be ‖ to $O'B$, and thus the pole of that diagram can easily be found and the corresponding equilibrium polygon, beginning at A, will pass through p and B.

(This general case includes those of §§ 341 and 342.)

379. Arch-Rib of two Hinges; by Prof. Eddy's Method.* [It is understood that the hinges are at the ends.] Required the location of the *special equilibrium polygon.* We here suppose the rib homogeneous (i.e., the modulus of Elasticity E is the same throughout), that it is a "curved prism" (i.e., that the moment of inertia I of the cross-section is constant), that the piers are on a level, and that the rib-curve is symmetrical about a vertical line. Fig. 424. For each point m of the rib curve we have an x and y (both known, being the co-ordinates of the point), and also a z (intercept between rib and special equilibrium polygon) and a z' (intercept

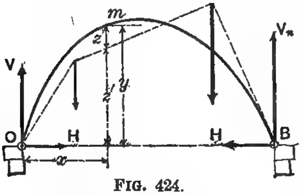

FIG. 424.

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* P. 25 of Prof. Eddy's book; see reference in preface of this work.

between the spec. eq. pol. and the axis X (which is OB).

The first condition given in § 378 for Class C may be transformed as follows, remembering [§ 367 eq. (3)] that $M = Hz$ at any point m of the rib (and that EI is constant).

$$\frac{1}{EI}\int_0^B Myds = 0, \text{ i.e., } \frac{H}{El}\int_0^B zyds = 0 \therefore \int_0^B zyds = 0$$

$$\left.\begin{array}{c}\text{but}\\ z = y - z'\end{array}\right\} \therefore \int_0^B (y - z')yds = 0; \text{ i.e., } \int_0^B yyds = \int_0^B yz'ds \; . \; (1)$$

In practical graphics we can not deal with infinitesimals; hence we must substitute Δs a small finite portion of the rib-curve for ds; eq. (1) now reads $\Sigma_0^B\, yy\, \Delta s = \Sigma_0^B\, yz'\, \Delta s$.

But if we take *all the* Δs's *equal*, Δs is a common factor and cancels out, leaving as a final form for eq. (1)

$$\Sigma_0^B(yy) = \Sigma_0^B(yz') \quad . \quad . \quad . \quad (1)'$$

The other two conditions are that the special equilibrium polygon begins at O and ends at B. (The subdivision of the rib-curve into an *even* number of *equal* Δs's will be observed in all problems henceforth.)

379a. Detail of the Construction. Given the arch-rib $O\,B$, Fig. 425, with specified loading. Divide the curve into

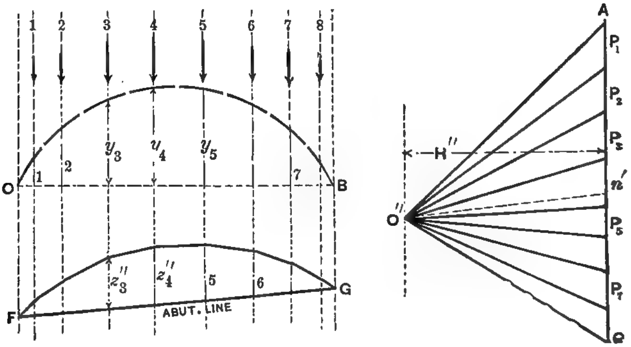

FIG. 425.

eight equal Δs's and draw a vertical through the middle of each. Let the loads borne by the respective Δs's be P_1, P_2, etc., and with them form a vertical load-line $A\,C$ to some convenient scale. With any convenient pole O'' draw a trial force diagram $O''\,AC$, and a corresponding trial equilibrium polygon $F\,G$, beginning at any point in the vertical F. Its ordinates z_1'', z_2'', etc., are proportional to those of the special equil. pol. sought (whose abutment line is OB) [§ 374a (2)]. We next use it to determine n' [see § 374a]. We know that OB is the "abutment-line" of the required special polygon, and that ∴ its pole must lie on a horizontal through n'. It remains to determine its H, or pole distance, by equation (1)′ just given, viz.: $\Sigma_1^8\,yy = \Sigma_1^8 yz'$. First by § 375 find the value of the summation $\Sigma_1^8(yy)$, which, from symmetry, we may write $= 2\Sigma_1^4(yy) = 2[y_1y_1 + y_2y_2 + y_3y_3 + y_4y_4]$

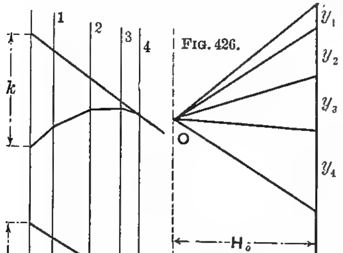

FIG. 426.

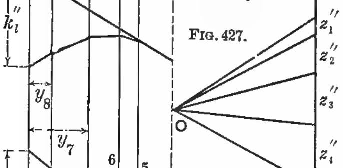

FIG. 427.

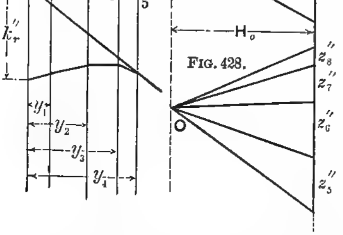

FIG. 428.

Hence, Fig. 426, we obtain

$$\Sigma_1^8\ (yy) = 2\ [H_0 k]$$

Next, also by § 375, see Fig. 427, using the same pole distance H_0 as in Fig. 426, we find

$$\Sigma_1^4\,(yz'') = H_0 k_1''\text{ ; i.e.,}$$

$$y_1z_1'' + y_2z_2'' + y_3z_3'' + y_4z_4'' = H_0k_1''.$$

Again, since $\Sigma_5^8\ (yz'') = y_8z_8'' + y_7z_7'' + y_6z_6'' + y_5z_5''$ which from symmetry (of rib)

$$= y_1z_8'' + y_2z_7'' + y_3z_6'' + y_4z_5'',$$

we obtain, Fig. 428, $\Sigma_5^8\,(yz'') = H_0k_r''$, (same H_0); and ∴

$\Sigma_1^8\,(yz'') = H_0\,(k_l'' + k_r'')$. If now we find that $k_l'' + k_r'' = 2k$, the condition $\Sigma_1^8 (yy) = \Sigma_1^8 (yz'')$ is satisfied, and the pole distance of our trial polygon in Fig. 425, is also that of the special polygon sought; i.e., the z'''s are identical in value with the z''s of Fig. 424. In general, of course, we do not find that $k_l''+k_r'' = 2k$. Hence the z'''s must all be increased in the ratio $2k : (k_l''+k_r'')$ to become equal to the z''s. That is, the pole distance H of the spec. equil. polygon must be

$H = \frac{k_l''+k_r''}{2k} H''$ (in which H'' = the pole distance of the trial polygon) since from §339 the ordinates of two equilibrium polygons (for the same loads) are inversely as their pole distances. Having thus found the H of the special polygon, knowing that the pole must lie on the horizontal through n', Fig. 425, it is easily drawn, beginning at O. As a check, it should pass through B.

For its utility see § 367, but it is to be remembered that the stresses as thus found in the different parts of the rib under a given loading, must afterwards be combined with those resulting from change of temperature and the shortening of the rib axis due to the tangential thrusts, before the actual stress can be declared in any part.

Note.—Variable Moment of Inertia. If the I of the rib section is different at different sections we may proceed as follows: For eq. (1), we now write $\int_0^B yy\frac{ds}{I} = \int_0^B yz'\frac{ds}{I}$. Taking the I of the crown section (say) as a standard of reference, denoting it by I', we may write for any other section $I = nI'$, where n is a variable ratio, or abstract number; whence eq. (1) becomes, after putting Δs for ds, $\frac{1}{I'}\sum_0^B yy\frac{\Delta s}{n} = \frac{1}{I'}\sum_0^B yz'\frac{\Delta s}{n}$. If now the length of each successive Δs, from the crown down, be made directly proportional to the number n at that part of the rib, the quantity $\Delta s \div n$ will have the same value in all the terms of each summation and may be factored out; and we then have a relation identical in form with eq. (1)', but *with the understanding* that the y's and z'''s concerned are those in the successive verticals drawn through the mid-points of the *unequal* Δs's, or subdivisions along the rib, obtained by *following the above plan* that each Δs is proportional to the value of the moment of inertia at that part of the rib. For instance, if the I of a section near the hinge O, or B, is three times that (I') at the crown, then the length of the Δs at the former point must be made three times the length of the Δs first assumed at the crown when the subdivision is begun. By a little preliminary investigation, a proper value for this crown Δs may be decided upon such that the total number of Δs's shall be sufficient for accuracy (sixteen or twenty in all)

380. Arch Rib of Fixed Ends and no Hinges.—Example of Class D. Prof. Eddy's Method.* As before, *E* and *I* are constant along the rib Piers immovable. Rib curve symmetrical about a vertical line. Fig. 429 shows such a rib under any loading. Its span is *OB*, which is taken as an axis *X*. The co-ordinates of any point *m′* of the rib curve are *x* and *y*, and *z* is the vertical intercept between *m′* and the special equilibrium polygon (as yet unknown, but to be constructed). Prof. Eddy's method will now be given for finding the special equil. polygon. The three conditions it must satisfy (see § 378, Class D, remembering that *E* and *I* are constant and that $M = Hz$ from § 367) are

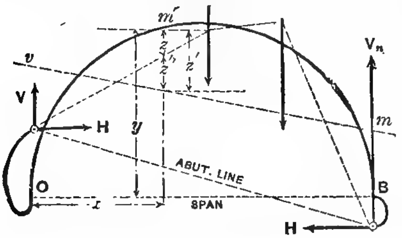

FIG. 429.

$$\int_0^B zds=0\,;\int_0^B xzds=\,;0 \text{ and } \int_0^B yzds=0 \quad . \quad . \quad (1)$$

Now suppose the auxiliary reference line (straight) *vm* to have been drawn satisfying the requirements, with respect to the rib curve that

$$\int_0^B z'ds=0\,; \text{and} \int_0^B xz'ds=0 \quad . \quad . \quad . \quad . \quad (2)$$

in which z' is the vertical distance of any point *m′* from *vm* and *x* the abscissa of *m′* from *O*.

From Fig. 429, letting z'' denote the vertical intercept (corresponding to any *m′*) between the spec. polygon and the auxiliary line *vm*, we have $z=z'-z''$, hence the three conditions in (1.) become

$$\int_0^B (z'-z'')ds=0\,; \text{ i.e., see eqs. (2) } \int_0^B z''ds=0 \quad . \quad . \quad . \quad (3)$$

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* P. 14 of Prof Eddy's book ; see reference in preface of this work

$$\int_0^B x\,(z'-z'')ds=0\text{ ; i.e., see eqs. (2) }\int_0^B xz''ds=o \qquad (4)$$

and $\int_0^B y(z'-z'')ds=\ ;0\ \therefore$ by transposition $\int_0^B yz'ds=\int_0^B yz''ds$. (5)

provided vm has been located as prescribed.

For graphical purposes, having subdivided the rib curve into an *even* number of small equal Δs's, and drawn a vertical through the middle of each, we first, by § 377, locate *vm* to satisfy the conditions

$$\Sigma_0^B(z')=0 \text{ and } \Sigma_0^B(xz')=0 \qquad . \qquad . \qquad (6)$$

(see eq. (2); the Δs cancels out); and then locate the special equilibrium polygon, with *vm* as a reference-line, by making it satisfy the conditions.

$$\Sigma_0^B(z'')=0\ .\ (7)\ ;\quad \Sigma_0^B(xz'')=0\ .\ (8)\ ;\quad \Sigma_0^B(yz'')=\Sigma_0^B(yz')\ .\ (9)$$

(obtained from eqs. (3), (4), (5) by putting $ds=\Delta s$, and cancelling).

Conditions (7) and (8) may be satisfied by an infinite number of polygons drawn to the given loading. Any one of these being drawn, as a trial polygon, we determine for it the value of the sum $\Sigma_0^B(yz'')$ by § 375, and compare it with the value of the sum $\Sigma_0^B(yz')$ which is independent of the special polygon and is obtained by § 375. [N.B. It must be understood that the quantities (lengths) x, y, z, z', and z'', here dealt with are those pertaining to the verticals drawn through the middles of the respective Δs's, which must be sufficiently numerous to obtain a close result, and not to the verticals in which the loads act, necessarily, since these latter may be few or many according to circumstances, see Fig. 429]. If these sums are not equal, the pole distance of the trial equil. polygon must be altered in the proper ratio (and thus change the z'''s in the inverse ratio) necessary to make these sums equal and thus satisfy condition (9). The alteration of the z'''s, all in the same ratio, will not interfere with conditions (7) and (8) which are already satisfied.

381. Detail of Construction of Last Problem. Symmetrical Arch-Rib of Fixed Ends.—As an example take a span of the St. Louis Bridge (assuming I constant) with "live load" covering the half span on the left, Fig. 430, where the vertical

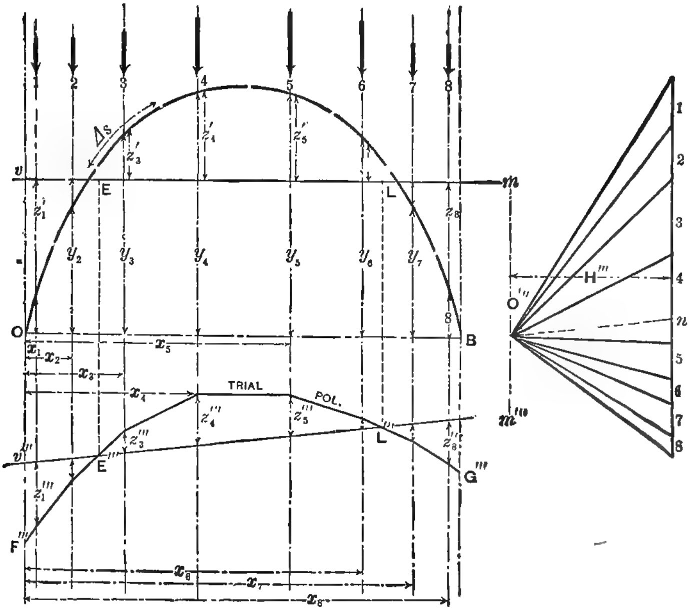

Fig. 430.

scale is much exaggerated for the sake of distinctness*. Divide into eight equal Δs's. (In an actual example sixteen or twenty should be taken.) Draw a vertical through the middle of each Δs. P_1, etc., are the loads coming upon the respective Δs's.

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* Each arch-rib of the St. Louis bridge is a built up or trussed rib of steel about 520 ft. span and 52 ft. rise, in the form of a segment of a circle. Its moment of inertia, however, is not strictly constant, the portions near each pier, of a length equal to one twelfth of the span, having a value of I one-half greater than that of the remainder of the arc.

First, to locate vm, by eq. (6); from symmetry it must be horizontal. Draw a trial vm (not shown in the figure), and if the $(+z')$'s exceed the $(-z')$'s by an amount z_0', the true vm will lie a height $\frac{1}{n}z_0'$ above the trial vm (or below, if vice versâ); $n =$ the number of Δs's.

Now lay off the load-line on the right (to scale), take any convenient trial pole O''' and draw a corresponding trial equil. polygon $F'''G'''$. In $F'''G'''$, by § 377, locate a straight line $v'''m'''$ so as to make $\Sigma_0^B(z''') = 0$ and $\Sigma_0^B(xz''') = 0$ (see Note (*b*) of § 377).

[We might now redraw $F'''G'''$ in such a way as to bring $v'''m'''$ into a horizontal position, thus: first determine a point n''' on the load-line by drawing $O'''n''' \parallel$ to $v'''m'''$, take a new pole on a horizontal through n''', with the same H''', and draw a corresponding equil. polygon; in the latter $v'''m'''$ would be horizontal. We might also shift this new trial polygon upward so as to make $v'''m'''$ and vm coincide. It would satisfy conditions (7) and (8), having the same z''''s as the first trial polygon; but to satisfy condition (9) it must have its z''''s altered in a certain ratio, which we must now find. But we can deal with the individual z''''s just as well in their present positions in Fig. 430.] The points E and L in vm, vertically over E''' and L''' in $v'''m'''$, are now fixed; they are *the intersections of the special polygon required, with vm.*

The ordinates between $v'''m'''$ and the trial equilibrium polygon have been called z''' instead of z''; they are proportional to the respective z'''s of the required special polygon.

The next step is to find in what ratio the (z''')'s need to be altered (or H''' altered in inverse ratio) in order to become the (z'')'s; i.e., in order to fulfil condition (9), viz.:

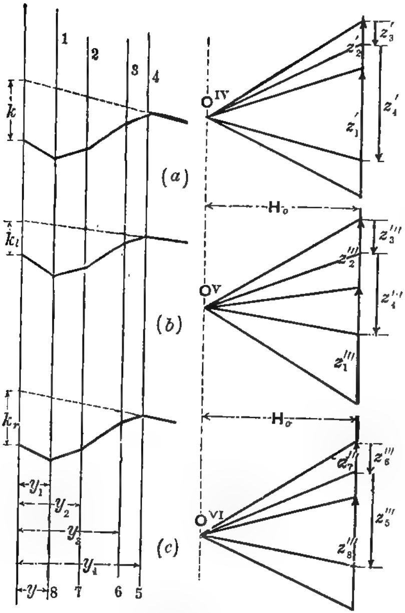

FIG. 431.

$$\Sigma_1^8(yz'')=\Sigma_1^8(yz') \quad . \quad (9)$$

This may be done precisely as for the rib with two hinges, but the negative (z''')'s must be properly considered (§ 375) See Fig. 431 for the detail. Negative z''s or z'''s point upward.

From Fig. 431a

$$\Sigma_1^4(yz')=H_0k$$

∴ from symmetry

$$\Sigma_1^8(yz')=2H_0k.$$

From Fig. 431b we have

$$\Sigma_1^4(yz''')=H_0k_1$$

and from Fig. 431c

$$\Sigma_5^8(yz''')=H_0k_r$$

[The *same pole distance* H_0 is taken in all these constructions] ∴ $\Sigma_1^8(yz''')=H_0(k_1+k_r)$.
If, then, $H_0\ (k_1+k_r) = 2H_0k$ condition (9) is satisfied by the z''''s. If not, the true pole distance for the special equil. polygon of Fig. 430 will be

$$H=\frac{k_1+k_r}{2k}\ .\ H'''$$

With this pole distance and a pole in the horizontal through n''' (Fig. 430) the force diagram may be completed for the required special polygon; and this latter may be constructed as follows: Beginning at the point E, in vm, through it draw a segment ∥ to the proper ray of the force diagram. In our present figure (430) this "proper ray" would be the ray joining the pole with the point of meeting of P_2 and P_3 on the load-line. Having this one segment of the special polygon the others are added in an obvious manner, and thus the whole polygon completed. It should pass through L, but not O and B.

For another loading a different special equil. polygon would result, and in each case we may obtain the *thrust*, *shear*, and *moment* of stress couple for any cross-section of the rib, by § 367. To the stresses computed from these, should be added (algebraically) those occasioned by change of temperature and by shortening of the rib as occasioned by the thrusts along the rib. These "temperature stresses," and stresses due to rib-shortening, will be considered in a subsequent paragraph. They have no existence for an arch-rib of three hinges.

Note.—If the moment of inertia of the rib section is *variable*, instead of dividing the rib axis into equal Δs's, we should make them *unequal*, following the plan indicated in the note on p. 464, the Δs being made *proportional to the values of the moment of inertia along the rib.* After such subdivision is made, and a vertical drawn through the mid-point of each Δs, the various y's, z''s, etc., in these verticals are dealt with in the same manner as just shown for the case of constant moment of inertia.

381a. Exaggeration of Vertical Dimensions of Both Space and Force Diagrams.—In case, as often happens, the axis of the given rib is quite a flat curve, it is more accurate (for finding M) to proceed as follows:

After drawing the curve in its true proportions and passing a vertical through the middle of each of the equal Δs's, *compute* the ordinate (y) of each of these middle points from the equation of the curve, and multiply each y by four (say). These quadruple ordinates are then laid off from the span upward, each in its proper vertical. Also multiply each load, of the given loading, by four, and then with these quadruple loads and quadruple ordinates, and the upper extremities of the latter as points in an exaggerated rib-curve, proceed to construct a special equilibrium polygon, and the corresponding force diagram by the proper method (for Class B, C, or D, as the case may be) for this exaggerated rib-curve.

The moment, Hz, thus found for any section of the exaggerated rib-curve, is to be divided by four to obtain the moment in the real rib, in the same vertical line. To find the thrust and shear, however, for sections of the real rib, besides employing tangents and normals of the real rib we must draw, and use, another force diagram, obtained from the one already drawn (for the exaggerated rib) by reducing its *vertical* dimensions (only), in the ratio of four to one. [Of course, any other convenient number besides four, may be adopted throughout.]

382. Stress Diagrams.—Take an arch-rib of Class *D*, § 378, i.e., of fixed ends, and suppose that for a given loading (including its own weight) the special equil. polygon and its force diagram have been drawn [§ 381]. It is required to indicate graphically the variation of the three stress-elements for any section of the rib, viz., the thrust, shear, and mom. of stress-couple. *I* is constant. If at any point *m* of the rib a section is made, then the stresses in that section are classified into three sets (Fig. 432). (See §§ 295 and 367) and from § 367 eq. (3) we see that the vertical intercepts between the rib and the special equil. polygon being proportional to the products *Hz* or moments of the stress-couples in the corresponding sections form a **moment diagram**, on inspection of which we can trace the change in this moment, $Hz = \frac{p_2 I}{e}$, and hence the variation of the stress per square inch, p_2, (as due to stress couple alone) in the outermost fibre of any section (tension or compression) at distance *e* from the gravity axis of the section), from section to section along the rib.

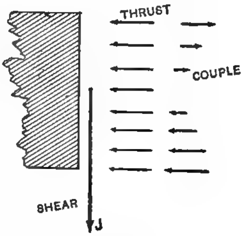

FIG. 432.

By drawing through *O* lines *On'* and *Ot'* parallel respectively to the tangent and normal at any point *m* of the rib axis [see Fig. 433] and projecting upon them, in turn, the proper ray (R_3 in Fig. 433) (see eqs. 1 and 2 of § 367) we obtain the values of the thrust and shear for the section at m. When found in this way for a number of points along the rib their values may be laid off as vertical lines from a horizontal axis, in the verticals containing the respective points, and thus a thrust diagram and a shear diagram may be formed, as constructed in Fig. 433. Notice that where the moment is a maximum or minimum the shear changes sign (compare § 240), either gradually or suddenly, according as the max. or min. occurs between two loads or in passing a load; see m', e. g.

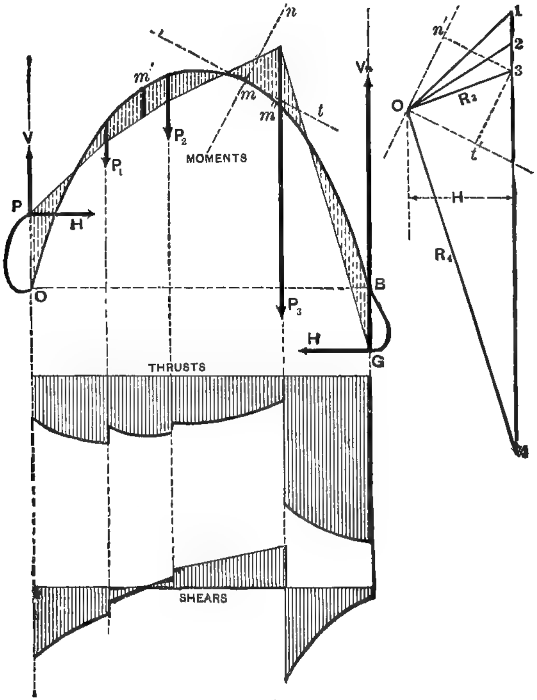

Fig. 433.

Also it is evident, from the geometrical relations involved, that at those points of the rib where the tangent-line is parallel to the "proper ray" of the force diagram, the thrust is a maximum (a local maximum) the moment (of stress couple) is either a maximum or a minimum and the shear is zero.

From the moment, $Hz = \frac{p_2 I}{e}$, $p_2 = \frac{Hze}{I}$

may be computed. From the thrust $= Fp_1$, $p_1 = \frac{\text{thrust}}{F}$, ($F$ = area of cross-section) may be computed. Hence the greatest compression per sq. inch (p_1+p_2) may be found in each section. A separate stress-diagram might be constructed for this quantity (p_1+p_2). Its max. value (after adding the stress due to change of temperature, or to rib-shortening, for ribs of less than three hinges), wherever it occurs in the rib, must be made safe by proper designing of the rib. The maximum shear J_m can be used as in §256 to determine thickness of web, if the section is I-shaped, or box-shaped. See § 295.

383. Temperature Stresses.—In an ordinary bridge truss and straight horizontal girders, free to expand or contract longitudinally, and in Classes A and B of § 378 of arch-ribs, there are no stresses induced by change of temperature; for the *form* of the beam or truss is under no constraint from the manner of support; but with the arch-rib of two hinges (hinged ends, Class C) and of fixed ends (Class D) having *immovable piers* which constrain the distance between the two ends to *remain the same* at all temperatures, stresses called "temperature stresses" are induced in the rib whenever the temperature, t, is not the same as that, t_0, when the rib was put in place. These may be determined, as follows, as if they were the only ones, and then combined, algebraically, with those due to the loading.

384. Temperature Stresses in the Arch-Rib of Hinged Ends.—(Class C, § 378.) Fig. 434. Let E and I be constant, with

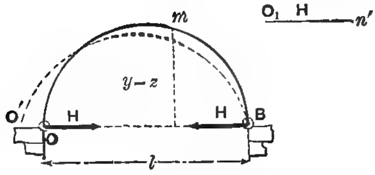

FIG. 434.

other postulates as in § 379. Let t_0 = temperature of erection, and t = any other temperature; also let l = length of span = OB (invariable) and η=co-efficient of linear expansion of the material of the curved beam or rib (see § 199). At temperature t there must be a horizontal reaction H at each hinge to prevent expansion into the form $O'B$ (dotted curve), which is the form *natural* to the rib for temperature t and without constraint. We may ∴ consider the actual form OB as having resulted from the unstrained form $O'B$ by displacing O' to O, i.e., producing a horizontal displacement $O'O = l\,(t-t_0)\eta$.

But $O'O = \Delta x$ (see §§ 373 and 374); (N.B. B's tangent has moved, but this does not affect Δx, if the axis X is horizontal, as here, coinciding with the span;) and the ordinate y of any point m of the rib is identical with its z or intercept between it and the spec. equil. polygon, which here consists of *one segment only*, viz.: OB. Its force diagram consists of a single ray $O_1\,n'$; see Fig. 434. Now (§ 373)

$$\Delta x = \frac{1}{EI}\int_0^B Myds\,; \text{ and } M=Hz = \text{in this case, } Hy$$

$$\therefore l\,(t-t_0)\,\eta=\frac{H}{EI}\int_0^B y^2ds\,; \left\{ \begin{matrix}\text{hence for graphics, and} \\ \textit{equal } \Delta s\text{'s, we have}\end{matrix}\right.$$

$$EIl(t-t_0)\eta=H\,\Delta s\,\Sigma_0^B\,y^2 \quad . \quad . \quad . \quad . \quad (1)$$

From eq. (1) we determine H, having divided the rib-curve into from twelve to twenty equal parts each called Δs.

For instance, for wrought iron, t and t_0, being expressed in Fahrenheit degrees, η = 0.0000066. If E is expressed in lbs. per square inch, all linear quantities should be in inches and H will be obtained in pounds.

$\Sigma_0^B y^2$ may be obtained by § 375, or may be computed. H being known, we find the moment of stress-couple = Hy, at any section, while the thrust and shear at that section are the projections of H, i.e., of O_1n' upon the tangent and normal. The stresses due to these may then be determined in any section, as already so frequently explained, and then combined with those due to loading.

385. Temperature Stresses in the Arch-Ribs with Fixed Ends.—See Fig. 435. (Same postulates as to symmetry, E and I constant, etc., as in § 380.) t and t_0 have the same meaning as in § 384.

Here, as before, we consider the rib to have reached its actual form under temperature t by having had its span forcibly shortened from the length natural to temp. t, viz.: $O'B'$, to the actual length OB, which the immovable piers compel it to assume. But here, *since the tangents at O and B are to be the same in direction under constraint as before,* the two forces H, representing the action of the piers on the rib, must be considered as acting on imaginary rigid prolongations at an unknown distance d above the span. To find H and d we need two equations.

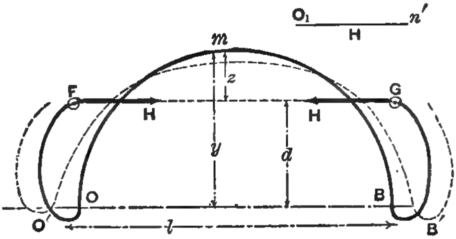

FIG. 435.

From § 373 we have, since $M=Hz=H(y-d)$,

$$\Delta x,\ \text{i.e.,}\ \overline{O'O}+\overline{BB'},\ \text{i.e.,}\ l(t-t_0)\eta,=\frac{H}{EI}\int_0^B (y-d)yds \quad . \ (2)$$

or, graphically, with equal Δs's

$$EIl\,(t-t_0)\eta=H\Delta s\left[\Sigma_0^B\, y^2-d\,\Sigma_0^B\, y\right] \quad . \ . \ . \quad (3)$$

Also, since there has been no change in the angle between end-tangents, we must have, from § 374,

$$\frac{1}{EI}\int_0^B Mds=0\,;\quad \text{i.e.,}\ \frac{H}{EI}\int_0^B zds=0\,;\text{i.e.,}\quad \int_0^B (y-d)\,ds=0$$

or for graphics, with *equal* Δs's, $\Sigma_0^B y = nd$ . . . (4)

in which n denotes the number of Δs's. From (4) we determine d, and then from (3) can compute H. Drawing the horizontal $F\,G$, it is the special equilibrium polygon (of but one segment) and the moment of the stress-couple at any section $= Hz$, while the thrust and shear are the projections of $H = O_1n'$ on the tangent and normal respectively of any point m of rib.

For example, in one span, of 550 feet, of the St. Louis Bridge, having a rise of 55 feet and fixed at the ends, the force H of Fig. 435 is $=$ 108 tons, when the temperature is 80° Fahr. higher than the temp. of erection, and the enforced span is 3¼ inches shorter than the span natural to that higher temperature. Evidently, if the actual temperature t is lower than that t_0, of erection, H must act in a direction opposite to that of Figs. 435 and 434, and the "thrust" in any section will be *negative*, i.e., a pull.

386. Stresses Due to Rib-Shortening.—In § 369, Fig. 407, the shortening of the element AE to a length $A'E$, due to the uniformly distributed thrust, p_1F, was neglected as producing indirectly a change of curvature and form in the rib axis; but such will be the case if the rib has *less than three hinges.* This change in the length of the different portions of the rib curve, may be treated as if it were due to a change of temperature. For example, from § 199 we see that a thrust of 50 tons coming upon a sectional area of $F = 10$ sq. inches in an iron rib, whose material has a modulus of elasticity $= E = 30{,}000{,}000$ lbs. per sq. inch, and a coefficient of expansion $\eta = .0000066$ per degree Fahrenheit, produces a shortening equal to that due to a fall of temperature $(t_0 - t)$ derived as follows: (See § 199) (units, inch and pound)

$$(t_0 - t) = \frac{P}{FE\eta} = \frac{100{,}000}{10 \times 30{,}000{,}000 \times .0000066} = 50°$$

Fahrenheit.

Practically, then, since most metal arch bridges of classes C and D are rather flat in curvature, and the thrusts due to ordinary modes of loading do not vary more than 20 or 30 per cent. from each other along the rib, an imaginary fall of temperature corresponding to an average thrust in any case of loading may be made the basis of a construction similar to that in § 384 or § 385 (according as the ends are *hinged*, or *fixed*) from which new thrusts, shears, and stress-couple moments, may be derived to be combined with those previously obtained for loading and for change of temperature.

387. Résumé.—It is now seen how the stresses per square inch, both shearing and compression (or tension) may be obtained in all parts of any section of a solid arch-rib or curved beam of the kinds described, by combining the results due to the three separate causes, viz.: the load, change of temperature, and rib-shortening caused by the thrusts due to the load (the latter agencies, however, coming into consideration only in classes C and D, see § 378). That is, in any cross-section, the stress in the outer fibre is, [letting T_h', T_h'', T_h''', denote the thrusts due to the three causes, respectively, above mentioned; $(Hz)'$, $(Hz)''$, $(Hz)'''$, the moments]

$$= \frac{T_h' \pm T_h'' - T_h'''}{F} \pm \frac{e}{I}\Big[(Hz)' \pm (Hz)'' \pm (Hz)'''\Big] \quad . \quad . \quad . \quad (1)$$

i.e., lbs. per sq. inch compression (if those units are used).

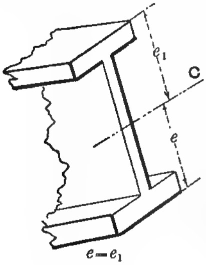

FIG. 436.

The double signs provide for the cases where the stresses in the outer fibre, due to a single agency, may be tensile. Fig. 436 shows the meaning of e (the same used heretofore) I is the moment of inertia of the section about the gravity axis (horizontal) C. F = area of cross-section. [$e_1 = e$; cross section symmetrical about C]. For a given loading we may find the maximum stress in a given rib, or design the rib so that this maximum stress shall be safe for the material employed. Similarly, the resultant shear (total, not per sq. inch) = $J' \pm J'' \pm J'''$ is obtained for any section to compute a proper thickness of web, spacing of rivets, etc.

388 The Arch-Truss, or braced arch. An open-work truss, if of homogeneous design from end to end, may be treated as a beam of constant section and constant moment of inertia, and if curved, like the St. Louis Bridge and the Coblenz Bridge (see § 378, Class D), may be treated as an arch-rib.* The moment of inertia may be taken as

$$I = 2\,F_1\left(\frac{h}{2}\right)^2$$

where F_1 is the sectional area of one of the pieces ∥ to the curved axis midway between them, Fig. 437, and h = distance between them.

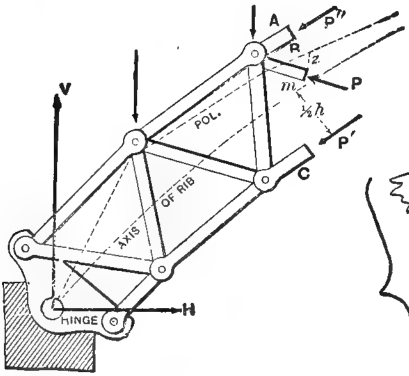

FIG. 438.

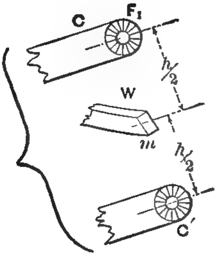

FIG. 437.

Treating this curved axis as an arch-rib, in the usual way (see preceding articles), we obtain the spec. equil. pol. and its force diagram for given loading. Any plane $\urcorner$ to the rib-axis, where it crosses the middle m of a "web-member," cuts three pieces, A, B and C, the total compressions (or tensions) in which are thus found: For the point m, of rib-axis, there is a certain moment $= Hz$, a thrust $= T_h$, and a shear $= J$, obtained as previously explained. We may then write $P \sin \beta = J$ . . . . (1) and thus determine whether P is a tension or compression; then putting $P' + P'' \pm P \cos \beta = T_h$ . . . . . 2 (in which P is taken with a plus sign if a compression, and minus if tension); and

---

*The St Louis Bridge is not strictly of constant moment of inertia, being somewhat strengthened near each pier

$$(P' - P'')\frac{h}{2} = Hz \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

we compute P' and P'', which are assumed to be *both compressions* here. β is the angle between the web member and the tangent to rib-axis at m, the middle of the piece. See Fig. 406, as an explanation of the method just adopted.

CIRCULAR RIBS AND HOOPS.

389. Deflections and Changes of Slope of Curved Beams. Analytical Method. For finding these quantities we may use eqs. (I.), (II.) and (III.) of § 374. For example, we have in Fig. 439, a curved beam of the form of the quadrant of a circle, fixed vertically at lower extremity B, and carrying a single concentrated load, P, at the free end O. [Its own weight neglected.]

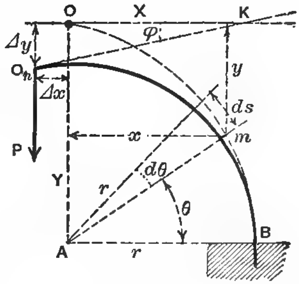

FIG. 439.

As a consequence of the loading, the extremity O is displaced to some position, O_n, but the bending is slight. Required, the projections Δx and Δy of this displacement and also the angle OKO_n or ϕ, which the tangent-line at O_n makes with its former (horizontal) position OX. The beam is homogeneous and of constant cross-section; i.e., E and I are constants.

To use the equations for Δx and Δy we must *take O as an origin* (since O is the point whose displacement is under consideration). Hence the co-ordinates x and y of any point, m, of the axis of the beam are as shown in the figure. Taking now polar co-ordinates, as shown, we note that $x = r \cos \theta$; $y = r(1 - \sin \theta)$; and $ds = rd\theta$. We must also put down the following integral forms for reference; viz.: —

$$\int \sin \theta . d\theta = - \cos \theta ; \qquad \int \theta . \cos \theta . d\theta = \theta . \sin \theta + \cos \theta ;$$

$$\int \cos \theta . d\theta = + \sin \theta ; \qquad \int \sin^2 \theta . d\theta = \tfrac{1}{2} \theta - \tfrac{1}{4} \sin 2 \theta ;$$

$$\int \sin \theta . \cos \theta . d\theta = \tfrac{1}{2} \sin^2 \theta ; \qquad \int \cos^2 \theta . d\theta = \tfrac{1}{2} \theta + \tfrac{1}{4} \sin 2 \theta .$$

Taking the portion $O_n m$ (m being any point on curved axis of beam) as a free body, we have, for the moment of the stress couple at m, $M = Px, = Pr \cos \theta$, and hence derive, for the angle ϕ,

$$\phi, = \frac{1}{EI} \int_O^B M . ds = \frac{Pr^2}{EI} \int_B^O \cos \theta . d\theta = \frac{Pr^2}{EI} (\sin \theta) \Big]_0^{\frac{\pi}{2}} = \frac{Pr^2}{EI} \quad . \quad . \quad . \quad (1)$$

Also

$$\Delta y, = \frac{1}{EI} \int_O^B M . xds, = \frac{Pr^3}{EI} \int_B^O \cos^2 \theta . d\theta = \frac{Pr^3}{EI} \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4} . \frac{Pr^3}{EI} \quad (2)$$

and

$$\Delta x, = \frac{1}{EI} \int_O^B M . yds, = \frac{Pr^3}{EI} \left[\int_0^{\frac{\pi}{2}} \cos \theta . d\theta - \int_0^{\frac{\pi}{2}} \cos \theta . \sin \theta . d\theta \right] = \frac{Pr^3}{2\,EI} . \quad (3)$$

It must be understood that the elastic limit is not passed in any fiber and that the bending is very slight. A simple curved crane and a ship's davit are instances of this problem, provided the cross-section has the same moment of inertia, I, about a gravity axis perpendicular to the plane of the paper in Fig. 439, at all parts of the beam.

390. Semi-Circular Arch-Rib. Hinged at the Two Piers or Supports, and Continuous Between. Fig. 440. The supports are at the same level. The arch-rib, or curved beam, is homogeneous and has a constant I at all sections. (It is a "curved prism".) It is stipulated that no constraint is necessary in fitting the rib upon the hinges at the piers before any load is placed on the rib; that is, that the distance apart of the piers (which are *unyielding*) is just equal to the distance between the ends of the rib when entirely free from strain. In other words, after the rib is in position it is under no stress until a load is put upon it. Its own weight is neglected and the load is a concentrated one of 2 P lbs. placed at the "crown", B. As a consequence of the gradual placing of the load the crown B settles slightly, but on account of symmetry the tangent-line to the curved axis at B remains horizontal. Also the extremities O and A tend to spread further apart, but this is prevented by the fact that the piers are immovable (or we may express it "the span is invariable"). Hence the reaction at each hinge support will have a horizontal component H as well as a vertical component, V, lbs. Fig. 440 shows the axis of the rib. Taking the whole rib as a free body we easily find (by putting Σ vert. comps. $=$ zero, and from symmetry) that each $V=P$; the whole load being called $2P$; but for determining the value of H (same at each hinge; from Σ (horiz. comps.) $=$ zero) we must have recourse to the theory of elasticity; i.e., must depend on the following fact, viz.: — that in the gradual settling of point B under the load, B remains in the same vertical, and the tang. line at B remains horizontal, and hence (since O moves neither horizontally nor vertically in actual space) the horizontal projection of O's displacement relatively to B and B's tangent is *zero* (or $\Delta x=0$), while the vertical projection of O's displacement relatively to B and B's tangent (Δy) equals the distance B has settled in actual space. Here we must take O as origin for x and y (as in figure) for any point m between O and B; and note that the $x = r\,(1-\cos\theta)$, and $y = r\sin\theta$; while $ds = r.\,d\theta$.

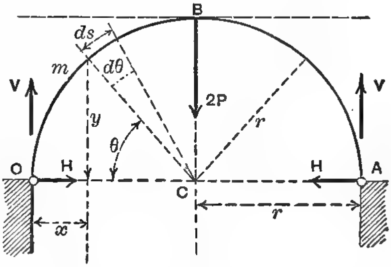

Fig. 440.

With Om as a free body (m being any point between O and B) we have for the moment of the stress couple at m, $M, = Vx - Hy, = Px - Hy$.

$$\Delta x, = \frac{1}{EI}\int_0^B Myds, =0\,;\ \therefore \int_0^{\frac{\pi}{2}}[P(r - r\cos\theta) - Hr\sin\theta]r^2\sin\theta d\theta = 0\,;$$

$$\therefore P\int_0^{\frac{\pi}{2}}\sin\theta.d\theta - P\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta d\theta - H\int_0^{\frac{\pi}{2}}\sin^2\theta.d\theta = 0\,;$$

and hence, (see integrals in § 389),

$$\left(P\left[-\cos\theta-\frac{\sin^2\theta}{2}\right]-H\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]\right)_0^{\frac{\pi}{2}}=0.$$

Inserting the limits, we have

$$P\left[-0+1-\tfrac{1}{2}+0\right]-H\left[\frac{\pi}{4}-0-0-(-0)\right]=0;$$

$$\therefore H=\frac{2\,P}{\pi}=\frac{\text{load}}{\pi}.$$

Also we may obtain, for the settlement of the crown, at B

$$\Delta y \text{ of } O \text{ relatively to } B,=\frac{1}{EI}\int_O^B Mxds=\frac{Pr^3}{EI}\left[\frac{3}{4}\frac{\pi}{}-\frac{1}{\pi}-2\right]$$

while the tangent-line at O, originally vertical, now makes with the vertical (on the outside)

$$\text{an angle }=\frac{1}{EI}\int_O^B Mds=\frac{Pr^2}{EI}\left[\frac{2}{\pi}+1-\frac{\pi}{2}\right].$$

This is a "*statically indeterminate structure*"; that is, one in which a solution is impossible by ordinary statics but must depend on the theory of the elastic change of form of the beam or body in question.

If the load were not placed at the crown, or highest point, we should be obliged to put

$$\frac{1}{EI}\int_O^A Myds=0$$

for the Δx of O relatively to A (instead of to B).

391. Cylindrical Pipe Loaded on Side. A cylindrical pipe of homogeneous material and small uniform thickness of pipe-wall, t, and length l, (so that the moment of inertia of the cross-section of wall is for present purposes $I=lt^3\div 12$) rests in a horizontal position on a firm horizontal floor and bears a concentrated load of $2P$ at the highest point, or crown, B. See Fig. 441. It is to be considered as a continuous curved beam or "hoop", without hinges. We neglect the weight of the pipe itself. The dotted circle shows the original unstrained form of the pipe-wall, or hoop, while the full line is its (slightly deformed) shape when it bears the load. The elastic limit is of course not to be passed. The upward force $2\,P$ at N is the reaction of the floor. *Required*, the maximum moment of stress-couple; and also the increase in the length of the horizontal diameter, and the decrease in that of the vertical diameter.

Consider as a free body the upper left-hand quadrant of the hoop, viz., OB, in Fig. 442, cutting just on the left of the load at B, a horizontal section being made at O. At each end of this body we must indicate a stress-couple, a shear, and a thrust. But at O it is evident, after a little consideration, that the shear (which would be horizontal) must be *zero;* there being at O, $\therefore$ only a thrust T_0 and a stress-couple of unknown moment M_0. At the other section the shear must be equal to one half of the load $2P$ (from considerations of symmetry) i.e., J at $B = P$; while the thrust at B is soon shown to be zero (since Σ (horiz. compons.) must = zero, and this thrust if it existed would be the only horiz. force besides those forming the stress-couple at B). At B, therefore, we find only a stress-couple, of an unknown moment M_B, and a shear J_B of direction shown in Fig. 442. By writing Σ (vert. compons.) = zero for this free body we find that the thrust, T_0, at O, must have a value P.

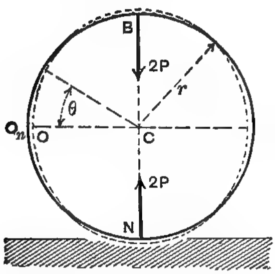

FIG. 441.

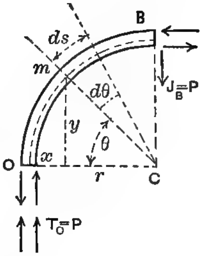

FIG 442.

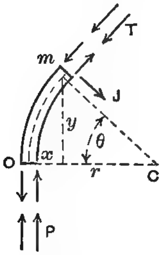

FIG. 443.

To determine M_0 we make use of the fact (evident from Fig. 441) that in the deformed condition of the "hoop" the tangent-lines at points O and B are still vertical and horizontal, respectively; in other words that the angle between them has not changed, i.e., is still 90°. Hence the value of ϕ, or *change of angle* between tangents at O and B is *zero*. Apply this fact to Fig. 442. Take O as origin for the x and y of any point m on OB (using θ later). From a consideration of the free body Om shown in Fig. 443 we have for the stress-couple-moment M at any section m the value $M = Px - M_0$. We have also $x = r(1 - \cos\theta)$; $y = r\sin\theta$; and $ds = rd\theta$.

Since $$\phi, = \frac{1}{EI}\int_O^B M\,ds, = 0, \quad \therefore \int_O^B [Pr^2 - Pr^2\cos\theta - M_0 r]\,d\theta = 0;$$

i.e., $$\left(Pr^2[\theta - \sin\theta] - M_0 r\theta\right)_0^{\frac{\pi}{2}} = 0; \text{ or, } Pr^2\left[\frac{\pi}{2} - 1\right] - M_0 r\frac{\pi}{2} = 0;$$

whence, finally, we have $M_0 = Pr\left[1 - \frac{2}{\pi}\right]$ . . . . (1)

Now that M_0 is known, we may find M_B by taking moments about the lower section, O, in Fig. 442, with OB as free body whence $M_B = (2 \div \pi)\,Pr$, which is greater than M_0. Hence the equation for safe loading is $(R'I \div e) = 2Pr \div \pi$, where R' is the maximum safe unit-stress for the material, and e the distance of the extreme fiber from the gravity axis of a section. (If, however, the radius, r, of the cylinder is not large compared with the radial thickness of the section, see §§ 298 and 299.)

Evidently the horizontal diameter has been lengthened by an amount $2\,\Delta x$, if Δx denote the horiz. proj. of O's displacement relatively to B and B's tangent; and similarly, the shortening of the vertical diameter is $2\,\Delta y$, if Δy denote the vert. proj. of O's displacement with regard to B and B's tangent-line.

Hence

$$\Delta x = \frac{1}{EI}\int_{\theta}^{B} Myds = \frac{1}{EI}\int_{0}^{\frac{\pi}{2}}[Pr^3 \sin\theta . d\theta - Pr^3 \cos\theta \sin\theta d\theta - M_0 r^2 \sin \theta d\theta];$$

from which we have, with $M_0 = Pr\,[1 - (2 \div \pi)]$,

$$\Delta x = \frac{Pr^3}{EI} \cdot \frac{4 - \pi}{2\,\pi}; \text{ while}$$

$$\Delta y = \frac{1}{EI}\int_{\theta}^{B} Mxds, = \frac{1}{EI}\int_{0}^{\frac{\pi}{2}}[Pr^3\,(1 - \cos\theta)^2\,d\theta - M_0 r^2\,(d\theta - \cos\theta d\theta)],$$

$$\text{i.e.,} \quad \Delta y = \frac{Pr^3}{EI} \cdot \frac{\pi^2 - 8}{4\,\pi} \cdot$$

It will be noted that the results obtained in this problem apply also to the case where the hoop is a circular link of a chain under a tension 2 P, except that the moments will be of opposite character and shears and thrusts of opposite direction. Also, the change of length 2 Δx of the horiz. diameter will be a *shortening*, that of the vertical diameter, a *lengthening*. (See Prof. Filkins' article on p. 99 of Vol. IV of the Transac. of Assoc. C. E. of Cornell Univ. and Engineering News, Dec. 1904, p. 547.)*

Numerical Example. Fig. 441. The length of a cast iron pipe is 10 ft., the thickness of wall ½ inch, and the radius of the pipe (measured to the middle of the thickness) is 6 inches. Required, the value of the safe load at crown, 2 P when the pipe is supported horizontally on a firm smooth bed or floor; the max. safe unit-stress being taken at the low figure 2000 lbs. per sq. inch.

Solution. We have only to substitute these values in $M_B = R'I \div e$ and obtain (since $I = l.\,t^3 \div 12$), $\frac{2000 \times 120 \times (\frac{1}{2})^3}{(\frac{1}{2}) \times (\frac{1}{2}) \times 12} = (0.6366\,P \times 6)$; hence *safe load* $= 2\,P, = 5236.$ lbs.; that is, 43.63 lbs. per running inch of pipe length.

If now the thickness be doubled, i.e., $t = 1''$, with other data unchanged, we find the safe load to be four times as great, i.e., 2 P = 20,944 lbs.; or 174.5 lbs. per running inch of pipe-length.

Although the load is called "concentrated" as regards the *end-view* of the pipe, it must be understood to be uniformly distributed along the length.

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* Experiments on chain links and circular rings have been made at the Engineering Experiment Station of the University of Illinois. (See bulletins of 1907.) The results show a remarkable agreement with theory.

CHAPTER XII.

FLEXURE OF BEAMS; BOTH SIMPLE AND CONTINUOUS. GEOMETRICAL TREATMENT.

392. By Geometrical Treatment is meant making use of the properties of geometrical figures to deduce algebraic relations. This *does not necessitate the use of drafting instruments*; but the graphic ideas involved greatly simplify the algebraic detail of finding deflections, angles, moments, shears, etc., in the case of horizontal beams originally straight and slightly bent under vertical loads and reactions. In the case of "continuous beams", or "girders", (p. 320), this mode of treatment leads to conceptions and methods which are remarkably clear and simple.

393. Angle Between End-Tangents of a Portion of a Bent Beam. If the cantilever beam of Fig. 443*a* (slender and originally straight) be loaded as shown, and the beam thus slightly bent, the two cross-sections, *AH* and *CD*, at the two ends of any dx of the axis of beam, are no longer parallel but become inclined at a small angle $d\phi$ which is also the angle between the normals to these sections, in their new (relative) position (see now Fig. 443*b*). *AH* now occupies the position *A′H′* (relatively to *CD*). The outer fiber *AC* (originally of length $= dx$) is longer by some amount $d\lambda$; and evidently the value of angle $d\phi$ may be written $= d\lambda \div e$. But, by the definition of the modulus of elasticity of the material, E, we have also

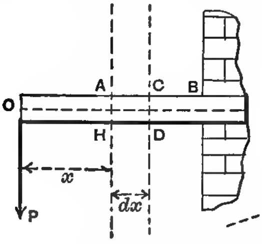


FIG. 443*a*.

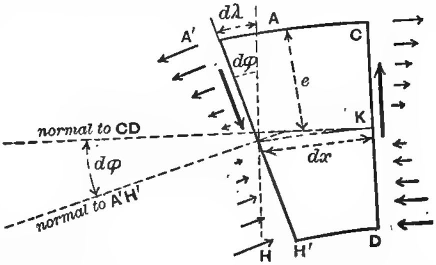


FIG. 443*b*.

$$E = p \div \frac{d\lambda}{dx}, \text{ (p. 209); whence } d\phi = \frac{p\,dx}{Ee} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now if M denote the moment of the stress-couple to which the tensions and compressions on the ends of the fibers in the section $A'H'$ are equivalent (M would equal Px in this simple case) we may combine the relation, (§ 229), $M = pI \div e$ with eq. (1) and thus derive, as

a fundamental relation: . . . $d\phi = \dfrac{Mdx}{EI}$ . . . . . . (2)

for the angle between two tangents to the elastic curve, one at each end of the elementary length, dx, of the curve; since the two normals to the sections $A'H'$ and $C\ D$ in Fig. 443b are tangents to the ends of the short length dx of the elastic curve. (This value of the angle $d\phi$ is in π-measure; i.e., *radians*.)

It follows, therefore, that when the cantilever of Fig. 443a is gradually bent from its original condition (in which the tangent lines at the two extremities O and B were coincident, i.e., made with each other an angle of zero) into its final form, by the gradual application of the load P at O, the angle between the tangent to elastic curve at O_n (the final position of O) and that at B (which tangent, in this case, has not moved) will have a value obtained by summing up all the small values of $d\phi$, one for each of the dx's between O and B (these dx's making up the length of curve between those points).

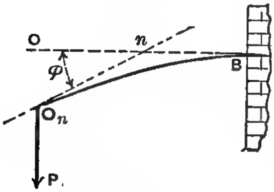


FIG. 444.

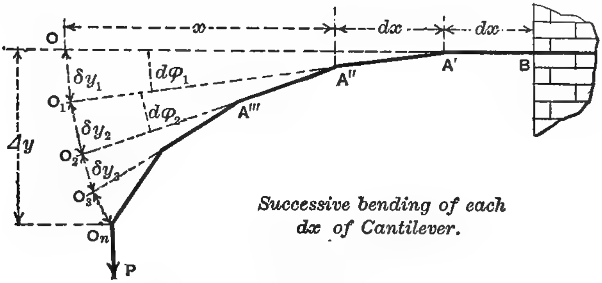
Or, **in general**, if O_n and B are **any two points** of an elastic curve (of axis of bent beam, originally straight and now only slightly bent, x being measured along the beam) we have for the

angle between the tangents at O_n and B $\Big\} = \phi = \displaystyle\int_o^B \frac{Mdx}{EI}$ . . . . (3)

(See Fig. 444 for case of cantilever.) This may be called the *angle between end-tangents* of any portion of such elastic curve. The beam must be *continuous* between these two points and only slightly bent. Usually the beam in question is homogeneous and then E may be taken outside of the integral sign. Also, if the beam be prismatic in form (i.e., sides parallel to a central axis, originally straight) the moment of inertia,

I, of the cross-section is the same for each dx, and may be placed outside of the $\int$ sign.

394. (Relative) Displacement of any Point, O, of Elastic Curve of a Bent Beam. In the case of the simple cantilever of Fig. 443a let us consider that the axis of the beam, originally straight and in position OB, passes gradually into its final form or elastic curve $O_n\ A'''\ A''\ .\ .\ .\ B$ by the *successive* change of form of each small block, or elementary length dx; beginning at the end B. When the section at A' turns through its angle $d\phi_1$, as due to the lengthening and shortening of the fibers forming the block (i.e., to the stress-couple in section A', of moment M') it carries with it all the portion OA' (still straight) into position O_1A' so that the extremity O describes a small distance (practically vertical) $\overline{OO_1} = \overline{OA'}\ .\ d\phi_1$. Similarly when, next in order, the section at A'' turns through its small angle $d\phi_2$, the left-hand end of the beam describes a further small distance $\overline{O_1O_2} = \overline{OA''}\ .\ d\phi_2$; and so on; until finally the extremity O has arrived at its final position O_n, having executed a total (vertical) displacement $\overline{OO_n}$, which will be called Δy.



Successive bending of each dx of Cantilever.

FIG. 444a.

If, now, any one of the elementary vertical displacements (like $\overline{O_1O_2}$, as typical) be called δy, we note that Δy is the sum of all these small δy's, each of which is practically a small circular arc described with a radius x swinging through a small angle $d\phi$, (the successive x's being successively smaller for the δy's lower in the series), so that $\delta y = xd\phi$; hence

$$\Delta y, = \int \delta y, = \int x d\phi. \text{ But, from eq. (2), } d\phi = Mdx \div EI;$$

$$\therefore \left\{ \begin{matrix} \text{Displacement of point } O \\ \text{relatively to } B\text{'s tangent} \end{matrix} \right\} = \Delta y = \int_0^B \frac{Mxdx}{EI} \quad . \quad . \quad . \quad (4)$$

(N.B. *In the use of this relation the x of each dx must be measured from the point O whose displacement is desired.*)

Although the special case of the cantilever has been in mind in the figure used in this connection, this result in eq. (4) may be generalized by stating that it gives the displacement Δy of any point O from the tangent-line drawn at any other point, B, of the elastic curve formed by the axis of a beam originally straight and slightly bent under the action of vertical forces and reactions. In order to use it, the value of the moment M of the stress-couple in each successive dx must be expressed as a function of x. If, in addition, the beam has a constant moment of inertia, I, of the cross-sections, the "I" may be taken outside of the sign of integration. An integration is then generally possible. (For example, in the above cantilever, for M we should write Px.)

395. Deflections and Slopes of Straight Homogeneous Prismatic Beams Slightly Bent under Vertical Loads and Reactions. (Beam Horizontal.)

If the beam is a *prism* and homogeneous, both E and I are constant along its length and may be taken outside of the integral sign in eqs. (3) and (4), and these two equations may now be applied to a portion of a beam situated between any two points O and B of the elastic curve assumed by the (originally straight) axis of the beam (Fig. 444b) under some load. The tangent-lines at O_n and B were originally coincident, and hence the angle between these tangents when the beam is bent is the *total change* in angle between the tangents, and consequently may be written $\phi = \frac{1}{EI}\int_0^B Mdx$ and is O_nCO in the figure. Again, if a vertical be drawn through the point O_n to B's tangent-line OB, the length $\overline{OO_n}$ is evidently O's displacement relatively to B's tangent-line, since originally the point O_n was situated in B's tangent itself.

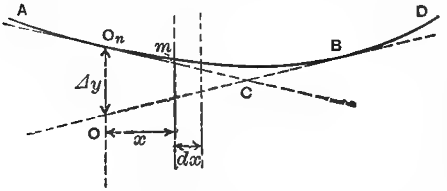


FIG. 444b.

That is, $\overline{OO_n}$, or Δy, $= \frac{1}{EI}\int_0^B Mxdx$, in which M is the moment of the stress-couple in the cross-section at any distance, x, from O. Note that in general M is a *variable;* also that the x must be measured from the point O whose displacement is under consideration.

Example I. *Simple cantilever* (Fig. 444_1), *built in horizontally at B and bearing a concentrated load $= P$ lbs. at the free extremity.* Both E and I are constant (homogeneous prism). Find the deflection $\overline{OO_n}$ and the slope ϕ.

Solution. From the free body O_nm (m being any point between O and B) we have $M = Px$ as mom. of stress-couple at m.

$$\phi = \frac{1}{EI}\int_O^B M\,dx = \frac{P}{EI}\int_0^l x\,dx = \frac{Pl^2}{2EI} = \text{the "slope" at } O_n. \quad \text{For } \Delta y$$

we have $$\Delta y, = \overline{OO_n}, = \frac{1}{EI}\int_0^B Mx\,dx = \frac{P}{EI}\int_0^l x^2dx = \frac{Pl^3}{3EI}.$$

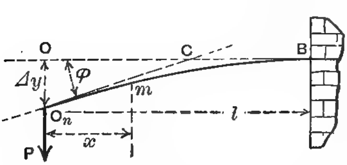


Fig. 444_1.

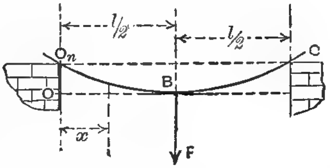


Fig. 444_2.

Example II. *Prismatic beam on two end-supports. Concentrated load P, lbs., in middle,* Fig. 444_2. The two supports being at same level we note that from symmetry the tangent at the middle point B of the elastic curve is horizontal. Hence the displacement $\overline{OO_n}$ of the extremity O from this tangent is equal to the deflection of B itself below the horizontal line O_nC. To find $\overline{OO_n}$ or Δy,

$$\Delta y = \frac{1}{EI}\int_O^B Mx\,dx = \frac{1}{EI}\int_O^B \left[\frac{P}{2}\,x\right] x\,dx = \frac{P}{2EI}\int_{x=0}^{x=\frac{l}{2}} x^2dx = \frac{Pl^3}{48\,EI}.$$

Example III. *Prismatic beam on two end-supports at same level, the load being uniformly distributed over the whole span, l.* Fig. 444_3. That is, $W = wl$,

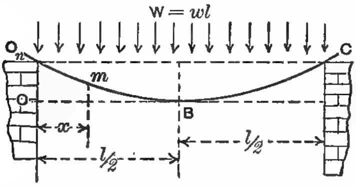


Fig. 444_3.

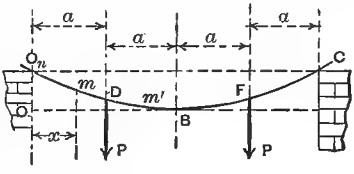


Fig. 444_4.

w being the load per running inch. As before, the tangent-line at middle point B of the elastic curve must be horizontal, so that the displacement of extremity O_n from this tangent will also give the deflection of B from the horizontal O_nC. Measuring x from O (as must always be done in these cases) we note that M at any point $m = \frac{W}{2}x - \frac{wx^2}{2}$.

$$\overline{OO_n} = \frac{1}{EI}\int_O^B\left[\frac{Wx^2}{2}dx - \frac{wx^3}{2}dx\right] = \frac{1}{EI}\left[\frac{Wx^3}{6} - \frac{wx^4}{8}\right]_0^{\frac{l}{2}} = \frac{5}{384}\cdot\frac{Wl^3}{EI}.$$

Example IV. *Prismatic beam on end-supports, bearing two equal loads, each $= P$, symmetrically placed on the span.* Fig. 444_4. Required, the deflection of the middle point, B, of the elastic curve, below the horizontal O_nC. Length $= 4a$.

Solution. In previous problems of this article the expression for M, the mom. of stress-couple for any point m between the points O and B, has been a single function of x, applying to all such points m. But in the present problem, having found the reaction at O to be $= P$ lbs., we note by considering a free body O_nm (where m is any point between O_n and D) that the value of M is $M = Px$; whereas if the free body extends into the portion DB the expression for M (the free body being now O_nm') is $M = Px - P(x - a)$ which reduces to $M = Pa$, a *different* function of x; (in fact a constant). Therefore, in making the summation $OO_n = (1 \div EI)\int Mx\,dx$ for all the dx's between O and B, this summation must be divided into two parts, viz.: one from O to D, involving for x the limits $x = 0$ and $x = a$; and the other from D to B, for which the limits for x are $x = a$ and $x = 2a$. Hence

$$\overline{OO_n} = \frac{1}{EI}\int_0^B Mx\,dx = \frac{1}{EI}\left[\int_{x=0}^{x=a} Px^2dx + \int_{x=a}^{x=2a} Pa\,.\,x\,dx\right] = \frac{11}{6}\cdot\frac{Pa^3}{EI}. \quad (1)$$

(The student should verify all details of this operation, noting that each summation or integral contains the proper value of M, *as a function of x*, for the proper portion of the elastic curve. As before, it should be said that on account of symmetry the tangent-line at the middle point B is horizontal, and parallel to O_nC. *Otherwise $\overline{O_nO}$ would not be equal to the deflection of B.*)

396. Non-prismatic Beam. Variable Moment of Inertia, I. If the I is variable, (e.g., if the beam tapers) it must be retained on the right of the integral sign in the expressions for ϕ and Δy and then expressed as a function of x before the integration can be proceeded with. In some cases I may be constant within the limits of definite portions of the beam and then the procedure is simple. For instance, if the beam in Fig. 444_4 has a constant value, $= I_1$, for the portions OD and FC, and a larger (but constant) value, of I_2, $= \frac{3}{2} I_1$, for all the sections from D to F, the following takes the place of eq. (1) above:

$$\overline{OO_n} = \frac{P}{EI_1}\int_0^a x^2dx + \frac{Pa}{EI_2}\int_a^{2a} xdx = \frac{4}{3}\cdot\frac{Pa^3}{EI_1} \quad . \; . \; . \; . \; (1a)$$

397. Properties of Moment Diagrams (Moment-Areas and Centers of Gravity). Prismatic Beams in Horizontal Position. Vertical Loads and Reactions. In Fig. 445 let AD be the bent condition (i.e., elastic curve) of the axis of a straight prismatic homogeneous beam supported on supports at, or nearly at, the same level (so that all tangent lines to the elastic curve deviate but slightly from the horizontal. That is, the bending is slight). Also, let $A''D''B''' O'''$ be the corresponding *moment diagram* (as defined and illustrated on pp. 265 to 309). For instance, for any point m of the elastic curve the moment of stress-couple (or "bending moment") in that section of the bent beam is represented (to scale) by the ordinate $m''m'''$ or M, in the same vertical as m.

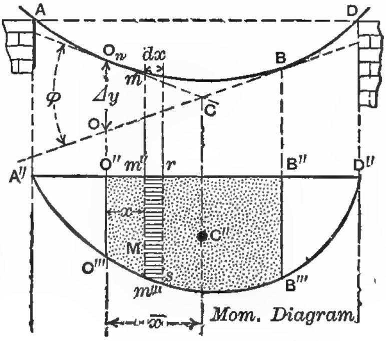


Fig. 445.

If now a small horiz. distance dx, or $m''r$, be laid off from m'' and a vertical $r\,..\,s$ be drawn through r, the product $M\,.\,dx$ would be proportional to, and may be represented by, the area of the vertical strip $m''rsm'''$. Now dx being inches (say) and M being inch-lbs., this product might be called so many "sq. inch-lbs." of *moment-area* (as it will be called). But the angle ϕ between the tangent-lines drawn at any two points O_n and B of the elastic curve is equal to $\frac{1}{EI}\int_O^B Mdx$; and hence we may write

$$\phi = \left[\begin{matrix}\text{total `` moment-area ''}\\ \text{between } O \text{ and } B\end{matrix}\right] \div EI \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

or, for brevity, $\phi = (A_O^B) \div EI$ . . . . (1)

This "moment-area," then, between O and B is the product of the base $O''B''$ (inches) by the average moment between O and B regarded as the average altitude of the figure, $O''B''B'''O'''$, this altitude being inch-lbs.

Again, if the elementary "moment-area" Mdx be multiplied by x, its horizontal distance from O'' (i.e. from O and O_n), and these products summed up for all the dx's between O and B, there results the expression $\int_O^B (M\,.\,dx)\,.\,x$ which may be written $(A_O^B)\,.\,\bar{x}$, where $\bar{x}$ denotes the horiz. distance of the *center of gravity* of the moment-area $O''B'''$ from $O''O'''$ (since, from the theory of the center of gravity, the sum of the products of each *strip* of an area by its x co-ordinate is equal to the product of the whole area by the distance of its center of gravity from the same axis). In the figure the center of gravity of the moment-area $O''B'''$ is shown at C'', and the corresponding $\bar{x}$ is marked.

But we have

$$\Delta y, \text{ or } \overline{OO_n}, = \left[\int_0^B Mxdx\right] \div EI = \left[\int_0^B Mdx \, . \, x\right] \div EI;$$

and hence we may write

$$\overline{OO_n}, \text{ or } \Delta y, = [(A_0^B) \, . \, \bar{x}] \div EI \quad . \quad . \quad . \quad . \quad (2)$$

which furnishes us with a simple means of determining the displacement of any point O_n in the elastic curve of the bent beam from the tangent-line at any other point B in that elastic curve.

Evidently, from equations (1) and (2) we have $\Delta y, = \overline{OO_n}, = \phi\bar{x}$; and can therefore state that the *intersection of the two tangent-lines, one drawn at O, the other at B, lies in the same vertical as the center of gravity of the intervening moment-area.*

(N.B. Instead of the product $(A_0^B) . \bar{x}$, we may, of course, use the algebraic sum of similar products for any component parts into which it may be convenient to subdivide the total moment-area.)

398. Examples of Use of Eqs. (1) and (2) of Preceding Paragraph.

Example I. Simple Cantilever. *Concentrated load at free end.* **Fig. 444$_1$. Constant E and I. (*Prism.*)** Here the moment-diagram for whole length is a ***triangle*** (§ 249) whose base is l inches and whose altitude is Pl inch-lbs.

Hence, with O and B taken as in **Fig. 445$_1$**, we note that

$$A_0^B = \left(l \, . \, \frac{Pl}{2}\right), \text{ and that } \bar{x} = \frac{2}{3}l.$$

$$\therefore \phi = \left[l \, . \, \frac{Pl}{2}\right] \div E \, . \, I. = \frac{Pl^2}{2\,EI}; \text{ for the slope}$$

at O_n; while $\overline{OO_n} = \frac{1}{EI}(A_0^B) . \bar{x}$,

$$\text{i.e., } \overline{OO_n} = \frac{1}{EI}\left[l \, . \, \frac{Pl}{2}\right] \cdot \frac{2}{3}l = \frac{Pl^3}{3 \, . \, EI}.$$

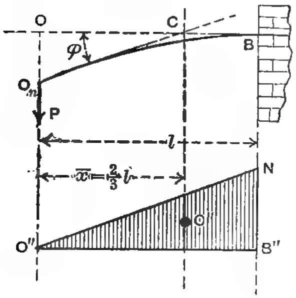


FIG. 445$_1$.

Example II. Prismatic Beam on Two End-Supports. *Load Uniformly distributed over the whole span or length, l; $W = wl$.* From p. 268 we know that the moment-diagram (Fig. 445$_2$) is a symmetrical segment of a parabola with axis vertical, and that the moment at the middle section is $Wl \div 8$. Also, from p. 12 of Notes, etc., in Mechanics, we find the $\bar{x}$ of the left-hand half of this moment-figure, measured from the left-hand extremity O_n, is $\frac{1}{2}l - \frac{3}{8}$ of $\frac{1}{2}l$; i.e., $\bar{x} = \frac{5}{8}$ of $\frac{1}{2}l$.

The area of this semi-parabolic-segment is two-thirds that of the circumscribing rectangle. From symmetry, the tangent-line drawn at B, the middle point of the elastic curve, is parallel to $O_n D$, so that the displacement $\overline{O_n O}$ of O from that tangent is equal to the deflection of B from $O_n D$. Hence

$$\overline{O_n O}, = \frac{(A_0^B) . \bar{x}}{EI}, = \left[\frac{2}{3} \cdot \frac{l}{2} \cdot \frac{Wl}{8} \cdot \frac{5}{8} \cdot \frac{l}{2}\right] \div EI = \frac{5}{384} \cdot \frac{Wl^3}{EI}.$$

Example III. *Prismatic Beam. Ends Supported. Two Concentrated Loads Equidistant from Supports.* Fig. 445_3.

Here, as before, from symmetry the tangent at B is horizontal, parallel to

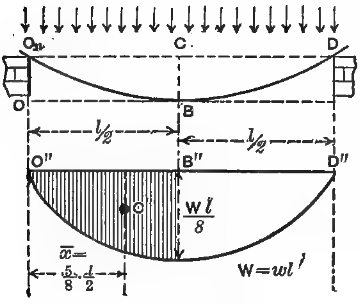


FIG. 445_2.

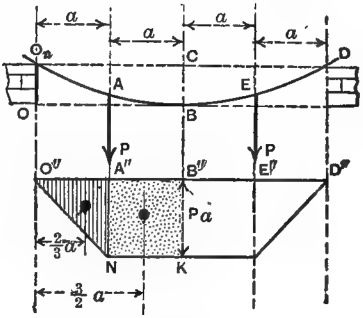


FIG. 445_3.

O_nD; so that O_nO equals the deflection of B from C (its position before loading of beam). Each load P is in middle of a half-span. Required $\overline{O_nO}$; i.e., $CB = ?$

In this case the moment-diagram is easily shown to consist of a triangle at each end with a central rectangle of altitude $= Pa$ (inch-lbs.). To find $\overline{O_nO}$ we need the product $(A_O^B) . \bar{x}$. But this A_O^B consists of the triangle $O''A''N$ with its center of gravity distant $\frac{2}{3}$ of a from O and of the rectangle $A''B''KN$ whose center of gravity is at a distance of $\frac{3}{2}$ of a from O. Utilizing, therefore the principle stated in the N.B. of § 397, we write

$$\overline{O_nO}_O = \frac{(A)_O^B . \bar{x}}{EI} = \frac{1}{EI}\left[a \cdot \frac{Pa}{2} \cdot \frac{2a}{3} + a . Pa \frac{3a}{2} \right] = \frac{11}{6} \cdot \frac{Pa^3}{EI}.$$

Example IV. *Prismatic Beam on End Supports. Single Eccentric Load, P.* Fig. 445_4. Here a tangent drawn to the elastic curve at the load-point B, not being horizontal, is *not parallel to* O_nC_n, and hence $\overline{O_nO}$ does *not* = the deflection, δ, of B.

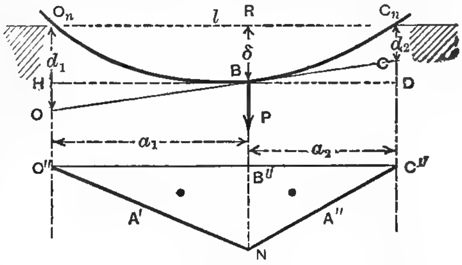


FIG. 445_4.

However, the displacements ($= d_1$ and d_2), of O from B's tangent and of C from B's tangent, are easily found, the moment-diagram $O''NC''$ having been drawn, in which $\overline{B''N} = (Pa_1a_2 \div l)$ inch-lbs. (§ 260). Call the "moment-area" of triangle $O''B''N$, A'; and that on right of load, viz. of $C''B''N$, A''.

Then, from eq. (2) of § 397, we may write

$$EId_1 = A'\bar{x}_1 ; \text{ and } EId_2 = A''\bar{x}_2 .$$

If now we draw a horizontal line, HD, through the point B of the elastic curve, we note, from the similar triangles thus formed, the proportion

$\frac{d_1 - \delta}{\delta - d_2} = \frac{a_1}{a_2}$. From these three equations d_1 and d_2 may be eliminated and δ

obtained; (since $A = \frac{1}{2} Pa_1^2a_2 \div l$, and $A'' = \frac{1}{2} Pa_2^2a_1 \div l$; while $\bar{x}_1 = \frac{2}{3} a_1$, and $\bar{x}_2 = \frac{2}{3} a_2$.)

We thus obtain $\delta = (\frac{1}{3} Pa_1^2a_2^2) \div (EI.l)$ . . . . (3)

This is for the load-point. For the *maximum* deflection see next example.

Example V. *Maximum Deflection of Prismatic Beam. End Supports. Single Eccentric Load.* Fig. 445_5. To locate the lowest point D of elastic curve and determine its deflection, d, below the horizontal O_nB.

Draw a tangent at D, also at B whose distance n from D is, as yet, unknown. Note that the tangent at D is *horizontal.*

The moment-diagram is a triangle of altitude M'; $(M' = Pab \div l)$; denote the moment at D by m'. We have $m' = (n \div b) . M'$. Now the angle $\phi = d' \div l$, and

$$d' = (A_U^B) . \bar{x} \div EI = \frac{1}{2} M'l\,(a + \tfrac{2}{3}[\tfrac{1}{2}(a+b) - a]) \div EI.$$

$\therefore 6EI\phi = M'(2a + b)$. But $\phi_1 = (A_D^B) \div EI, = n . m' \div 2EI$, and $\phi = \phi_1$; $\therefore$, finally, we have $n = \sqrt{\frac{1}{3}b(2a+b)}$, which locates the point R.

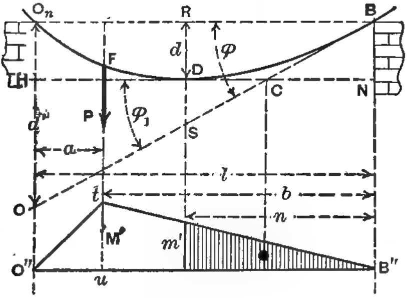


FIG. 445_5.

Now note that the intersection C lies in the vertical through the center of gravity of the shaded triangle (§ 397). Hence $\overline{CD} = \frac{1}{3}n$ and therefore from similar triangles $\overline{DS} = \frac{1}{3}\overline{RS}$. But $\overline{RD}, = d, = \overline{RS} - \overline{DS}$, and $\overline{DS} = \phi_1 . \overline{CD} = \phi . \frac{1}{3}n$. Hence $d = \frac{2}{3}\phi n$ and finally by substitution, and with M' placed $= Pab \div l$, we have (with $b > a$)

$$d = \frac{1}{9} \cdot \frac{Pab}{EI.l}[2a + b]\sqrt{\tfrac{1}{3}b(2a+b)} \quad . \quad \left\{ \begin{matrix} \text{same as} \\ \text{on page 258} \end{matrix} \right\}$$

399. The "Normal Moment Diagram." If a portion, OB, of a horizontal beam carrying loads, be conceived separated from the remainder of beam and placed on two supports at its extremities O and B, while carrying the loads [say P_1 and P_2] originally lying between O and B, the corresponding moment-diagram, $O'''TB'''$ of Fig. 446 may be called the "**normal-moment diagram**" for portion OB (of original beam) and its load. If V_0 is the pier reaction at left, we have for any section t (say between P_1 and P_2) x ft. from O, the moment of stress-couple [call it M_n or "**normal moment**"]

$$M_n = V_0x - P_1(x - a) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now consider OB in its original condition (see lower part of Fig. 446) when forming part of a much longer beam supported in *any manner*. If we consider OB, *now*, as a "free body," we must put in, besides the loads P_1 and P_2, a shear J_0 and a stress-couple of moment M_0 in section at O, and J_B and couple of moment M_B at B. The moment in any section t of OB is now $M = M_0 + J_0x - P_1(x - a)$. Let V = difference between J_0 and V_0,

i.e., $$J_0 = V_0 + V;$$

then $$M = M_0 + Vx + [V_0x - P_1(x - a)] \quad . \quad (2)$$

i.e. [see (1)], $$M = M_0 + Vx + M_n \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

Hence the moment M ($= kw$ in Fig. 446) of any section of OB is made up of a constant part M_0, a part proportional to x, and a third part equal to the "*normal moment*" of that section. Therefore, if, in the moment-diagram $O'B'B''wO''$ for OB we join O'' and B'' by a straight line, and also draw a horizontal through O'', the vertical intercepts [such as uw] between the line $O''B''$ [or "chord"] and the broken line $O''wB''$ are the *normal moments* for OB and its load, and the area (mom.-area) of the figure formed by these intercepts is equal to that of the *normal moment diagram*.

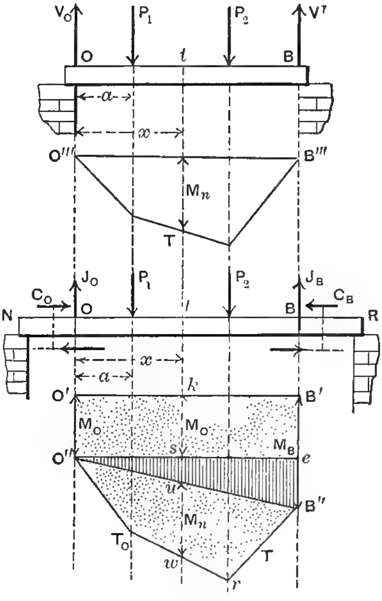


FIG. 446.

It is also evident that the center of gravity of the figure $O''B''w$ lies in the same vertical as that of the *normal moment diagram*.

(In the next paragraph the trapezoid $O'B'B''O''$ will be divided into two triangles, instead of into a triangle and a rectangle.)

400. The Theorem of Three Moments. Let O, B, and C, Fig. 446a, be any three points in the elastic curve of a homogeneous, *continuous*, and *prismatic* beam, originally straight and horizontal but now slightly bent under vertical forces (some of which are reactions of supports; no loads or forces are shown in the figure).

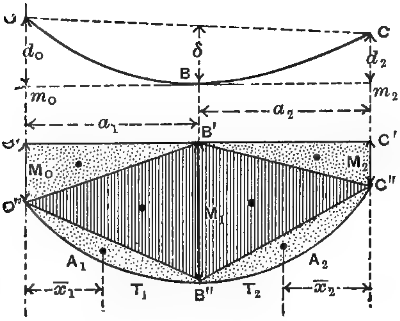


FIG. 446a.

Let M_0, M_1, and M_2 be the moments of the couples in sections O, B, and C. The moment-diagram for portion OBC is $O'C'C''T_2B''T_1O''$. Join $O''B'$ and $C''B'$; also $O''B''$ and $C''B''$. At the point B of elastic curve draw a tangent m_0m_2 and join OC. Then $\overline{Om_0}$, or d_0, is the displacement of point O from B's tangent, and $d_2 = \overline{Cm_2}$, is the displacement of C from the same tangent; while δ is the *deflection of B from the straight line joining O and C.*

Now the vertical displacement $d_0 =$ [mom.-area $O'B'T_1 \times$ distance of its cent. grav. from OO''] $\div EI$. But the moment-figure $O'B'T_1$, under OB, is composed of the two triangles shown and the "*normal moment-diagram*" *for OB*, viz.: $O''B''T_1$, whose mom.-area may be called A_1 and whose center of gravity is $\bar{x}_1$ ft. from OO'', while the corresponding distances for the triangles are $\frac{1}{3}a_1$ and $\frac{2}{3}a_1$.

Hence, from eq. (2), § 397, we have:

$$EId_0 = \tfrac{1}{2}M_0a_1 \cdot \tfrac{1}{3}a_1 + \tfrac{1}{2}M_1a_1 \cdot \tfrac{2}{3}a_1 + A_1\bar{x}_1; \quad . \quad . \quad (1)$$

and similarly, with corresponding notation, for the right-hand portion, or segment, BC, of OBC (denoting the "normal mom.-area" $C''B''T_2$ by A_2 and reckoning $\bar{x}_2$, etc., from CC''),

$$EId_2 = \tfrac{1}{2}M_2a_2 \cdot \tfrac{1}{3}a_2 + \tfrac{1}{2}M_1a_2 \cdot \tfrac{2}{3}a_2 + A_2\bar{x}_2. \quad . \quad . \quad (2)$$

If now a straight line be conceived to be drawn through B parallel to OC, we have, from the similar triangles so formed, (as in Fig. 445_4), $(d_0 - \delta) \div a_1 = (\delta - d_2) \div a_2$. Combining this with eqs. (1) and (2) we have finally

$$\frac{M_0 a_1}{6} + \frac{M_1(a_1 + a_2)}{3} + \frac{M_2 a_2}{6} + \frac{A_1 \bar{x}_1}{a_1} + \frac{A_2 \bar{x}_2}{a_2} = EI\delta\left[\frac{1}{a_1} + \frac{1}{a_2}\right] \quad (4)$$

which is the "**Theorem of Three Moments.**"

E is the modulus of elasticity of material of beam, I the "moment of inertia" of its cross-section; M_0, M_1, M_2, the moments of stress-couples ("bending-moments") at O, B, and C, respectively. Distances a_1 and a_2 are shown in Fig. 446a, while A_1, A_2, $\bar{x}_1$, and $\bar{x}_2$ are as above; δ being the deflection of point B from the straight line joining O and C.

N.B. It should be carefully noted that eq. (4) does not apply unless the part of beam from O to C is *continuous* and *prismatic;* also that in its derivation, the elastic curve is considered *concave upward* throughout; hence if a negative number is obtained for M_0, M_1, or M_2, in any example by the use of eq. (4), it implies that at that section the beam is *convex* upward, instead of concave; in other words that the upper fibers are in tension and the lower in compression (instead of the reverse, as in Fig. 446a).

401. Values of $A_1\bar{x}_1$ and $A_2\bar{x}_2$ in Special Cases. The Theorem of Three Moments involves the use of the (imaginary) *normal mom.-area* of each of the two portions (left and right "spans", or "panels"), OB and BC; i.e. of the products $A_1\bar{x}_1$ and $A_2\bar{x}_2$, where $\bar{x}_1$ is measured from the *left* end of the *left* panel, and $\bar{x}_2$ from the *right* end of the *right* panel. We are now to determine values of $A_1\bar{x}_1$ and $A_2\bar{x}_2$ for several ordinary cases of loading.

Case I. *Single Central Concentrated Load, P,* Fig. 446_1. Here, for a left-hand panel,

$$A_1\bar{x}_1 = \frac{Pl}{4} \cdot \frac{l}{2} \cdot \frac{l}{2},$$

$= \frac{Pl^3}{16}$; and for a right-hand panel,

$$A_2\bar{x}_2 = \frac{Pl^3}{16}.$$

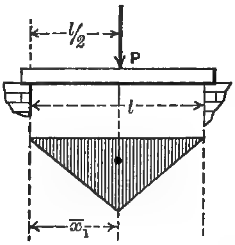


FIG. 446_1.

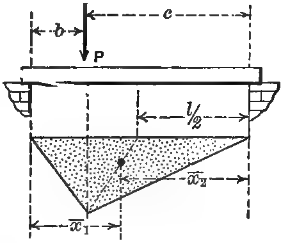


FIG. 446_2.

Case II. *Single Non-central Concentrated Load, P.* Fig. 446_2. For this case as a left-hand panel,

$$A_1\bar{x}_1 = \left[\frac{Pbc}{l} \cdot \frac{l}{2}\right]\left[\frac{l}{2} - \frac{1}{3}\left(\frac{l}{2} - b\right)\right] = \frac{Pbc\,(l + b)}{6}$$; while, as a right-hand panel, $A_2\bar{x}_2 = \frac{Pbc\,(l + c)}{6}$

Case III. *Two* (or more) *Concentrated Loads.* Fig. 446_3.
$A_1\bar{x}_1 = \frac{1}{6}[P'b'c'\ (l + b') + P''b''c''\ (l + b'')]$; and for each load more than two add a proper term in the bracket.

For $A_2\bar{x}_2$ interchange b' and c', b'' and c'', etc.

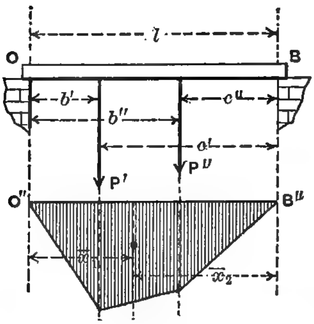


FIG. 446_3.

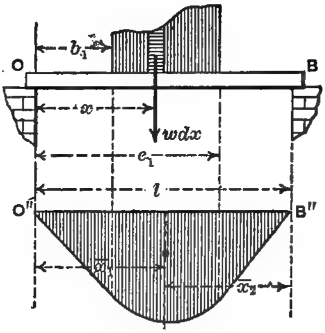


FIG. 446_4.

Case IV. *Any Continuous Load over a Part or the Whole of the Span;* of w lbs. per linear foot, w being variable or constant. Fig. 446_4. The load on a length dx (of loaded part) is wdx lbs.; comparing which with the P of Case II, (or one of the P's of Case III), we note that x corresponds to b, and $l-x$ to c;

hence
$$A_1\bar{x}_1 = \frac{1}{6}\int_{x=b_1}^{x=c_1} wdx\,(l - x)\,x\,(l + x) = \frac{1}{6}\int_{b_1}^{c_1} wx\,(l^2 - x^2)\,dx.$$

If w is variable it must first be expressed in terms of x. (For $A_2\bar{x}_2$ we measure x from the right-hand end, B.)

Case V. *Uniformly Distributed Load over Whole Span;* (i. e., w is *constant*). Let W, $= wl$, = whole load, lbs. Fig. 446_5.

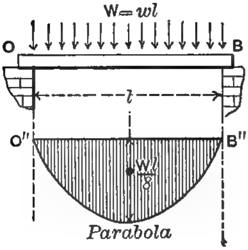


FIG. 446_5.

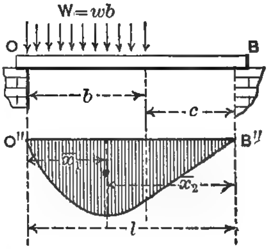


FIG. 446_6.

$$A_1\bar{x}_1 = A_2\bar{x}_2$$
$$= \tfrac{2}{3} \cdot l \cdot \frac{Wl}{8} \cdot \frac{l}{2}$$
$$= \frac{Wl^3}{24}.$$

Case VI. *Uniformly Distributed Load Adjoining one End of Span;* (left end for example). Fig. 446_6. Total load $= W = wb$. Applying method of Case IV, with $c_1 = b$, and $b_1 = 0$, we have $A_1\bar{x}_1 = \frac{1}{12} Wb\,(l^2 - \frac{1}{2}b^2)$. Also from Case IV, now measuring x from B, $A_2\bar{x}_2 = \frac{1}{24} W\,(l^2 - c^2)\,(l + c)$.

Case VII. *Uniformly Distributed Load Not Adjoining either End of the Span.* Fig. 446_7. Whole load $= W = w(e-b)$. By Case IV, we find

$$A_1\bar{x}_1 = \frac{W(e+b)}{12}\left[l^2 - \frac{e^2+b^2}{2}\right];$$

$$A_2\bar{x}_2 = \frac{W(k+f)}{12}\left[l^2 - \frac{k^3+f^2}{2}\right].$$

It is now seen how $A_1\bar{x}_1$ and $A_2\bar{x}_2$ may be obtained for any loading.

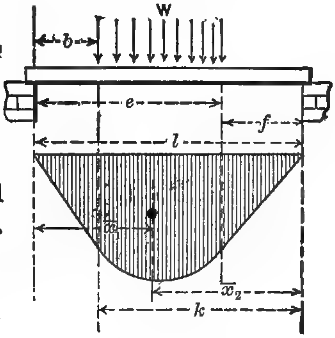


FIG. 446_7.

402. Continuous Girders Treated by the Theorem of Three Moments.

This theorem is of special advantage in solving continuous beams (p. 271); and examples will now be given.

Example I. Fig. 447_1. A straight, homogeneous, *prismatic* beam or girder, 35 feet long, is placed upon three supports at the *same level*, forming two spans of 15′ and 20′; two concentrated loads in the left span, a uniformly distributed load on part of right span. Required the maximum moment, and maximum shear. (Neglect weight of beam.)

Take O, B, and C, as the three sections where the three moments M_0, M_1, and M_2 are situated [respectively] used in the theorem of § 400. But both M_0 and M_2 are *zero* in this case, and δ (deflection of point B from line joining O and C) is also zero (since the supports are on the same level). Hence M_1 (i.e., at B) is the only unknown quantity in applying the theorem of § 400 (eq. (4)) to this problem.

Taking the $A_1\bar{x}_1$ from Case III, and $A_2\bar{x}_2$ from Case VII (with $f=0$), of § 398, we obtain (using the *foot* and *ton* as units),

$$0 + \frac{M_1(15+20)}{3} + 0 + \frac{1}{6\times 15}[6\times4\times11\times19+8\times10\times5\times25]$$

$$+\frac{16\times16}{12\times20}\left[20^2 - \frac{16^2}{2}\right] = 0;\ \text{and}\ \therefore M_1 = -\,39.2\ \text{ft.-tons.}$$

The negative sign shows that at section B the elastic curve is *convex* on its upper side (see N. B. in § 400). To follow up the solution from this point, let us draw the actual moment-diagram somewhat differently from that in Fig. 446*a*, (which see), where the actual moment for any section is measured from a continuous horizontal line, $O'C'$, as an axis. Let the "chords" $O''B''$ and $B''C''$ of the two normal moment figures, $O''B''T_1$ and $C''B''T_2$, be made a continuous horizontal line by an upward shifting (*each in its own vertical*) of the intercepts of those figures. The intercepts $O''O'$, $B''B'$, and $C''C'$, and the four triangles involved with them, now extend upward from that horizontal line. But in our present problem both M_0 and M_2 are *zero;* hence the two upper triangles disappear and the two inner triangles project above the horizontal line, with M_1 as a common base. The actual moments of the points of the elastic curve are now measured (in general) by the vertical intercepts between the lower boundary of the normal moment figures and the upper edges of the two triangles; but since in the present case M_1 is negative, M_1 must be laid off below the (new) horizontal line so that the lines $O''B'$ and $C''B'$ will *cross* the lower boundaries of the normal figures; *the actual moments being now measured by the vertical intercepts between these two oblique lines and the curved (or broken) boundaries of the normal figures.*

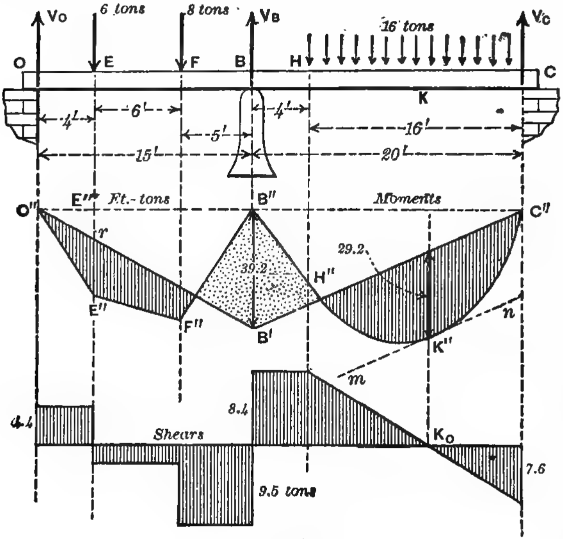


FIG. 447_1.

This re-arrangement has been observed in the moment-diagram $O''B''C''$ of Fig. 447_1, where *line-shaded* areas correspond to the parts of the elastic curve which are *concave upward;* and the *dot-shaded* areas, to parts *convex upward* (or upper fibers in tension).

Before determining the shears, J, along the beam, we must first determine the reactions at the support, viz. V_O, V_B, and V_C. Consider the portion OB as a "free body", cutting just on left

of support B, and put $\sum$ (moments) $= 0$ about the neutral axis of section at B; deriving
$6 \times 11 + 8 \times 5 - 39.2 - V_O \times 15 = 0$; and $\therefore\ V_O = 4.5$ tons.

Similarly, with BC as free body, taking moments about B, $16 \times 12 - 39.2 - V_C \times 20 = 0$; and $\therefore\ V_c = 7.6$ tons; and hence, since $V_O + V_B + V_C = 30$ tons, $V_B = 17.9$ tons. The shear-diagram is now easily formed (see Fig. 447_1); the maximum shear being evidently 9.55 tons, occurring in the section just on the *left* of support B.

We note that the shear changes sign three times, corresponding to the three (local) maximum moments (at E, B, and K). To locate, and determine, M_K, note that the change of sign of the shear at K_0 is *gradual* and that hence the shear is exactly zero at K; which requires that the load between K and the support C be equal to the reaction at C, (from the free body concerned). Since w along HC is one ton per foot, the distance KC must be 7.64 tons $\div$ 1.00, $= 7.6$ ft. From this free body, KC, we now find, by moments, that $M_K = 29.2$ ft.-tons.

As to the other maximum moment, i.e., at E, we note that the moment at E in the normal moment-figure would be $E''' E'' = 28.2$; from which by deducting $\frac{4}{15}$ of M_B (i.e., of 39.2) we obtain M_E, $= 17.7$ ft.-tons. Hence, the greatest moment to be found in the beam is that at B, viz. 39.2 ft.-tons, and upon this depends the choice of a safe and economical beam.

Example II. Fig. 447_2, Continuous prismatic beam OC. Three supports at same level. Find maximum moment, etc., under given loading, the 12 tons being uniformly distributed over whole of right-hand span. Neglect weight of beam.

M max. $= M_B = 16.6$ ft.-tons. Ans.

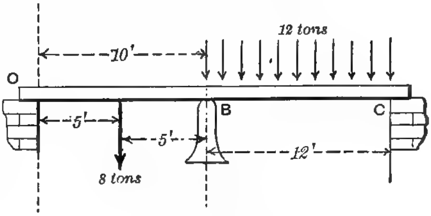


Fig. 447_2.

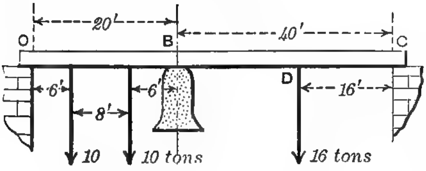


Fig. 447_3.

Example III. Continuous prismatic beam on three supports O, B, C, at same level. Three concentrated loads. Neglect weight of beam. Find M_B and maximum moment, etc.

$M_B = -92.6$ ft.-tons. Max. $M = 116$ ft.-tons, at D. Ans.

Example IV. Continuous prismatic beam, 40 ft. long and extending over *four* supports at the *same level.* The loading is *symmetrical,* as shown (Fig. 447$_4$). Here we note that from symmetry M_B must $= M_C$; also M_O and M_D each $= 0$. Applying the *Three-Moment Theorem* to O, B, and C, (with $\delta = 0$) we find

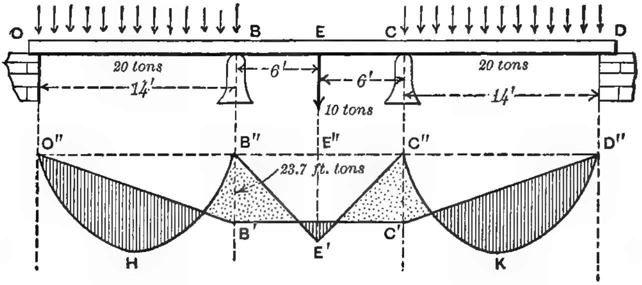


FIG. 447$_4$.

$$0 + \frac{1}{3} M_B \times 26 + \frac{1}{6} M_B \times 12 + \frac{1}{24} \cdot \frac{20}{14} \times \overline{14}^3 + \frac{10}{16} \cdot \frac{1}{12} \times \overline{12}^3 = 0;$$

and $\therefore M_B = -23.7$ ft.-tons, ($= M_C$, also).

Completing the mom.-diagram we find that $M_B (= M_C)$, or 23.7, is greater than any other moment along the beam; ∴ M max. $= 23.7$ ft.-tons. The reactions of the supports are found to be: V_O (and V_D) $= 8.3$ tons; and V_B (and V_C) $= 16.6$ tons. Evidently the elastic curve is *convex up*, over both B and C. The maximum shear is 11.8 tons, close on the left of support B, (or close on right of C).

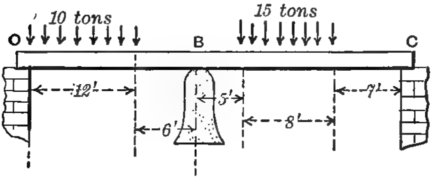


FIG. 447$_5$.

Example V. Continuous prismatic beam, 38 ft. long, on three supports *at the same level.* Fig. 447$_5$. Uniformly distributed loads over portions of the length. Find the maximum moment and maximum shear, (J).

$$\left.\begin{array}{l}\text{Max. } M = -39.6 \text{ ft.-tons (at } B\text{)};\\ \text{Max. } J = 10.2 \text{ tons (close at right of } B\text{)}\end{array}\right\} \text{ANS.}$$

Example VI. Fig. 447$_6$. Continuous prismatic beam, 50 ft. long, on *four* supports *at same level;* but the arrangement of loading and span-lengths is *non-symmetrical.* Fig. 447$_6$. Find the max. M and max. J. In this case M_O and M_D are each $= 0$, but M_B is not $= M_C$. We are therefore compelled to apply the Theorem of Three Moments *twice*, viz.: first to the three points O, B, and C; and then to the three points B, C, and D; whence we have

$$0 + \frac{M_B\,(14 + 20)}{3} + \frac{M_C \times 20}{6} + \frac{16 \times 6 \times 8\,(14 + 6)}{6 \times 14} + \frac{20 \times \overline{20}^3}{24 \times 20} = 0;$$

$$\frac{M_B \times 20}{6} + \frac{M_C\,(20 + 16)}{3} + 0 + \frac{20 \times \overline{20}^3}{24 \times 20} + \frac{15 \times 7 \times 9\,(16 + 7)}{6 \times 16} = 0;$$

or, $$68 M_B + 20 M_C + 1097.1 + 2000 = 0; \quad \ldots \ldots \quad (1)$$

and $$20 M_B + 72 M_C + 2000 + 1358.3 = 0; \quad \ldots \ldots \quad (2)$$

two simultaneous equations, for determining M_B and M_C.

Solving, we find $M_B = -34.6$, and $M_C = -37.0$, ft.-tons.

The three "normal mom.-diagrams" having been drawn to the common base $O''B''C''D''$ in the figure, we lay off $B''B'$ downward from B'' and $= 34.6$ ft.-tons; and $C''C'$, also downward, $= 37.0$; and draw the straight lines $O''B'$, $B'C'$, and $C'D''$; thus completing the mom.-diagram, in which, as before, the differently shaded portions show whether the elastic curve is concave up (line-shading), or convex up (dot-shading), in the corresponding part of beam.

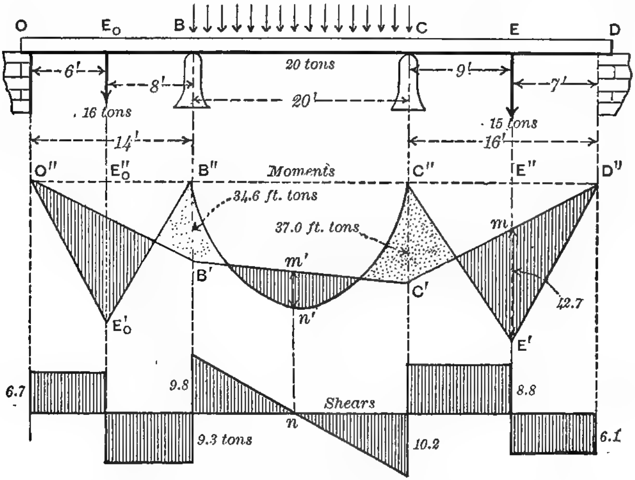


FIG. 447$_6$.

The four reactions of supports are then found, viz.: V_O, V_B, V_C, and V_D, $= 6.7$, 19.2, 19.0, and 6.1 tons, respectively. Shears are now easily found and are shown in the shear diagram, the max. J being 9.8 tons, occurring just on the right of support B. The max. M is found to occur at E, and to have a value of 42.7 ft.-tons.

402a. Continuous Beam with One Span Unloaded. In Fig. 448 we have a continuous prismatic beam supported at three points O, B, and C at the same level; but the support at O is *above* the beam, instead of below; since that end tends to *rise*, there being *no load in the left-hand span*. (Case of a *drawbridge* with the left-end "latched down"). Neglect weight of beam. By the Theorem of Three Moments applied to O, B, and C, with M_0 and $M_C = 0$, and M_B unknown, we find $M_B = -4.9$ ft.-tons. In forming the moment-diagram here, we note that since there is no load from O to B the lower edge of the "normal mom.-diagram" for OB coincides with its upper edge, i.e., with the axis itself, viz. $O''B''$. The "normal mom.-diagram" for BC is a triangle, with $B''C''$ as base. Laying off $B''B' = 4.9$ *downward* from B', and joining $O''B'$ and $B'C''$, we complete the actual (shaded) mom.-diagram, as shown. Max. moment is found under the load and $= +$ 9.55 ft.-tons.

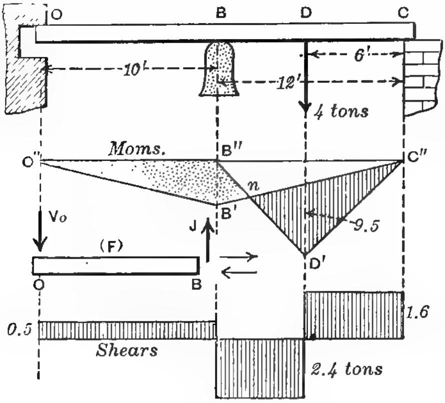


Fig. 448.

To find the reaction at O, take OB as "free body", cutting close on left of B. (See (F) in the figure. Note the position of stress-couple at right-hand end of this body.) By moments about B we have $V_O \times 10 - 4.9 = 0$, whence $V_O =$ (say) 0.5 tons. The other reactions and the shear diagram are now easily determined.

403. Supports out of Level. In the foregoing examples the quantity δ has been zero in each instance of the application of the Theorem of Three Moments; but when such is not the case the quantities E and I are brought into play. In this connection it must be remembered that any unequal settling of the supports (originally at same level) after the beam has been put in place, may cause considerable changes in the values of the various moments and shears, and consequent stresses in the material. (See lower half of p. 323.)

404. Continuous Beam with "Built-in" Ends. Fig. 449. As a case for illustration take the prismatic beam in Fig. 449, "*built-in*" or clamped, horizontally, at B and at C; at the same level. A load P is placed as shown. On account of the mode of support the tangents to the elastic curve at B and C will be horizontal and are coincident; so that portions of the curve near the ends are convex up. Now conceive the beam to be sustained at B by a simple

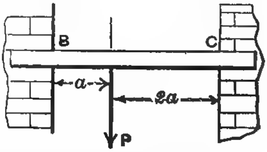


Fig. 449.

support underneath and to extend toward the left a length a_0 at the end of which a support, O, is placed *above* the beam, and at *same level as B* and C (allowing for thickness of beam). This makes an additional span (with $M_O =$ zero); and the tangent to elastic curve at B will no longer be horizontal. But *it may be made as nearly horizontal as we please, by taking a_0 small enough* (supposing no limit to strength of beam). When $a_0 =$ *zero* the tangent at B will be in its actual position (horizontal). We may therefore apply the Theorem of Three Moments (§ 400) to O, B, and C, [noting that there is no load on $O\,B$], if we write both M_O, and a_0, $= 0$, whence (see also Case II of § 401),

$$0 + \frac{M_B\,(0 + 3a)}{3} + \frac{M_C\,3a}{6} + 0 + \frac{P2a\,.\,a(3a + 2a)}{6 \times 3a} = 0.$$

Similarly, by conceiving the beam extended to the right, a distance a' to a point D, for another support, etc., we may apply the theorem to the three points, B, C, and D in like fashion, with M_D and $a' = 0$, obtaining

$$\frac{M_B\,3a}{6} + \frac{M_C\,(3a + 0)}{3} + 0 + \frac{P2a\,.\,a\,(3a + a)}{6 \times 3a} + 0 = 0.$$

Elimination gives $M_B = -\frac{4}{9}Pa$, and $M_C = -\frac{2}{9}Pa$, ft.-tons.

405. Deflections found by the Theorem of Three Moments for Prismatic Beams. Since this theorem contains δ (see Fig. 446a) the deflection of the point B of the elastic curve of a continuous *prismatic* beam from the line joining the two others, O and C, we may use the theorem in many cases for determining deflections *when the three moments are known.*

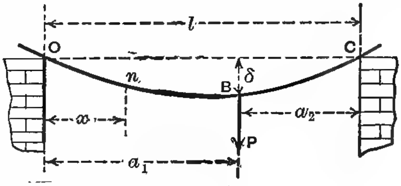


FIG. 450

Example I. Fig. 450. Case of two end-supports and a single non-central load, P; (with weight of beam neglected). Taking O, B, C, as the three points (i.e., OB is the left-hand, and BC the right-hand, span: *with no load on either span*) we have, with M_O and $M_C = 0$, and $M_B = Pa_1a_2 \div l$,

$$0 + \frac{Pa_1a_2}{l}\,\frac{l}{3} + 0 + 0 + 0 = E\,I\delta\left[\frac{1}{a_1} + \frac{1}{a_2}\right]; \therefore \delta = \frac{Pa_1{}^2a_2{}^2}{3EIl}.$$

Example II. If n is any point between O and B, at x ft. from O, and O, n, and C are taken as the three points for the theorem, we may find δ_n, the deflection of n below OC. That is, with M_O and $M_C = 0$ and $M_n = P(a_2 \div l)\,x$,

$$0 + \frac{Pa_2x\,.\,l}{3l} + 0 + 0 + \frac{Pa_2(a_1 - x)[(l-x) + a_2]}{6(l-x)}$$
$$= EI\delta_n\left[\frac{1}{x} + \frac{1}{l-x}\right]$$

or, after reduction,

$$\delta_n = \frac{Pa_2}{6EI\,.\,l}\,[l^2 - a_2{}^2 - x^2]x \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now if $a_1 > a_2$, we may find the distance x', from O, of the point of maximum deflection, by putting $\frac{d\,(\delta_n)}{dx} = 0$; whence is obtained

$$x' = \sqrt{\tfrac{1}{3}\,(l^2 - a_2{}^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and this substituted in (4) gives

$$\text{max. deflection,} = \frac{Pa_2}{9EI\,.\,l}\,(l^2 - a_2{}^2)\,\sqrt{\tfrac{1}{3}\,(l^2 - a_2{}^2)} \quad . \quad . \quad . \quad (6)$$

(Compare with pp. 258 and 494.)

(The following example is the one referred to at the foot of page 514.)

Example.—A hollow sphere of mild steel, of thickness 2 in. and internal radius of $r_0 = 4$ in., contains fluid at a pressure of 2 tons/in.^{2} Find max. stress and max. strain; with $E = 15{,}000$ tons/in.^{2} and $k = 0.30$. Here $n = 1.5$; and by substitution in eq. (30) we obtain max. hoop stress to be $q_0 = -2.26$ tons/in.^{2} (tension), while from eqs. (22) and (23) the tangential, or hoop strain, at inner surface is found to be $\varepsilon_2 = -0.000145$ (elongation), and the radial strain to be $\varepsilon_1 = +0.000224$ (shortening).

The latter strain, ε_1, is seen to be the greater and the ideal "*equivalent simple stress*" (see § 405*b*) is $E\varepsilon_1 = +3.35$ tons/in.^{2}, compression, i.e., *much larger than the actual max. stress* (2.26) *in this case.* On the "elongation theory" (see § 405*b*), the 3.36, and not the 2.26, tons/in.^{2}, is the figure that, for safety, should not pass a prescribed unit stress as inferred from compressive tests with an ordinary testing machine.

CHAPTER XIII.

THICK HOLLOW CYLINDERS AND SPHERES.

405a. General Relations between Stress and Strain.—(*Elastic limit not passed.*) If a small cube of homogeneous and isotropic material, dx inches long on each edge, is subjected to a compressive stress of p_1 lbs./in.^{2} on two opposite faces, not only is its length in direction of the stress diminished, and by an amount $d\lambda$, but its lateral dimensions are *increased* by an amount $d\lambda''$ which is a certain fraction (from 0.20 to 0.35 for metals) of $d\lambda$. This ratio, or fraction, is called *Poisson's Ratio,* and will be denoted by k. Thus, in Fig. 450a we have such a cube, AD being its unstrained form. Axes 1 and 2 are in the plane of the paper while axis 3 is ⅂ to the paper. On the left and right faces is shown acting the compressive unit-stress p_1 lbs./in.^{2}, $A'D$ being the form of the cube under this stress. If now E represent the modulus of elasticity (Young's) of the material, we have (see p. 203) denoting the ratio $d\lambda \div dx$, or relative decrease in length, by ε_1, $\varepsilon_1 = p_1 \div E$; so that if $d\lambda''$ is the *increase* in length of the vertical edges we have $d\lambda'' \div dx$ (call this ratio ε_2) $= -kp_1 \div E$; while the relative increase of length in the horizontal edges ⅂ to paper will be an equal amount, viz., $\varepsilon_3 = -kp_1 \div E$. These ratios ε_1, ε_2, and ε_3 are called the *strains* along the three axes 1, 2, and 3, respectively, and are abstract numbers. Hence the three strains produced by the stress p_1 *acting alone* are

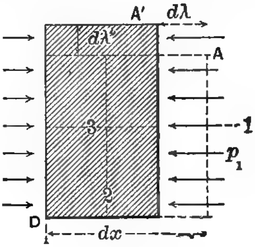


FIG. 450a.

$$\varepsilon_1 = \frac{p_1}{E}; \quad \varepsilon_2 = -\frac{kp_1}{E}; \quad \text{and} \quad \varepsilon_3 = -\frac{kp_1}{E}. \quad . \quad . \quad (1)$$

Now if while p_1 is still in action a compressive stress of p_2 lbs./in.^{2} acts on the two horizontal faces, and also a compressive stress of p_3 on the two vertical faces which are parallel to the paper, the *total strain* in the direction of axis 1 (that is, the relative shortening of the cube in that direction) will be, by superposition, $\varepsilon_1 = \frac{p_1}{E} - \frac{k(p_2 + p_3)}{E}$; and similarly, in the directions of the other two axes, we have

$$\varepsilon_2 = \frac{p_2}{E} - \frac{k(p_1 + p_3)}{E}; \quad \text{and} \quad \varepsilon_3 = \frac{p_3}{E} - \frac{k(p_1 + p_2)}{E}. \quad . \quad (2)$$

(This form of stress-strain relation is due to Grashof.) Note that if either p_1, p_2, or p_3 is a *tensile* stress, a negative number must be substituted for it; and that if a negative number is obtained for ε_1, ε_2, or ε_3, in any problem, it indicates a lengthening instead of a shortening. Similarly, if the condition is imposed that the strain ε_2 (say) shall be a relative *elongation* of 0.00020, -0.00020 must be substituted for it in above relation.

405b. "Elongation Theory" of Safety.—In all preceding chapters the criterion of safety has been that the unit-stress in the element of the elastic body where the stress is highest, regardless of stress on the side faces, should not exceed a certain value, or working stress, $=R'$ lbs./in.^{2}, as determined upon by a consideration of the stress at "elastic limit;" this "elastic limit" being itself determined by the ordinary experiments on "simple" tension or compression of rods of the material in question, there being *no stress* on the *sides* of the rod. In such experiments, however, an element with four faces parallel to the axis is subjected to stress, say p, on two (end) faces only; and the question naturally arises whether the elastic limit would be reached for the same value of p as before, in case there were also present tensile or compressive stress acting on the *side faces* of the element. Experiments which would throw much light on this point are unfortunately wanting, and some authorities, notably on the continent of Europe, contend that the extreme limit of safety, as regards state of stress in isotropic materials, is when the greatest relative *strain* (elongation or shortening), say ε_1, is as great as would be produced at the elastic limit in an experiment involving only "simple" tension, or compression (as above described), in an ordinary testing machine. This view would make the greatest "strain," or deformation (change of form), the criterion of safety instead of the greatest stress. Now if a stress of "simple" tension, $=p'$, (no side stresses) acts on an element, the highest strain produced is in the direction of this stress and has a value $\varepsilon'=p'\div E$, since Young's modulus, E, is determined by experiments of this very nature; that is, $p'=E\varepsilon'$. Hence if the greatest strain in an element in some compound state of stress, as in § 405a, is ε_1, and it is desired to place it equal to $\frac{3}{4}$ (say) of the ε_1 in simple tension at elastic limit, we may write $\varepsilon_1=\frac{3}{4}\varepsilon'=\frac{3}{4}(p'\div E)$; or $E\varepsilon_1=\frac{3}{4}p'$. If now we denote $\frac{3}{4}p'$ by p'' we may write $E\varepsilon_1 = p''$ and describe p'', or $E\varepsilon_1$, as the *ideal* tensile stress which would produce a strain, or relative elongation, equal to ε_1 *in case there were no side stresses;* Cotterill calls this ideal stress ($E\varepsilon_1$) the "*equivalent simple stress.*"

For instance, if on an element of the shell of a cylindrical steam-boiler of soft steel the "hoop stress" (p. 537) is p_1 on two end faces and the stress on two of the other faces is p_2, $= \frac{1}{2}p_1$, (the stress on the remaining two opposite faces being p_3 = practically zero in this connection) we have for the strain in direction 1, by eq. (2), $E\varepsilon_1 = p_1 - k(\frac{1}{2}p_1 + 0)$. . . . (2′)

Let p'', $= -15{,}000$ lbs./in.^{2}, tension, be the safe working stress for the metal in simple tension; with $E = 30{,}000{,}000$ lbs./in.^{2}, and Poisson's ratio $= k = 0.30$. Then according to the view of preceding chapters the greatest safe value for the stress p_1 would be $-15{,}000$ lbs./in.^{2} But according to the new view now being illustrated the safe value of p_1 must be determined by limiting the strain ε_1 to a value which would be produced by 15,000 lbs./in.^{2} in simple tension, i.e., -0.00050, (which $= p'' \div E$); which amounts to the same thing as requiring that the "equivalent simple stress" shall $= 15{,}000$ lbs./in.^{2} Hence, substituting in eq. (2′), we have

$$-15{,}000 = p_1(1 - \tfrac{1}{2}\,.\,(0.30)); \quad \text{i.e.,} \quad p_1 = -17{,}600 \text{ lbs./in.}^2$$

tension; which is considerably greater than the 15,000 allowed by the older theory. The relation thus brought out in this case that the tenacity of a material is *increased* by the presence of lateral tension "can hardly be considered as intrinsically probable, and such direct experimental evidence as exists is against the supposition" (Cotterill).

But in many cases the results of this "elongation theory" are more probable than those based on the older theory; hence the former is much favored by continental writers.

405c. Thick Hollow Cylinder. Stresses and Strains.—Fig. 450*b* shows a longitudinal section of a thick hollow cylinder of homogeneous and isotropic material (say steel or iron) provided with end stoppers (no friction nor leakage); and Fig. 450*c* a transverse section, giving dimensions. r_0 is the inner

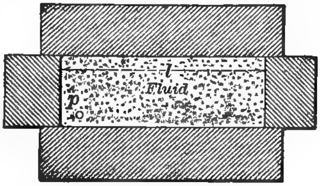


FIG. 450*b*.

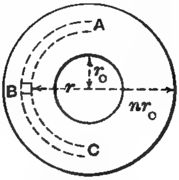


FIG. 450*c*.

radius and nr_0 the outer radius (n is a ratio). Fig. 450*c* also shows (dotted lines) an elementary hoop, or shell, of inner radius r and outer radius $r+dr$. The interior of the cylinder is filled with fluid under a high pressure, p_0 lbs./in.^{2}; and it is required to determine the stresses and strains in a cubic element in *any* elementary hoop or ring such as ABC, Fig. 450*c*. Let the half hoop ABC of Fig. 450*c* be considered as a "free body" in Fig. 450*d*, showing also at 3 a small cubic element, as mentioned above. The compressive stress on the inner surface of the hoop is p (radial), exerted on it by the adjacent inner hoop; while on the outer surface of the elementary hoop, and exerted on it by the adjacent outer hoop, is the compressive stress $p+dp$. The thickness of the hoop is dr. The stress on the edges, A and C, of this free body (half hoop), will be taken as compressive at first, of intensity q lbs./in.^{2} Let the hoop or thin shell have a length$=l$, ┐ to paper i.e., longitudinally (see Fig. 450*b*). Now for this free body put $\Sigma X=0$ and we have (see pp. 525 and 526)

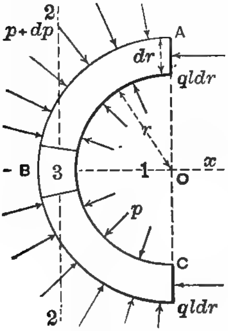


FIG. 450*d*.

$$(p+dp)(2r+2dr)l-2ql\,.\,dr-p(2rl)=0 \quad . \quad .(3)$$

i.e.,

$$pr+r\,.\,dp+p\,.\,dr+dp\,.\,dr-q\,.\,dr-pr=0 \quad . \quad . \quad (4)$$

and hence, omitting the term $dp\,.\,dr$ of the second order,

$$r\,.\,dp+p\,.\,dr=q\,.\,dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is a differential equation of stress. Next consider the relations of stress and strain to be found in the small cube at 3, Fig. 450*d*. It is subjected to a compressive stress p along the radial axis 1; to a compressive stress q a long the tangential axis 2; while on the front and back faces the stress is $p_3=$zero, parallel to axis 3 (┐ to paper). Let now ε_1 denote the radial strain, ε_2 the tangential, and ε_3 the axial strain, this latter being parallel to the axis of the cylinder. All of these strains are supposed to be *shortenings* for the present; and from the circumstances of the case the third strain ε_3 (axial) is considered constant (i.e., the same for all values of the variable r), since the cylinder is under no constraint as to longitudinal change of form.

We may therefore write the relations (see § 405a)

$$E\varepsilon_1=p-k\,.\,q, \ (6); \quad E\varepsilon_2=q-k\,.\,p, \ (7); \quad E\varepsilon_3=0-k(p+q), \ (8)$$

From (8) we have $q=-p-(E\varepsilon_3 \div k)$, which in (5) gives

$$r\,.\,dp+2p\,.\,dr=-(E\varepsilon_3 \div k)dr \quad . \quad . \quad . \quad . \quad (9)$$

Now multiply by r (integrating factor) and denote $E\varepsilon_3 \div k$ by A, an unknown constant, (unknown since it contains the strain ε_3) and we have

$$r^2.\,dp+2pr\,.\,dr=-Ar\,.\,dr; \quad . \quad . \quad . \quad (10)$$

that is to say,

$$d[r^2p]=-Ar\,.\,dr; \quad . \quad . \quad . \quad (11)$$

which may be integrated, giving,

$$r^2p=-\tfrac{1}{2}Ar^2+C; \quad . \quad . \quad (12)$$

where C is a constant of integration. The two constants A and C may now be determined by substituting in (11) the values r_0 and p_0 which the two variables r and p assume at the inside surface of the cylinder. Similarly, at the outside surface r and p have the values nr_0 and 0 (atmospheric pressure relatively small and hence neglected); which being placed in (11) give rise to a second equation, which like the first contains constants only. From these two equations we easily find

$$A=\frac{2p_0}{n^2-1}; \quad \text{and} \quad C=\frac{n^2r_0{}^2p_0}{n^2-1};$$

and hence finally, from equations (12) and (8),

$$p=\frac{p_0}{n^2-1}\left[\frac{n^2r_0{}^2}{r^2}-1\right]. \quad (13); \text{ and } q=-\frac{p_0}{n^2-1}\left[\frac{n^2r_0{}^2}{r^2}+1\right]. \quad (14)$$

From (13) and (14) we may find the stresses p and q for any value of the variable distance, r, from the axis. The negative sign for q shows that it is in reality a *tensile* stress, the reverse of the character assigned to it at first; i.e., it is a "negative compressive" stress for this case of fluid pressure acting *inside* the cylinder. Both p and q have maximum values at the inner surface and diminish toward the outside.

Example.—With inner radius $r_0=4$ in.; and outer, $=5$ in. (hence $n=5/4$ or 1.25); and $p_0=800$ lbs./in.^{2}; we find q max. (or q_0), at inner surface, to be 3644 lbs./in.^{2}; while at outer surface $q=-2844$ lbs./in.^{2}, tension. If the metal is cast iron we may put $k=0.23$ (see p. 230) and $E=15{,}000{,}000$ lbs./in.^{2}, and thus obtain for the radial strain at the inner surface, $\varepsilon_1=+0.000109$, indicating a shortening; and for the "hoop" strain (or tangential strain) at the same place, $\varepsilon_2=-0.000255$, i.e., a relative *elongation* of about $2\frac{1}{2}$ parts in 10,000. (The student should verify all the details of this example, carefully noting the proper signs to be used).

405d. Thick Hollow Cylinder under External Fluid Pressure.—If the cylinder is surrounded by fluid under high pressure, p_n lbs./in.^{2}, the internal pressure p_0 being practically zero (atmosphere) in comparison, we must determine the constants A and C of eq. (12) on the basis that $p=0$ for $r=r_0$ and that $p=p_n$ for $r=nr_0$; whence, finally, we obtain

$$p=\frac{n^2p_n}{n^2-1}\left(1-\frac{r_0{}^2}{r^2}\right); \quad . \quad . \quad . \quad (15) \qquad q=+\frac{n^2p_n}{n^2-1}\left(1+\frac{r^2_0}{r^2}\right). \quad . \quad . \quad . \quad (16)$$

for the stresses at any distance r from the axis. Here *both* p and q are compressive stresses; the latter increasing, and the former diminishing, toward the center. Evidently if the cylinder were not hollow, but *solid*, r_0 would $=0$, and $n=\infty$; and both p and q would be constant, $=p_n$, at all points.

405e. Approximate equalization of the tensile hoop stress in successive rings of a thick hollow cylinder under internal fluid pressure may be brought about by using two or more separate cylinders of which the outer ones are successively "shrunk on" before the fluid is introduced. For instance, with only two cylinders, the outer one is first heated to such an extent that it barely fits over the inner one, the latter being cold. When the compound cylinder has cooled the outer one has shrunk and is in a state of hoop *tension*, while the inner one is in a state of hoop *compression*. The radial pressure between them, at their surface of contact, is an internal or bursting pressure for the outside cylinder, and an external or collapsing pressure for the inside cylinder. The original and final temperatures being known, we are able to make use of the foregoing equations [(6) to (16)] to compute all stresses and strains thus induced before the fluid is introduced into the interior of the smaller cylinder. When the internal pressure is eventually produced, the hoop stresses in the smaller cylinder, initially compressive, will be converted into moderate tensions and the tensile stresses in the external cylinder will be increased; but the maximum tension is not so great as if the complete cylinder had been originally continuous. (See Prof. Ewing's Strength of Materials).

In the case of thick hollow cylinders subjected to the severe internal pressures needed in the manufacture of lead pipe, to produce "flow" of the metal, it is well known (Cotterill) that a permanent increase in the internal diameter takes place, showing that in the inner layers of the cylinder the elastic limit has been passed. In this way an approach is made to equalization in the hoop stresses of all the layers and the above formulæ no longer hold; but the cylinder as a whole is not injured, having simply adapted itself better to its function. Cast-iron hydraulic press cylinders are sometimes subjected to the high internal pressure of 3 tons/in.^{2} If the cylinder is short, its resistance is doubtless much increased by connection with the end plates, or "domes."

405f. Equation of Continuity for Thick Hollow Cylinder under Stress.—(Cotterill.) In Fig. 450*e* we have in *ABCD* the form and position of a "cubical" element (of an elementary hoop) between the two radial planes *ABO* and *DCO*, in its condition of *no stress*. After it is subjected to stress it will still be found *between the same radial planes* and will occupy some position $A'B'C'D'$. The radial thickness $\overline{AB}=dr$, or t, will have changed to t', BC has changed to $B'C'$, etc. With axes 1 and 2 as shown, 1 being radial, and 2 tangential (or circumferential), we note that the whole circumference of which BC is a part has shortened from a length $2\pi r$ to $2\pi s$, so that $2\pi s=2\pi r(1-\varepsilon_2)$, ε_2 being the value of the tangential strain, or "hoop" strain, at distance r from center O; and similarly $2\pi s'=2\pi r'(1-\varepsilon'_2)$, where ε_2' is the hoop strain at distance r' (i.e., $r+dr$) from O. But ε_2 varies with r, so $\varepsilon_2'=\varepsilon_2+t\,.\,\frac{d\,\varepsilon_2}{dr}$. Hence, subtracting,

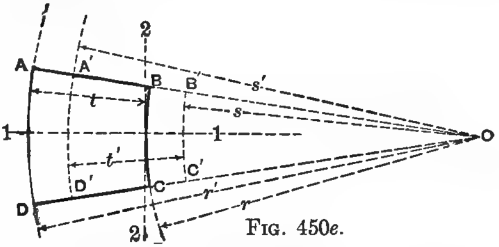


FIG. 450e.

$$s'-s=(r'-r)(1-\varepsilon_2)-r't\frac{d\,\varepsilon_2}{dr}. \qquad . \; . \; . \; . \; . \; . \quad (17)$$

i.e.,
$$t'=t-t\,\varepsilon_2-r't\frac{d\,\varepsilon_2}{dr};\quad\text{or,}\quad t(1-\varepsilon_1)=t-t\varepsilon_2-r't\frac{d\,\varepsilon_2}{dr}; \qquad . \quad (18)$$

whence
$$\varepsilon_1-\varepsilon_2=r\frac{d\,\varepsilon_2}{dr}; \qquad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (19)$$

which is the "equation of continuity of substance," of the cylinder.

Since in the foregoing cases of thick hollow cylinder, under bursting or collapsing fluid pressure, both ε_1 and ε_2 may be expressed as functions of r, it is a simple matter for the student to show that (19) is verified in those cases, and that hence *the solutions given are adequate.*

Evidently eq. (19) also holds in the case of the thick hollow sphere, where there is a hoop strain ε_3, ⊥ to paper in Fig. 450, equal to that, ε_2, along axis 2.

405g. Thick Hollow Sphere under Internal Fluid Pressure. — (For thin-walled spheres see p. 536). As thick hollow spheres are sometimes used to hold fluids under high pressure, and the halves of such spheres frequently form the ends of thick hollow cylinders, the following treatment will be of practical value. In Fig. 450*f* we have a cross-section of the sphere through the center. The inner radius is r_0; the outer, nr_0 (where n is a ratio); while the (variable) r is the inner radius of any thin spherical shell, of thickness dr, an infinite number of which make up the complete sphere. Though each of these shells is under a "hoop tension," when the internal fluid pressure is in action, we shall at first deal with this stress as if compressive, for uniformity in applying the principles of § 450*a*.

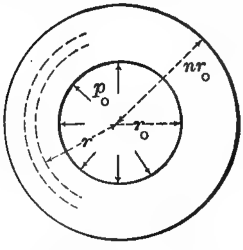


FIG. 450*f*.

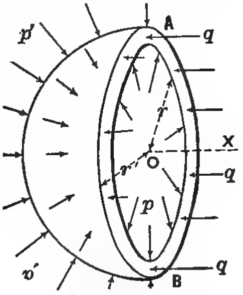


FIG. 450*g*.

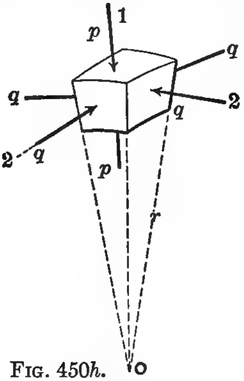


FIG. 450*h*.

Consider as a free body any hemispherical shell as shown in Fig. 450g. The pressure (radial) on the inside, from the adjacent shell, is p lbs./in.^{2}; and that from the adjacent shell on the outside is $p+dp$, or p'. The radius of the outside will be called r', $(=r+dr)$. The "hoop compression" on the thin edge of the shell is q lbs./in.^{2} These three quantities, p, r, and q, are *variables*, i.e., refer to any infinitesimal shell of the sphere. The stresses affecting any small "cubical" block in any shell are p, radial; q, tangential; and q_3, $=q$, along a tangent ┐ to the first mentioned.

Taking an axis X through the center and ┐ to sectional plane AB, and putting Σ (X-components) $=0$, for equilibrium, we have

$$\pi r'^2 p' - \pi r^2 p - 2\pi r \,.\, dr \,.\, q = 0; \qquad \text{i.e.,} \qquad r'^2 p' - r^2 p = 2qr \,.\, dr.$$

But the difference, $r'^2p' - r^2p$, is nothing more than the increment accruing to the product r^2p when r takes the increment dr, and is therefore the *differential* of the quantity or product r^2p; hence we may write

$$d(r^2p) = 2qr \,.\, dr; \quad \text{or,} \quad r^2 \,.\, dp + 2pr \,.\, dr = 2qr \,.\, dr \;.\;\;.\;\;. \qquad (20)$$

We next consider the relations of stress and strain in the small cubical element shown in Fig. 450h, having four radial and two tangential faces. The unit stresses on the faces are as there shown; p on the two tangential, and q on the four radial, faces. Radial strain $=\varepsilon_1$ and hoop strain (tangential) $=\varepsilon_2=$ same for any tangent. It has already been proved in § 405f that

$$\varepsilon_1 - \varepsilon_2 = r \,.\, \frac{d\varepsilon_2}{dr}; \;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \qquad (21)$$

and we also have, from § 405a, $E\varepsilon_1 = p - 2kq$; . . . . (22)

and $E\varepsilon_2 = q - k(p+q)$ . . . . (23)

From the four equations, (20), (21), (22) and (23), containing the five variables p, q, ε_1, ε_2, and r, we may by elimination and integration finally determine p and q, each as a function of r; as follows: (C', C'', C_1, C_2, etc., will denote constants of integration, or, constant products involving E and k.

From (22) and (23) we obtain a value for $\varepsilon_1 - \varepsilon_2$ which in (21) gives rise to an expression for $p-q$. Another expression for $p-q$ is obtained from (20). Equating these two expressions we derive $-(1+k)dp = 2Ed\varepsilon_2$; i.e., by integration,

$$-(1+k)p = 2E \,.\, \varepsilon_2 + C_1 \;.\;.\;.\;.\;.\;.\;.\;. \qquad (24)$$

By elimination of ε_2 between (24) and (23) we obtain q in terms of p which substituted in (20) produces

$$r^2 \,.\, dp + 3pr \,.\, dr = C_3 r \,.\, dr \;\;.\;.\;.\;. \qquad (25)$$

Eq. (25) is made integrable by multiplying by r (integrating factor); whence

$$r^3 \,.\, dp + 3pr^2 \,.\, dr = C_3 r^2 \,.\, dr$$

The left-hand member is evidently $d[r^3p]$. Therefore $d[r^3p] = C_3r^2 \,.\, dr$; or, integrating,

$$r^3 p = \tfrac{1}{3}C_3 r^3 + C_4; \qquad \text{that is,} \qquad p = C' + \frac{C''}{r^3}, \quad .\;. \qquad (26)$$

whence, also,

$$\frac{dp}{dr} = -\frac{3C''}{r^4}; \qquad \text{which in (20) gives rise to} \qquad q = C' - \frac{C''}{2r^3}. \quad . \qquad (27)$$

We may now determine the two constants C' and C'' substituting in eq. (26), first $p=p_0$ and $r=r_0$; and then $p=0$ with $r=nr_0$. The values of C' and C'' so obtained are placed in (26) and (27), resulting finally in the relations

$$p = \frac{p_0}{n^3-1}\left(\frac{n^3r_0^3}{r^3} - 1\right); \quad . \quad (28) \qquad \text{and} \qquad q = -\frac{p_0}{n^3-1}\left(\frac{n^3r_0^3}{2r^3} + 1\right). \quad .\;. \quad (29)$$

The negative sign of q shows that it is a tensile stress, or "hoop tension." It is evidently a maximum for $r=r_0$, this maximum value being

$$q_0, = q \text{ max.}, = -\frac{p_0}{n^3-1}\left(\frac{n^3}{2} + 1\right). \quad .\;.\;.\;.\;.\;. \qquad (30)$$

(For a numerical example see foot of p. 506).

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APPENDIX.

Note.—This appendix contains addenda and tables, and also several pages from a former edition of the book. In these last the numbering of the articles and of the figures has been left unchanged.

48a. Addendum to § 55. Mass.—In PHYSICS, the fundamental units are those of

SPACE, involving a unit of length (and thence of area and volume);
TIME, " a unit of time, usually the second;
MASS, " a unit of mass, which (by Government decree) may be the quantity of matter in a specified piece of platinum, or specified volume of water, etc. (a beam-balance being used to determine equal quantities of mass); while
FORCE involves a derived unit, being measured by its effect in accelerating the velocity of a moving mass, since it is proportional both to the mass and the acceleration. The unit force (called absolute unit) is the force necessary to produce unit acceleration in a unit of mass; so that to produce an acceleration $=p$ in a mass $=m$ requires a force $= F = mp$, and the force thus obtained is in absolute units. This is called the dynamic measure of a force.

Example.—In the C.G.S. system of units, required the constant force necessary to cause a mass of 400 grams to gain 200 velocity units in 2 sec; i.e., $p = 100$ centims. per sec., per sec. From $F = mp$ we have

$$F = 400 \times 100 = 40000 \text{ abs. units of force (or dynes, in C.G.S. system).}$$

In the ft.-lb.-sec. system the absolute unit is called a *poundal.*

In MECHANICS OF ENGINEERING, however, it is more convenient to regard the fundamental units to be those of

SPACE, as ft., metre, etc., area and volume corresponding;
TIME, as seconds, hours, etc.;
Force, as lbs., grams, kilograms, tons, etc., indicated by a *spring balance;* while for
MASS we assume a derived unit, a mode of measuring it being developed as follows:

If by experiment (block on smooth table, for instance) we find that a constant force P (lbs., tons, kilos.) will maintain an acceleration $=p$ in the rectilinear motion (in line of force) of a body whose weight (by previous trial with a spring balance) is G (lbs., tons, or other unit); and if in a second experiment, by allowing the force G to act on the same body in vacuo, a free vertical fall with acceleration $=g$ is the result,—we find that the proportion (Newton's 2d Law) $P : G :: p : g$ is verified. This may be written $P = \frac{G}{g} \cdot p$, and may then be read: Force = mass × acceleration, if we call the quotient $G \div g$ the MASS of the body whose weight (by spring balance) is $= G$ at a locality where the acceleration of gravity $= g$; for this quotient will be the same at all localities on the earth's surface.

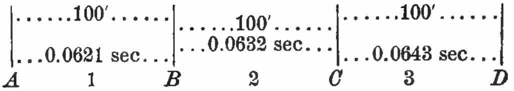
Example (same as above).—If a body whose weight $G = 400$ grams (force) is to have its velocity increased, in 2 sec., from 300 centims. per sec. to 500 centims. per sec., at a uniform rate, we must provide a constant force

$$P = \frac{400}{981} \times 100 = \frac{40000}{981} = 40.77 \text{ grams; or .040 kilos.}$$

This is called the *gravitation measure* of a force. Hence it is evident that to reduce absolute units (called *dynes* and *poundals*) in the C.G.S. and ft.-lb.-sec. systems, respectively) to ordinary practical units of force (lbs., tons, kilos., etc., of a spring balance), *we divide by the value of g proper to the system of units employed;* and *vice versâ.*

(Addendum to § 49a of page 49.) Numerical Example.—A set of light screens is set up at intervals of 100 feet apart in the horizontal path of a cannon-ball, with the object of determining its velocity, and also the rate of change (or negative acceleration) of that velocity, as due to the resistance of the air.

By electrical connection the time of passing each screen is noted, and the *intervals* of time are given in this diagram for four of the screens, *A*, *B*, *C*, and *D*.



From these data it is required to compute, as nearly as the circumstances allow, the velocity and acceleration (negative) of the ball at various points (the ball moves from left to right).

Solution.—In passing from *A* to *B* the ball has an *average* velocity of 1610 ft. per second, obtained by dividing the distance of 100 feet by the time of passage, 0.0621 second. Similarly we find the average velocity between *B* and *C* to be 1582 ft. per second, and that between *C* and *D* to be 1554 ft. per second.

As the velocity is not changing very rapidly, we may claim that the ball actually possesses the velocity $v_1 = 1610$ ft. per second at the point 1, midway between *A* and *B*, or very near that point; and similarly the velocity $v_2 = 1582$ ft. per second at point 2, midway between *B* and *C*; and $v_3 = 1554$ ft. per second at point 3, midway between *C* and *D*.

Hence the total gain of velocity from 1 to 2 is $1582–1610 = -28$ ft. per second; and the time in which this gain is made is one half of the 0.0621 second plus one half of the 0.0632 second, i.e., 0.0626 second. Therefore an approximate value for the *average* acceleration between points 1 and 2 is found by dividing the -28 ft. per second gain in velocity by the time 0.0626 second occupied in acquiring the gain. This gives -447 ft. per second per second *average* acceleration for portion 1...2 of path, and since screen *B* lies at the middle of this portion, the actual acceleration of the ball's motion as it passes the screen *B* is very nearly equal to this, viz.: -447 ft. per second per second (or "ft. per square second").

By a similar process the student may compute the acceleration at screen *C*. Of course the reason why these results are merely approximate is that the spaces and times concerned, though relatively small, are not infinitesimal.

[A recent English writer calls a unit of velocity a "*speed*;" and a unit of acceleration, a "*hurry*."]

TABLE OF HYPERBOLIC SINES AND COSINES (See p. 48).

u	cosh u	sinh u	u	cosh u	sinh u
0.00	1.0000	0.			
0.05	1.0013	0.0500	2.05	3.9484	3.8196
.10	1.0050	.1002	2.10	4.1443	4.0219
.15	1.0112	.1506	2.15	4.3507	4.2342
.20	1.0201	.2013	2.20	4.5679	4.4571
.25	1.0314	.2526	2.25	4.7966	4.6912
0.30	1.0453	0.3045	2.30	5.0372	4.9369
.35	1.0619	.3572	2.35	5.2905	5.1952
.40	1.0811	.4108	2.40	5.5569	5.4662
.45	1.1030	.4653	2.45	5.8373	5.7510
.50	1.1276	.5211	2.50	6.1323	6.0502
0.55	1.1551	0.5782	2.55	6.4426	6.3645
.60	1.1855	.6367	2.60	6.7690	6.6947
.65	1.2188	.6967	2.65	7.1123	7.0417
.70	1.2552	.7586	2.70	7.4735	7.4063
.75	1.2947	.8223	2.75	7.8533	7.7894
0.80	1.3374	0.8881	2.80	8.2527	8.1919
.85	1.3835	0.9561	2.85	8.6728	8.6150
.90	1.4331	1.0265	2.90	9.1146	9.0596
.95	1.4862	1.0995	2.95	9.5791	9.5268
1.00	1.5431	1.1752	3.00	10.0677	10.0179
1.05	1.6038	1.2539	3.05	10.5814	10.5340
1.10	1.6685	1.3356	3.10	11.1215	11.0765
1.15	1.7374	1.4208	3.15	11.6895	11.6466
1.20	1.8107	1.5097	3.20	12.2866	12.2459
1.25	1.8884	1.6019	3.25	12.9146	12.8758
1.30	1.9709	1.6984	3.30	13.5748	13.5379
1.35	2.0583	1.7991	3.35	14.2689	14.2338
1.40	2.1509	1.9043	3.40	14.9987	14.9654
1.45	2.2488	2.0143	3.45	15.7661	15.7343
1.50	2.3524	2.1293	3.50	16.5728	16.5426
1.55	2.4619	2.2496	3.55	17.4210	17.3923
1.60	2.5775	2.3757	3.60	18.3128	18.2855
1.65	2.6995	2.5075	3.65	19.2503	19.2243
1.70	2.8283	2.6456	3.70	20.2360	20.2113
1.75	2.9642	2.7904	3.75	21.2723	21.2488
1.80	3.1075	2.9422	3.80	22.3618	22.3394
1.85	3.2583	3.1013	3.85	23.5072	23.4859
1.90	3.4177	3.2682	3.90	24.7113	24.6911
1.95	3.5855	3.4432	3.95	25.9773	25.9581
2.00	3.7622	3.6269	4.00	27.3082	27.2899

266. The Four x-Derivatives of the Ordinate of the Elastic Curve —If y = func. (x) is the equation of the elastic curve for any portion of a loaded beam, on which portion the load per unit of length of the beam is w = either zero, (Fig. 234) or = constant, (Fig. 235), or = a continuous func. (x) (as in the last §), we may prove, as follows, that w = the x-derivative of the shear. Fig. 269. Let N and N' be two consecutive cross-sections of a loaded beam, and let the block between them, bearing its portion, wdx, of a distributed load, be considered free. The elastic forces consist of the two stress-couples (tensions and compressions) and the two shears, J and $J + dJ$, dJ being the shear-increment consequent upon x receiving its increment dx. By putting Σ(vert. components) = 0 we have

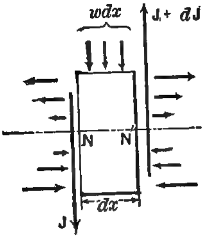


FIG. 269.

$$J+dJ-wdx-J=0 \therefore w=\frac{dJ}{dx}$$

Q. E. D. But J itself = $dM \div dx$, (§ 240) and $M = [d^2y \div dx^2]\, EI$. By substitution, then, we have the following relations:

y=func.(x)=ordinate at any point of the elastic curve (1)

$\frac{dy}{dx} = \alpha$ =slope at any point of the elastic curve . . (2)

$EI\,\frac{d^2y}{dx^2} = M$ = ordinate (to scale) of the moment curve (3)

$EI\,\frac{d^3y}{dx^3}$ = the shear, $J = \left\{\begin{array}{l}\text{the ordinate (to scale)}\\ \text{of the shear diagram}\end{array}\right\}$ . . (4)

$EI\,\frac{d^4y}{dx^4} = w = \left\{\begin{array}{l}\text{the load per unit of length}\\ \text{of beam = ordinate (to scale)}\\ \text{of a curve of loading.}\end{array}\right\}$. (5)

If, then, the equation of the elastic curve (the neutral line of the beam itself; a reality, and not artificial like the other curves spoken of) is given; we may by successive differentiation, for a prismatic and homogeneous beam so that both E and I are constant, find the other four quantities mentioned.

As to the converse process, (i.e. having given w as a function of x, to find expressions for J, M and y as functions of x) this is more difficult, since in taking the x-anti-derivative, an unknown constant must be added and determined. The problem just treated in § 264, however, offers a very simple case since w is the same function of x, *along the whole beam*, and there is therefore but one elastic curve to be determined.

We ∴ begin, numbering backward, with

$$EI\,\frac{d^4y}{dx^4} = -\gamma bx \left\{ \begin{array}{l} \text{since } w = \gamma bx \text{, see} \\ \text{last § and Fig. 268} \end{array} \right\} \quad . \quad . \quad . \quad (5a)$$

[N. B.—This derivative $(dJ \div dx)$ is negative since dJ and dx have contrary signs.]

$$\therefore\ (shear =) EI\frac{d^3y}{dx^3} = -\gamma\, b\,\frac{x^2}{2} + Const.$$

But writing out this equation for $x=0$, *i.e.* for the point O, where the shear$=R_0$, we have $R_0 = 0 + Const.$ ∴ $Const. = R_0$, and hence write

$$EI\,\frac{d^3y}{dx^3} = -\gamma\, b\frac{x^2}{2} + R_0 \quad . \quad (Shear) \quad . \quad (4a)$$

Again taking the x-anti-derivative of both sides

$$(Moment =)\ E\,I\,\frac{d^2y}{dx^2} = -\gamma\, b\frac{x^3}{6} + R_0 x + (Const. = 0) \quad . \quad (3a)$$

[At O, $x=0$ also M, ∴ $Const. = 0$]. Again,

$$EI\,\frac{dy}{dx} = -\gamma b\,\frac{x^4}{24} + R_0\frac{x^2}{2} + C'$$

At O, where $x=0$ $dy \div dx = a_0 =$ the unknown slope of the elastic line at O, and hence $C' = EI\,a_0$

$$\therefore\ EI\,\frac{dy}{dx} = -\gamma b\,\frac{x^4}{24} + R_0\frac{x^2}{2} + EI\,a_0 \quad . \quad . \quad . \quad (2a)$$

Passing now to y itself, and remembering that at O, both y and x are zero, so that the constant, if added, would= zero, we obtain (inserting the value of R_0 from last §)

$$EIy = -\gamma\, b\, \frac{x^5}{120} + \gamma\, b\, l^2 \frac{x^3}{36} + EI\alpha_0 x \quad . \quad (1a)$$

the equation of the elastic curve. This, however, contains the unknown constant α_0 = the slope at O. To determine α_0 write out eq. ($1a$) for the point B, Fig. 268, where x is known to be equal to l, and y to be = zero, solve for α_0, and insert its value both in ($1a$) and ($2a$). To find the point of max. y (i.e., of greatest deflection) in the elastic curve, write the slope, i.e. $dy \div dx$, = zero [see eq. $2a$] and solve for x; four values will be obtained, of which the one lying between 0 and l is obviously the one to be taken. This value of x substituted in ($1a$) will give the maximum deflection. The location of this maximum deflection is neither at the centre of action of the load $\left(x = \frac{2}{3}\, l\right)$ nor at the section of max. moment ($x = l \div \sqrt{3}$.)

The qualities of the left hand members of equations (1) to (5) should be carefully noted. *E. g.*, in the inch-pound-second system of units we should have:

1. y (a linear quantity) = (so many) inches.
2. $dy \div dx$ (an abstract number) = (so many) abstract units.
3. M (a moment) = (so many) inch-pounds.
4. J (a shear, i.e., force) = (so many) pounds.
5. w (force per linear unit) = (so many) pounds per running inch of beam's length.

As to the quantities E, and I, individually, E is pounds per sq. in., and I has four linear dimensions, i.e. (so many) bi-quadratic inches.

287. Cantilevers of Uniform Strength.—Beams built in at one end, horizontally, and projecting from the wall without support at the other, should have the forms given below, for the given cases of loading, if all cross-sections are to be **Rectangular** and the weight of beam neglected. Sides of sections horizontal and vertical. Also, the sections are symmetrical about the axis of the piece. b and h are the dimensions at the wall. $l=$ length. No proofs given.

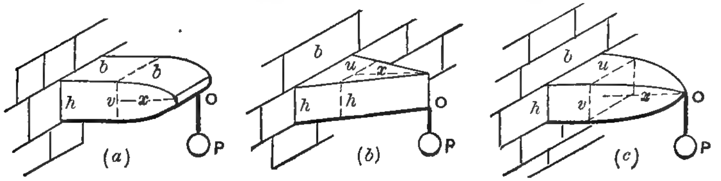


FIG. 290.

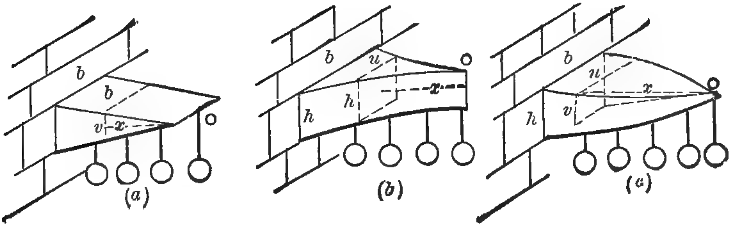


FIG. 291.

Width constant. Vertical outline parabolic. Single end load. } Fig. 290, (*a*). $(\frac{1}{2}v)^2=(\frac{1}{2}h)^2\frac{x}{l}$ (1)

Height constant. Single end load. Horizontal outline triangular. } Fig. 290, (*b*). $(\frac{1}{2}u)=(\frac{1}{2}b)\frac{x}{l}$. (2)

Constant ratio of height v to width u. Both outlines cubic parabolas. } Fig. 290. (*c*).

$$(\tfrac{1}{2}v)^3=(\tfrac{1}{2}h)^3\frac{x}{l} \quad . \quad (3)$$

$$(\tfrac{1}{2}u)^3=(\tfrac{1}{2}b)^3\frac{x}{l} \quad . \quad (3)'$$

Uniform Load. Width constant. Vertical outline triangular.	Fig. 291, (*a*).	$(\frac{1}{2}v)=(\frac{1}{2}h)\frac{x}{l}$	(4)
Uniform Load. Height constant. Horiz. outline is two parabolas meeting at *O* (vertex) with geomet. axes ‖ to wall.	Fig. 291, (*b*).	$\frac{1}{2}u=(\frac{1}{2}b)\frac{x^2}{l^2}$	(5)
Uniform Load. Both outlines semicubic parabolas. Sections similar rectangles.	Fig. 291, (*c*).	$(\frac{1}{2}u)^3=(\frac{1}{2}b)^3\frac{x^2}{l^2}$	(6)
		$(\frac{1}{2}v)^3=(\frac{1}{2}h)^3\frac{x^2}{l^2}$	(6)′

289.—Beams and cantilevers of circular cross-sections may be dealt with similarly, and the proper longitudinal outline given, to constitute them "bodies of uniform strength." As a consequence of the possession of this property, with loading and mode of support of specified character, the following may be stated; that to find the equation of safe loading *any cross-section whatever may be employed.* This refers to tension and compression. As regards the shearing stresses in different parts of the beam the condition of "uniform strength" is not necessarily obtained at the same time with that for normal stress in the outer fibres.

DEFLECTION OF BEAMS OF UNIFORM STRENGTH.

290. Case of § 283, the double wedge, **but symmetrical,** i.e., $l_1=l_0=\frac{1}{2}l$, Fig. 292. Here we shall find the use of the

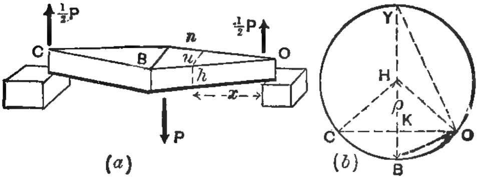


FIG. 292.

form $\frac{EI}{\rho}$ (of the three forms for the moment of the *stress couple*, see eqs. (5), (6) and (7), §§ 229 and 231) of the most direct service in determining the form of the elastic curve *OB*, which is symmetrical, and has a common tangent at *B*, with the curve *BC*. First to find the radius of curvature, ρ, at any section *n*, we have for the free body *nO*, $\Sigma(\text{moms.}_n=0)$, whence

$$-\frac{EI}{\rho}+\tfrac{1}{2}Px=0\text{; but}\left\{\begin{array}{c}\text{from eq.}\\ (3)\ \S\ 283\end{array}\right\}\ x=\frac{u}{b}\tfrac{1}{2}l\text{ and }I=\tfrac{1}{12}uh^3$$

$$\text{we have }\tfrac{1}{12}\frac{E}{\rho}uh^3=\tfrac{1}{4}P\frac{ul}{b}\text{ and }\therefore\ \rho=\tfrac{1}{3}\frac{bh^3}{l}\cdot\frac{E}{P}\qquad(1)$$

from which all variables have disappeared in the right hand member; i.e., ρ is **constant**, the same at all points of the elastic curve, hence the latter is the arc of a circle, having a horizontal tangent at *B*.

To find the deflection, *d*, at *B*, consider Fig. 292, (*b*) where $d=\overline{KB}$, and the full circle of radius $\overline{BH}=\rho$ is drawn.

The triangle *KOB* is similar to ***YOB***,

and ∴ $\overline{KB}:\overline{OB}::\overline{OB}:\overline{YB}$

But $OB=\tfrac{1}{2}l$, $KB=d$ and $YB=2\rho$

$$\therefore\ d=\frac{(\tfrac{1}{2}l)^2}{2\rho}\text{, and }\therefore\text{, from eq. (1), }d=\tfrac{3}{8}\cdot\frac{Pl^3}{bh^3E}\qquad(2)$$

From eq. (4) §233 we note that for a beam of the same material but *prismatic* (parallelopipedical in this case,) having the same dimensions, *b* and *h*, at all sections as at the middle, deflects an amount $=\frac{1}{48}\frac{Pl^3}{EI}=\tfrac{1}{4}\frac{Pl^3}{bh^3E}$ under a load P in the middle of the span. Hence the tapering beam of the present § has only $\frac{2}{3}$ the stiffness of the prismatic beam, for the same b, h, l, E, and P.

291. Case of § 281 (Parabolic Body), With $l_1=l_0$, i.e., Symmetrical.—Fig. 293,(*a*). Required the equation of the neutral line OB. For the free body nO, $\Sigma(\text{moms.}_n)=0$ gives us

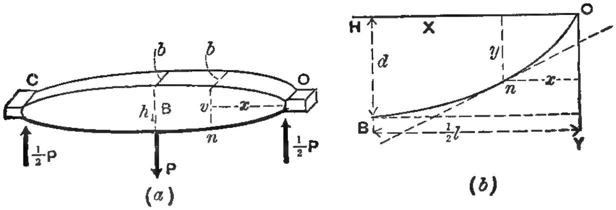


FIG. 293.

$$EI\frac{d^2y}{dx^2}=-\tfrac{1}{2}Px \quad . \quad . \quad . \quad (1)''$$

Fig. 293, (*b*), shows simply the geometrical relations of the problem, position of origin, axes, etc. OnB is the neutral line or elastic curve whose equation, and greatest ordinate d, are required. (The right hand member of eq. (1)″ is made negative because $d^2y \div dx^2$ is negative, the curve being concave to the axis X in this, the first quadrant.)

Now if the beam were prismatic, I, the "moment of inertia" of the cross-section would be constant, i.e., the same for all values of x, and we might proceed by taking the x-anti-derivative of each member of (1)″ and add a constant; but it is *variable* and is

$$=\tfrac{1}{12}b\,v^3=\tfrac{1}{12}\cdot\frac{bh_1^3}{(\frac{1}{2}l)^{\frac{3}{2}}}\,x^{\frac{3}{2}}, \quad \text{(from eq. 3, § 281, putting } l_0=\tfrac{1}{2}l\text{)}$$

hence (1)″ becomes

$$\tfrac{1}{12}E\frac{bh_1^3}{(\frac{1}{2}l)^{\frac{3}{2}}}\,x^{\frac{3}{2}}\cdot\frac{d^2y}{dx^2}=-\tfrac{1}{2}Px \quad . \quad . \quad . \quad . \quad . \quad (1)''$$

To put this into the form Const. $\times\frac{d^2y}{dx^2}=$ func. of (x), we need only divide through by $x^{\frac{3}{2}}$, (and for brevity denote $\frac{1}{12}Ebh_1^3 \div (\frac{1}{2}l)^{\frac{3}{2}}$ by A) and obtain

$$A\frac{d^2y}{dx^2} = -\tfrac{1}{2}Px^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \qquad (1)$$

We can now take the x-anti-derivative of each member, and have

$$A\frac{dy}{dx} = -\tfrac{1}{2}P(2x^{+\frac{1}{2}}) + C \quad . \quad . \quad . \quad . \qquad (2)'$$

To determine the constant C, we utilize the fact that at B, where $x=\frac{1}{2}l$, the slope $dy \div dx$ is zero, since the tangent line is there horizontal, whence from (2)′

$$0=-P\sqrt{\tfrac{l}{2}}+C \quad \therefore C=P\sqrt{\tfrac{l}{2}}$$

$\therefore$ (2)′ becomes $A\dfrac{dy}{dx}=P[\sqrt{\tfrac{1}{2}l}-x^{\frac{1}{2}}]$ . . . . . . (2)

$$\therefore \qquad Ay=P\,[\sqrt{\tfrac{1}{2}l}.x-\tfrac{2}{3}\,x^{\frac{3}{2}}]+[C'=0] \quad . \quad . \quad . \qquad (3)$$

($C'=0$ since for $x=0$, $y=0$). We may now find the deflection d (Fig. 293(b)) by writing $x=\frac{1}{2}l$ and $y=d$, whence, after restoring the value of the constant A,

$$d=\tfrac{1}{2}\frac{Pl^3}{Ebh_1^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

and is twice as great [being$=2.\dfrac{Pl^3}{4Ebh_1^3}$]* as if the girder were parallelopipedical. In other words, the present girder is only half as stiff as the prismatic one.

292. Special Problem. (I.) The symmetrical beam in Fig. 294 is of rectangular cross-section and constant width $=b$.

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* See § 233, putting $I=\frac{1}{12}bh^3$ in eq. (4).

but the height is constant only over the extreme quarter spans, being $=h_1=\frac{1}{2}h$, i. e., half the height h at mid-span. The convergence of the two truncated wedges forming the middle quarters of the beam is such that the prolongations of the upper and lower surfaces *would meet over the supports* (as should be the case to make $h=2h_1$). Neglecting the weight of the beam, and placing a single load in middle, it is required to find the equation for safe loading; also the equations of the four elastic curves; and finally the deflection.

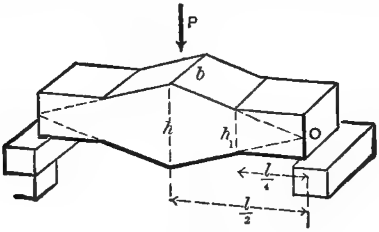

FIG. 294.

The solutions of this and the following problem are left to the student, as exercises. Of course the beam here given is not one of uniform strength.

293. Special Problem. (II). Fig. 295. Required the manner in which the width of the beam must vary, the height being constant, cross-sections rectangular, weight of beam neglected, to be a beam of uniform strength, if the load is uniformly distributed?

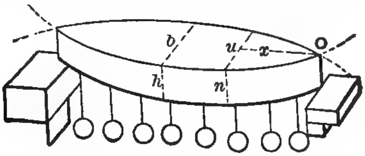

FIG. 295.

Note.—The following five pages originally formed the concluding part of the chapter on " Arch-ribs" of this book; and gives a graphic treatment of straight girders considered as a particular case of curved beams (or arch ribs).

HORIZONTAL STRAIGHT GIRDERS.

389. Ends Free to Turn.—This corresponds to an arch-rib with hinged ends, but it must be understood that there is no hindrance to horizontal motion. (Fig. 439.) In treating a straight beam, *slightly* bent under *vertical forces only* (as in this case with no horizontal constraint), as a

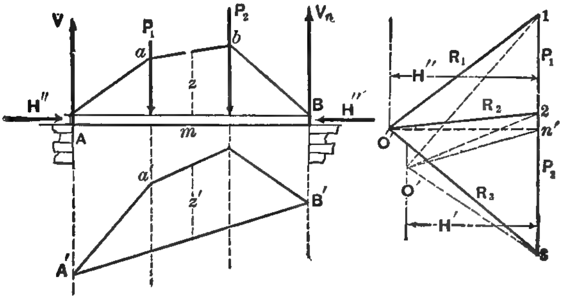

FIG. 439.

particular case of an arch-rib, it is evident that since the pole distance must be zero, the special equil. polygon will have all its segments vertical, and the corresponding force diagram reduces to a single vertical line (the load line). The first and last segments must pass through A and B (points of no moment) respectively, but being vertical will not intersect P_1 and P_2; i.e., the remainder of the special equilibrium polygon *lies at an infinite distance* above the span AB. Hence the actual spec. equil. pol. is useless.

However, knowing that the shear, J, and the moment M (of stress couple) are the only quantities pertaining to any section m (Fig. 439) which we wish to determine (since there is no thrust along the beam), and knowing that an imaginary force H'', applied horizontally at each end of the beam, would have no influence in determining the shear and moment at m as due to the new system of forces, we may therefore obtain the shears and moments graphically from *this new system* (viz.: the loads P_1, etc.; the vertical reactions V and V_n, and the two equal and opposite H'''s). [Evidently, since H' has no moment about the neutral axis (or gravity axis here), of m, the moment at m will be unaffected by it; and since H'' has no component ┐ to the beam at m, the shear at m is the same in the new system of forces, as in the old, before the introduction of the H'''s.]

Hence, lay off the load-line 1 . . 2 . . 3, Fig. 439, and construct an equil. polyg. which shall pass through A and B and have any convenient arbitrary H'' (force) as a pole distance. This is done by first determining n' on the load-line, using the auxiliary polygon $A'a'B'$, to a pole O' (arbitrary), and drawing $O'n'$ ‖ to $A'B'$. Taking O'' on a horizontal through n', making $O''n'=H''$, we complete the force diagram, and equil. pol. AaB. Then, *z being the vertical intercept* between m and the equil. polygon, we have: **Moment at** $m=M=H''z$ (or$=H'z'$ also), and shear at m, or J,$=$**2 . .** n', i.e., $=$ projection of the proper ray R_2, or O'' . . **2,** upon the vertical through m. Similarly we obtain M and J at any other section for the given load. (See

§§ 329, 337 and 367). The moment of inertia need not be constant in this case.

390. Straight Horizontal Prismatic Girder of Fixed Ends at Same Level.—No horizontal constraint, hence no thrust. I constant. Ends at same level, with end-tangents horizontal. We may consider the whole beam free (cutting close to the walls) putting in the unknown upward shears J_0 and J_n and the two stress couples of unknown moments M_0 and M_n at these end sections. Also, as in § 388, an arbitrary H'' horizontal and in line of beam at each extremity. Now (See Fig. 33) the couple at O and the force H'' are equivalent to a single horizontal H'' at an unknown vertical distance c below O; similarly at the right hand end. The special polygon FG is to be determined for this new system, since the moment and shear will be the same at any section under this new system as under the real system. The conditions for determining it are as follows: Since the end-tangents are fixed, $\Sigma M\Delta s=0$ ∴ $\Sigma_0^B z\Delta s=0$ and since

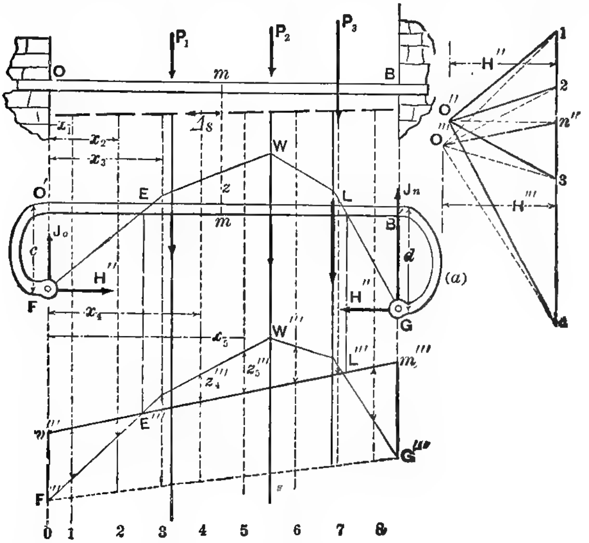

FIG. 440.

O's displacement relatively to B's tangent is zero we have $\Sigma Mx\Delta s=0$ $\therefore$ $\Sigma H''zx\Delta s=0$ $\therefore$ $\Sigma xz\Delta s=0$. See § 374. Hence for **Equal** Δs's, $\Sigma(z)=0$ and $\Sigma(xz)=0$. Now for any pole O''' draw an equil. pol. $F'''G'''$ and in it (by § 377; see Note) locate $v'''m'''$ so as to make $\Sigma(z''')=0$ and $\Sigma(xz''')=0$. Draw verticals through the intersections E''' and L''', to determine E and L on the beam, these are the points of inflection (i.e., of zero moment), and are points in the required special polygon FG.

Draw $O'''n''$ ∥ to $v'''m'''$ to fix n''. Take a pole O'' on the horizontal through n'', making $O''n''=H''$ (arbitrary), draw the force diagram O'' 1234 and a corresponding equilibrium polygon beginning at E. It should cut L, and will fulfil the two requirments $\Sigma_0^B(z)=0$ and $\Sigma_0^B(xz)=0$, with reference to the axis of the beam $O'B'$. The moment of the stress-couple at any section m will be $M=H''z$, and the shear J = the projection of the "proper ray" of the force diagram O'' . . 1, 2, etc., upon the vertical (not in the trial diagram O'''. . 1, 2, etc.). As far as the moment is concerned the trial polygon $F'''G'''$ will serve as well as the special polygon FG; i.e., $M=H'''z'''$ as well as $H''z$, H''' being the pole-distance of O'''; but for the shear we must use the rays of the final and not the trial diagram.

The peculiarity of this treatment of straight beams, considered as a particular case of curved beams, consists in the substitution of an imaginary system of forces involving the two equal and opposite, and *arbitrary* H's, for the real system in which there is no horizontal force and consequently no "special equilibrium polygon," and thus determining all that is desired, i.e., the moment and shear at any section.

In the polygon FG the student will recognize the "moment-diagram" of the problems in Chaps. III and IV.

He will also see why the shear is proportional to the slope $\frac{dM}{dx}$ of the moment curve in those chapters. For example, the "slope" of the second segment of the polygon FG, that segment being ∥ to O'' 2, is tang. of angle $2O''n''=\overline{2n''}\div\overline{O''n''}$=shear $\div$ H''

and similarly for any other segment; i.e., the tangent of the inclination of the "moment curve," or line, is proportional to the shear.

It is also interesting to notice with the present problem of a straight beam, that in the conditions

$$\Sigma(z\Delta s)=0 \text{ and } \Sigma(z\Delta s)x=0,$$

or locating the polygon FG, each Δs is ┐ to its z, and that consequently each $z\Delta s$ is the area of a small vertical strip of area between the beam and the polygon, and $(z\Delta s)x$ is the "moment" of this strip of area, about O' the origin of x. Hence these conditions imply; *first*, that the area EWL between the polygon and the axis of the beam on one side is equal to that $(O'FE+LB'G)$ on the other side, and, *secondly*, that the centre of gravity of EWL lies in the same vertical as that of $O'FE$ and $LB'G$ combined. Another way of stating the same thing is that, if we join FG, the area of the trapezoid $FO'B'G$ is equal to that of the figure $FEWLG$, and their centres of gravity lie in the same vertical. A corresponding statement may be made (if we join $F'''G'''$) for the trapezoid $F'''v'''m'''G'''$ and figure $F'''E'''W'''L'''G'''$.

LOGARITHMS (BRIGGS').

N	0	1	2	3	4	5	6	7	8	9	Dif.
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	19
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	9
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8

N. B.—Naperian log = Briggs' log × 2.302.
Base of Naperian system = e = 2.71828.

LOGARITHMS (BRIGGS').

N	0	1	2	3	4	5	6	7	8	9	Dif.
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

N. B.—Naperian log = Briggs' log × 2.302.
Base of Naperian System = *e* = 2.71828.

Trigonometric Ratios (Natural); including "*arc*," by which is meant "radians," or "π-measure," or "circular measure;" e.g., arc 100° = 1.7453293, $= \frac{100}{180}$ of π.

arc	de-gree	sin	cosec	tan	cotan	sec	cos		
0	0	0	infin.	0	infin.	1.0000	1.0000	90	1.5708
0.0175	1	0.0175	57.299	0.0175	57.290	1.0001	0.9998	89	1.5533
.0349	2	.0349	28.654	.0349	28.636	1.0006	.9994	88	1.5359
.0524	3	.0523	19.107	.0524	19.081	1.0014	.9986	87	1.5184
.0698	4	.0698	14.336	.0699	14.301	1.0024	.9976	86	1.5010
.0873	5	.0872	11.474	.0875	11.430	1.0038	.9962	85	1.4835
0.1047	6	0.1045	9.5668	0.1051	9.5144	1.0055	0.9945	84	1.4661
.1222	7	.1219	8.2055	.1228	8.1443	1.0075	.9925	83	1.4486
.1396	8	.1392	7.1853	.1405	7.1154	1.0098	.9903	82	1.4312
.1571	9	.1564	6.3925	.1584	6.3138	1.0125	.9877	81	1.4137
.1745	10	.1736	5.7588	.1763	5.6713	1.0154	.9848	80	1.3963
0.1920	11	0.1908	5.2408	0.1944	5.1446	1.0187	0.9816	79	1.3788
.2094	12	.2079	4.8097	.2126	4.7046	1.0223	.9781	78	1.3614
.2269	13	.2250	4.4454	.2309	4.3315	1.0263	.9744	77	1.3439
.2443	14	.2419	4.1336	.2493	4.0108	1.0306	.9703	76	1.3264
.2618	15	.2588	3.8637	.2679	3.7321	1.0353	.9659	75	1.3090
0.2793	16	0.2756	3.6280	0.2867	3.4874	1.0403	0.9613	74	1.2915
.2967	17	.2924	3.4203	.3057	3.2709	1.0457	.9563	73	1.2741
.3142	18	.3090	3.2361	.3249	3.0777	1.0515	.9511	72	1.2566
.3316	19	.3256	3.0716	.3443	2.9042	1.0576	.9455	71	1.2392
.3491	20	.3420	2.9238	.3640	2.7475	1.0642	.9397	70	1.2217
0.3665	21	0.3584	2.7904	0.3839	2.6051	1.0712	0.9336	69	1.2043
.3840	22	.3746	2.6695	.4040	2.4751	1.0785	.9272	68	1.1868
.4014	23	.3907	2.5593	.4245	2.3559	1.0864	.9205	67	1.1694
.4189	24	.4067	2.4586	.4452	2.2460	1.0946	.9135	66	1.1519
.4363	25	.4226	2.3662	.4663	2.1445	1.1034	.9063	65	1.1345
0.4538	26	0.4384	2.2812	0.4877	2.0503	1.1126	0.8988	64	1.1170
.4712	27	.4540	2.2027	.5095	1.9626	1.1223	.8910	63	1.0996
.4887	28	.4695	2.1301	.5317	1.8807	1.1326	.8829	62	1.0821
.5061	29	.4848	2.0627	.5543	1.8040	1.1434	.8746	61	1.0646
.5236	30	.5000	2.0000	.5774	1.7321	1.1547	.8660	60	1.0472
0.5411	31	0.5150	1.9416	0.6009	1.6643	1.1666	0.8572	59	1.0297
.5585	32	.5299	1.8871	.6249	1.6003	1.1792	.8480	58	1.0123
.5760	33	.5446	1.8361	.6494	1.5399	1.1924	.8387	57	0.9948
.5934	34	.5592	1.7883	.6745	1.4826	1.2062	.8290	56	0.9774
.6109	35	.5736	1.7435	.7002	1.4281	1.2208	.8192	55	0.9599
0.6283	36	0.5878	1.7013	0.7265	1.3764	1.2361	0.8090	54	0.9425
.6458	37	.6018	1.6616	.7536	1.3270	1.2521	.7986	53	0.9250
.6632	38	.6157	1.6243	.7813	1.2799	1.2690	.7880	52	0.9076
.6807	39	.6293	1.5890	.8098	1.2349	1.2868	.7771	51	0.8901
.6981	40	.6428	1.5557	.8391	1.1918	1.3054	.7660	50	0.8727
0.7156	41	0.6561	1.5243	0.8693	1.1504	1.3250	0.7547	49	0.8552
7330	42	.6691	1.4945	.9004	1.1106	1.3456	.7431	48	0.8378
7505	43	.6820	1.4663	.9325	1.0724	1.3673	.7314	47	0.8203
.7679	44	.6947	1.4396	.9657	1.0355	1.3902	.7193	46	0.8028
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